Constructive Versions of Quantum Mechanics

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Abstract. The standard formulation of quantum mechanics is essentially non-constructive, since it is based on continuous unitary groups and number fields $\mathbb{R}$ and $\mathbb{C}$. This descriptive flaw does not allow one to study some fine details of the structure of quantum systems and sometimes leads to artifacts.

In [1–3], we considered a modification of quantum mechanics based on permutation representations of finite groups in Hilbert spaces over cyclotomic fields. This *permutation quantum mechanics* (PQM) "can accurately reproduce all of the results of conventional quantum mechanics" [4] in the permutation invariant *standard subspace* of the Hilbert space. Unitary evolution in PQM is generated by a permutation of *ontic* elements, which form a basis of the Hilbert space. By decomposing the permutation into a product of disjoint cycles, we can split the Hilbert space into a direct sum of subspaces, in each of which the evolution generated by a cyclic permutation occurs independently. Thus, in an N-dimensional Hilbert space, it suffices to consider the evolutions generated by cycles of length N. Such a cycle generates the group $\mathbb{Z}_N$. Since any projective representation of a cyclic group is trivial, to describe quantum mechanical phenomena it is necessary to consider the product $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$, where $\tilde{\mathbb{Z}}_N \,(\simeq \mathbb{Z}_N)$ is the Pontryagin dual group to $\mathbb{Z}_N$. Note that we have only changed the description slightly, without introducing any additional external information: if X and Z are matrices representing generators of $\mathbb{Z}_N$ and $\tilde{\mathbb{Z}}_N$, respectively, then Z is simply the diagonal form of X, obtained by the Fourier transform.

In fact, we have come to the Weyl–Schwinger version of quantum mechanics, which is sometimes called *finite quantum mechanics* (FQM). FQM arose as a result of Weyl's correction of Heisenberg's canonical commutation relation, which cannot be realized in finite-dimensional Hilbert spaces. Weyl's canonical commutation relation has the form

$$XZ = \omega ZX,\ \omega = \mathrm{e}^{2\pi\mathbf{i}/N},$$

where X and Z are the matrices mentioned above. Weyl proved that the X and Z are generators of a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ in the N-dimensional Hilbert space. The orthonormal bases associated with the matrices X and Z are *mutually unbiased bases*, a concept introduced by Schwinger.

FQM, constructive by its nature, requires mathematical tools that differ significantly from those used in traditional continuous theory: number theory, Galois field theory, complex Hadamard matrices, finite geometries, etc.

At the same time, in FQM, it is possible to pose and solve problems that are important for fundamental quantum theory and quantum informatics, but which are difficult or even impossible to formulate within the framework of standard quantum mechanics. Let us give examples of problems in which the structure of the decomposition of the dimension of the Hilbert space into prime numbers is essential, which does not make sense in continuous quantum mechanics:

- decomposition of a quantum system into smaller subsystems;
- calculation of sets of mutually unbiased bases (sets of orthonormal bases in Hilbert space, measurements in which give maximum information about the quantum state);
- construction of symmetric information-complete positive operator-valued measures (SIC-POVM, a symmetric set of vectors in a Hilbert space, important for quantum measurement theory and related to Hilbert's 12th problem).

Modern problems of quantum physics and quantum informatics require a detailed analysis of the "fine structure" of quantum systems, which cannot be carried out using traditional approximate methods of quantum mechanics. However, exact methods are complex and often involve open (unsolved) mathematical problems. In these circumstances, a natural approach is to use computer calculations based on the methods of computer algebra and computational group theory.

References

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