First differential approximation for ODE systems with parameters

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Abstract. Using the example of Jacobi oscillator, the first differential approximation for various numerical methods for systems of ordinary differential equations is constructed. Computational experiments show a good coincidence of the residuals on the first integrals of the system and their first differential approximations. The presented methods make it possible to carry out effective calculations by means of computer algebra systems.

In the 60s of the last century, N. N. Yanenko [1] formulated a method for differential approximation of a difference scheme. The main idea of this method is to replace the study of the properties of a difference scheme with the study of a problem with differential equations occupying an intermediate position between the original differential problem and its difference scheme. In the works of N. N. Yanenko and his students, as a result, concepts such as the approximation viscosity of the difference scheme and the first differential approximation (FDA) of the difference scheme were formulated.

The paper considers systems of ordinary differential equations (ODE) depending on the parameters. Examples of such systems can be both systems with parameters and systems with the first integrals. The first integrals retain their values on the solutions of the original system and these values can be considered as parameters of the original system.

There are a large number of numerical methods for solving ODE. The FDA allows to obtain information about the quality of the selected numerical method for a specific system using only symbolic calculations.

We consider a first-order ODE system resolved with respect to the first derivatives and present an algorithm to calculate the FDA. The algorithm is a set of simple operations with formal power series.

Consider Jacobi oscillator as an example. By definition, the Jacobi functions $p = \operatorname{sn} t$, $q = \operatorname{cn} t$, $r = \operatorname{dn} t$ are a particular solution of a nonlinear autonomous system with the initial conditions are $p = 0$, $q = r = 1$, $k = 1/2$ for $t = 0$. This

autonomous system has two quadratic integrals of motion:

$$\begin{cases} p_t - qr = 0, \\ q_t + pr = 0, \\ r_t + k^2 pq = 0, \\ p^2 + q^2 = C_1, \\ k^2 p^2 + r^2 = C_2. \end{cases} \tag{1}$$

Consider the original Runge–Kutta method of 4 orders for Jacobi oscillator (1)

$$\begin{cases} \frac{\widehat{p}-p}{h} - qr + h\left(\frac{k^2pq^2}{2} + \frac{pr^2}{2}\right) + h^2\left(-\frac{k^2qr(2p-q)(2p+q)}{6} + \frac{qr^3}{6}\right) + \ldots = 0, \\ \frac{\widehat{q}-q}{h} + pr + h\left(-\frac{k^2p^2q}{2} + \frac{qr^2}{2}\right) + h^2\left(\frac{k^2pr(p-2q)(p+2q)}{6} - \frac{pr^3}{6}\right) + \ldots = 0, \\ \frac{\widehat{r}-r}{h} + k^2pq + h\left(\frac{k^2r(q-p)(p+q)}{2}\right) + h^2\left(\frac{k^4pq(p-q)(p+q)}{6} - \frac{2k^2pqr^2}{3}\right) + \ldots = 0. \end{cases} \tag{2}$$

Here, for the compactness of formulas, p, q, r are denoted by $p(t), q(t), r(t)$, $\widehat{p}, \widehat{q}, \widehat{r}$ are denoted by $p(t+h), q(t+h), r(t+h)$, and according to the degree of h, the first two terms are given. Let's add the first two integrals to the system (2).

$$\begin{cases} \dfrac{(\widehat{p}^2 + \widehat{q}^2) - (p^2 + q^2)}{h} = 0, \\ \dfrac{(k^2\widehat{p}^2 + \widehat{r}^2) - (k^2p^2 + r^2)}{h} = 0, \end{cases} \tag{3}$$

Substitute Taylor series expansion at the point t for the functions $p(t), q(t), r(t), p(t+h), q(t+h), r(t+h)$ in the system (2), (3)

$$\begin{cases} p_t - qr + h\left(\frac{k^2pq^2}{2} + \frac{pr^2+p_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ q_t + pr + h\left(-\frac{k^2p^2q}{2} + \frac{qr^2+q_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ r_t + k^2pq + h\left(-\frac{k^2r(p-q)(p+q)}{2} + \frac{r_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ 2\left(pp_t + qq_t\right) + h\left(pp_{tt} + p_t^2 + qq_{tt} + q_t^2\right) + \mathcal{O}(h^2) = 0, \\ 2k^2pp_t + 2rr_t + h\left(k^2\left(pp_{tt} + p_t^2\right) + rr_{tt} + r_t^2\right) + \mathcal{O}(h^2) = 0. \end{cases} \tag{4}$$

To obtain the correct order of the Runge-Kutta method, it is necessary to construct the FDA. The system (4) forms a differential ideal [2] with respect to the operations of addition, multiplication and differentiation of series equal to zero. The form of the series(4), resolved with respect to the first derivatives, allows you to replace the derivatives of functions in coefficients at h through the functions themselves.

It is possible to formulate the following FDA construction algorithm for Runge–Kutta methods, which ends when the first nonzero term in powers of h is expressed only through the functions themselves. As a consequence, the FDA view does not depend on the decomposition point in the Taylor series and is a canonical form.

In our talk, a program to construct the FDA for given ODE system will be presented. This program written in the SymPy computer algebra system and

available at `https://github.com/blinkovua/sharing-blinkov/tree/master/FDA_ODE`. For presented example, it give us FDA in the form

$$\begin{cases} p_t - qr + h^4\left(-\frac{k^4qr\left(3p^4+3p^2q^2-2q^4\right)}{240} - \frac{k^2qr^3(p-q)(p+q)}{80} + \frac{qr^5}{120}\right) + \mathcal{O}(h^5) = 0, \\ q_t + pr + h^4\left(-\frac{k^4pr\left(2p^4-3p^2q^2-3q^4\right)}{240} + \frac{k^2pr^3(p-q)(p+q)}{80} - \frac{pr^5}{120}\right) + \mathcal{O}(h^5) = 0, \\ r_t + k^2pq + h^4\left(-\frac{k^6pq\left(2p^4-3p^2q^2+2q^4\right)}{240} + \frac{k^4pqr^2(p-q)(p+q)}{80} + \frac{k^2pqr^4}{80}\right) + \mathcal{O}(h^5) = 0, \\ h^4\left(\frac{k^4pqr(p-q)(p+q)\left(p^2+q^2\right)}{24}\right) + \mathcal{O}(h^5) = 0, \\ h^4\left(\frac{k^2pqr(kp-r)(kp+r)\left(k^2p^2+r^2\right)}{24}\right) + \mathcal{O}(h^5) = 0. \end{cases} \tag{5}$$

Acknowledgments. This work is supported by the Russian Science Foundation (grant no. 20-11-20257).

References

[1] Yanenko N. N., Shokin Yu. I. The first differential approximation of difference schemes for hyperbolic systems of equations. Siberian Mathematical Journal, 1969, vol. 10, no. 5, pp. 868–880. https://doi.org/10.1007/BF00971662

[2] Kolchin E. R. Differential Algebra and Algebraic. Academic Press, New York, 1973. 446 P.

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