Constructive Versions of Quantum Mechanics

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Quantum mechanics in general

Starting point: energy = frequency $\boxed{E = h\nu}$

- State $\begin{cases} \text{pure: } |\psi\rangle, & \text{unit vector (ray) in complex Hilbert space} \\ \text{mixed: } \rho, & \text{Hermitian trace-one non-negative matrix} \end{cases}$

- Observable: operator $\mathcal{O}$

$$\begin{cases} \mathcal{O} = \mathcal{O}^*, & \text{Hermitian, real eigenvalues, standard in physics} \\ \mathcal{O}^{-1} = \mathcal{O}^*, & \text{unitary, eigenvalues on unit circle, Weyl, Schwinger,}\ldots \\ \mathcal{O}\mathcal{O}^* = \mathcal{O}^*\mathcal{O}, & \text{normal, unitarily diagonalizable, most general case} \end{cases}$$

- Evolution: cyclic group $U_t = \mathrm{e}^{-\mathrm{i}\frac{H}{\hbar}t} = \left(\mathrm{e}^{-\mathrm{i}\frac{H}{\hbar}}\right)^t$

$$|\psi_t\rangle = U_t\,|\psi_0\rangle$$

- Observation (measurement)

  $\mathrm{Prob} = \mathrm{tr}(\rho_{\mathrm{meter}}\rho_{\mathrm{system}})$, Born rule $\Longleftarrow$ Gleason theorem

  $\mathrm{tr}(A^*B) \equiv \langle \mathrm{vec}(A)\,|\,\mathrm{vec}(B)\rangle$ is the Frobenius inner product

States in permutation quantum mechanics

Standard space in permutation representation

- $\Omega = \{e_1, \ldots, e_{\mathcal{N}}\}$, ontic elements
- $\mathcal{H}_{\mathrm{ont}} = \mathrm{span}\{e_1, \ldots, e_{\mathcal{N}}\}$, ontic Hilbert space
- $\mathcal{P}(\mathsf{S}_{\mathcal{N}})$, permutation representation of symmetric group
- $\mathcal{H}_{\mathrm{ont}} = \mathcal{H}_{\mathrm{triv}} \oplus \mathcal{H}_{\mathrm{std}}$, invariant decomposition

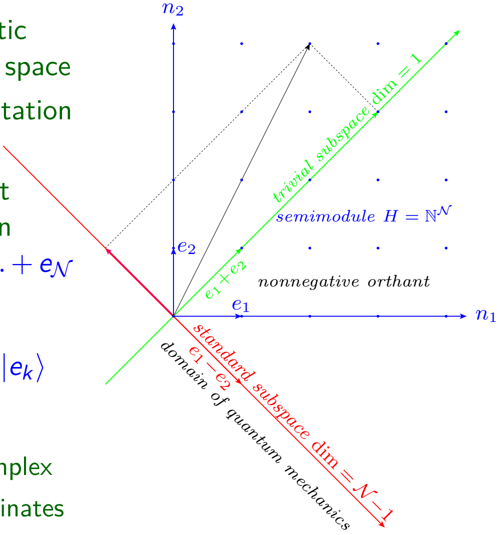
  $\mathcal{H}_{\mathrm{triv}} = \mathrm{span}\{\mathbb{e}\}$, $\mathbb{e} = e_1 + \ldots + e_{\mathcal{N}}$
- $\mathcal{H}_{\mathrm{ont}} \xrightarrow{\mathrm{P}_\star} \mathcal{H}_{\mathrm{std}}$, projection

  $\mathrm{P}_\star = \mathbb{1} - \frac{|\mathbb{e}\rangle\langle\mathbb{e}|}{\mathcal{N}}$, $|k_\star\rangle = \frac{1}{\sqrt{2}}\mathrm{P}_\star|e_k\rangle$
- $\|k_\star\|^2 = \frac{1}{2} - \frac{1}{2\mathcal{N}}$

  $\|k_\star - \ell_\star\|^2 = 1 \Longrightarrow$ regular simplex

  $\{|k_\star\rangle\}$, affine barycentric coordinates

  $\{|k_\star\rangle\} \cup \{\mathrm{P}_\star|\mathbb{e}\rangle\}$, projective frame



Quotes in support of the permutation QM idea I

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Finite Deformations of Quantum Mechanics

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Abstract

We investigate modifications of quantum mechanics (QM) that replace the unitary group in a finite dimensional Hilbert space with a finite group and determine the minimal sequence of subgroups necessary to approximate QM arbitrarily closely for general choices of Hamiltonian. This mathematical study reveals novel insights about 't Hooft's Ontological Quantum Mechanics, and the derivation of statistical mechanics from quantum mechanics. We show that Kornyak's proposal to understand QM as classical dynamics on a Hilbert space of one dimension higher than that describing the universe, supplemented by a choice of the value of a naturally conserved quantum operator in that classical evolution, can probably be a model of the world we observe.

Quotes in support of the permutation QM idea II

Gerard 't Hooft:

> *We postulate the existence of an ontological basis.*
> *It is an orthonormal basis of Hilbert space that is truly superior to the basis choices that we are familiar with. In terms of an ontological basis, the evolution operator for a sufficiently fine mesh of time variables, does nothing more than permute the states.*
>
> p. 66 in The Cellular Automaton Interpretation of Quantum Mechanics
> Springer, 2016

Decomposition of permutation evolution

$\mathsf{S}_{\mathcal{N}} \ni g = (c_1)\,(c_2)\cdots(c_K)$, permutation is a product of disjoint cycles

$$\Downarrow$$

$$\mathcal{P}(g) = \mathcal{P}(c_1) \oplus \mathcal{P}(c_2) \oplus \cdots \oplus \mathcal{P}(c_K) = \begin{pmatrix} \mathcal{P}(c_1) & 0 & \cdots & 0 \\ 0 & \mathcal{P}(c_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{P}(c_K) \end{pmatrix}$$

Single cycle generator of unitary evolution

$c \equiv c_N = (0, 1, \dots, N-1)$, cyclic permutation of length N

- $\mathcal{P}(c) = X = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$, Sylvester's "shift matrix"

  $\mathcal{B}_X = (|0\rangle, \dots, |N-1\rangle)$, ontic basis

- $F\mathcal{P}(c)\,F^{-1} = Z = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix}$, Sylvester's "clock matrix"

  $\mathcal{B}_Z = (|Z; 0\rangle, \dots, |Z; N-1\rangle) \equiv \mathcal{B}_X F^{-1}$, energy basis

  $F = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \cdots & \omega^{-(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \cdots & \omega^{-(N-1)(N-1)} \end{pmatrix}$, Fourier transform

  $\omega = e^{2\pi \mathbf{i}/N}$, Nth base primitive root of unity (algebraic integer)

$\mathcal{B}_X$ and $\mathcal{B}_Z$ are mutually unbiased bases (MUBs)

$$|Z;\ell\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle\, \omega^{k\ell}$$

$$\Downarrow$$

$$\langle Z;\ell\,|\,k\rangle = \frac{1}{\sqrt{N}} \omega^{-k\ell}$$

$$\Downarrow$$

$$\left|\langle Z;\ell\,|\,k\rangle\right|^2 = \frac{1}{N}$$

$\mathcal{B}_X$ and $\mathcal{B}_Z$ and path integral formulation of QM

- path contribution $\mathrm{e}^{\mathbf{i}S} \sim \mathrm{e}^{\mathbf{i}\int L dt} \rightsquigarrow \mathrm{e}^{\mathbf{i}L}$
- evolution operator $\mathrm{e}^{-\mathbf{i}H}$
- inverse Legendre transform

$$L = p\dot{q} - H \;\rightsquigarrow\; pq_{t_2} - H - pq_{t_1}$$

$$\mathrm{e}^{\mathbf{i}L} \rightsquigarrow \mathrm{e}^{\mathbf{i}\left(pq_{t_2} - H - pq_{t_1}\right)} = \underbrace{\mathrm{e}^{\mathbf{i}pq_{t_2}}}_{3}\,\underbrace{\mathrm{e}^{-\mathbf{i}H}}_{2}\,\underbrace{\mathrm{e}^{-\mathbf{i}pq_{t_1}}}_{1}$$

1. $\sum_{q_{t_1}} \mathrm{e}^{-\mathbf{i}pq_{t_1}}$, Fourier transform, change basis to p at time t_1, $\mathcal{B}_Z$
2. $\mathrm{e}^{-\mathbf{i}H}$, evolution in time $t_1 \to t_2$, action by Z
3. $\sum_{p} \mathrm{e}^{\mathbf{i}pq_{t_2}}$, inverse Fourier transform, change basis back to $\mathcal{B}_X$

Cycle structure and decomposition of a quantum system

- $\mathbb{Z}_N \simeq \langle c_N \rangle$
- $\mathbb{Z}_{n\ell} \simeq \mathbb{Z}_n \times \mathbb{Z}_\ell$ if $\gcd(n, \ell) = 1 \longrightarrow \mathcal{H}_{n\ell} \simeq \mathcal{H}_n \otimes \mathcal{H}_\ell$
- $N = p_1^{m_1} p_2^{m_2} \cdots p_K^{m_K} \longrightarrow G = \mathbb{Z}_{p_1^{m_1}} \times \mathbb{Z}_{p_2^{m_2}} \times \cdots \times \mathbb{Z}_{p_K^{m_K}}$

$$G \simeq \begin{cases} \mathbb{Z}_N, & \text{if all primes } p_1, \ldots, p_K \text{ are distinct} \\ A, & \text{an arbitrary abelian group otherwise:} \\ & \text{the fundamental theorem of finite abelian groups} \end{cases}$$

Elementary unit of study $\mathcal{H}_{p^m}$

The complementarity principle

Niels Bohr:

> *. . . for objective description and harmonious comprehension it is necessary in almost every field of knowledge to pay attention to circumstances under which evidence is obtained.*
>
> Atomic Theory and the Description of Nature.

- $N^2 - 1$ real numbers describe general quantum state: $\rho = \rho^*$, $\operatorname{tr} \rho = 1$
- $N - 1$ parameters can be given by one measuring setup: $p_1 + \ldots + p_N = 1$
- $N + 1 = (N^2 - 1)\,/\,(N - 1)$ setups may completely restore ρ
- Pair of setups A and B gives maximum information if $P_A = \{0, \ldots, 1, \ldots, 0\} \implies P_B = \{1/N, \ldots, 1/N\}$ and vice versa
  - Mutually unbiased bases (MUBs): $\mathcal{B}_A,\ \mathcal{B}_B$
  - Complementary observables: $\mathcal{O}_A,\ \mathcal{O}_B$

Maximum number of MUBs $N + 1$ is reached at $N = p^m$

Bohr's complementarity and Pontryagin's duality

a general view of the phase space

Phase space in general: $A \times \tilde{A}$

- $\tilde{A} := \mathrm{Hom}(A, \mathrm{U}(1))$, Pontryagin dual to
  locally compact abelian group A

- $A \simeq \tilde{\tilde{A}}$, Pontryagin theorem

- $A \simeq \tilde{A}$, for finite abelian group A (for $\mathbb{R}^n$ too)
  $\Longrightarrow$ finite (and classical) phase space: $A \times \tilde{A} \simeq A \times A$,
  which always has nontrivial
  projective representations or, equivalently, central extensions

Classical phase space: $\mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\text{central extension}} H_{2n+1}$ — Heisenberg group

Minimal example: Klein four-group: $V \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \left(\simeq \mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2\right)$

- Extension by group of roots of unity:

$$\mathbf{1} \to \{1, -1\} \to Q_8 \to V \to \mathbf{1}$$

$$Q_8 = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\}\,, \text{ quaternion group}$$

- 2D projective representation

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \mathrm{spec}(XZ) = \{\mathbf{i}, -\mathbf{i}\}$$

Pauli matrices:

$$\sigma_x = X, \quad \sigma_y = \mathbf{i}XZ, \quad \sigma_z = Z$$

Dirac matrices

$$\begin{aligned} &\gamma^0 = Z \otimes \mathbb{1}_2, \\ &\gamma^1 = -XZ \otimes X,\ \gamma^2 = -\mathbf{i}XZ \otimes XZ,\ \gamma^3 = -XZ \otimes Z \end{aligned}$$

generate representation of the Clifford algebra $\mathrm{Cl}_{1,3}(\mathbb{R})$

Heisenberg-Weyl group and Schwinger unitary basis

- Heisenberg's Canonical Commutation Relation: $[q, p] = \mathbf{i}\hbar$
- Weyl's CCR: $XZ = \omega ZX, \quad \omega = \mathrm{e}^{2\pi \mathbf{i}/N}$
- Heisenberg-Weyl group G_{HW}: $Y_{k\ell m} = \omega^k X^\ell Z^m, \;\; k, \ell, m = 0, 1, \ldots, N-1$

$$Y_{k\ell m}^N = \begin{cases} \mathbb{1}, & \text{if } N \text{ odd}, \\ (-1)^{\ell m}\, \mathbb{1}, & \text{if } N \text{ even}. \end{cases}$$

Clifford group $G_{\mathrm{C}\ell}$: normalizer G_{HW} in $\mathrm{U}(N)$, $\;G_{\mathrm{C}\ell} \simeq \mathrm{SP}(2, \mathbb{Z}_N) \rtimes G_{\mathrm{HW}}$

- Schwinger unitary basis: $\left\{X^\ell Z^m\right\}$

Mathematics associated with Weyl-Schwinger formalism

- complex Hadamard matrices
- projective representations of abelian groups
  and generalized Clifford algebras
- finite affine and projective geometries
- finite fields
- number theory

Problems

- mutually unbiased bases:

  $\left|\langle e_i \,|\, f_j\rangle\right|^2 = \dfrac{1}{N}$; finite projective plane

- SIC-POVM (Symmetric Informationally Complete Positive Operator Valued Measure): $\dfrac{1}{N}\displaystyle\sum_{k=0}^{N^2-1} |a_k\rangle\langle a_k| = \mathbb{1},\ \left|\langle a_k \,|\, a_\ell\rangle\right|^2 = \dfrac{1}{N+1}$; finite affine plane; 12th Hilbert problem

- entanglement in finite QM

- quantum tomography

- quantum mereology

- Wigner and Weyl functions in finite QM

- coherent states in finite quantum systems

All this requires computer algebra and computational group theory

Program for constructing unextendible sets of MUBs

Input:
a set of complex
Hadamard matrices

$\xrightarrow[N=2,3,\ldots,6,\ldots,10,11]{\text{C program, 3.3GHz PC}}$

Output:
unextendible sets
of MUBs

Maximal number of MUBs in problematic dimensions: $\mathfrak{M}_6 = \mathfrak{M}_{10} = 3$

Times of hardest tasks: $T_{10} = 1$ h 23 min 32 sec, $T_{11} = 9$ min 53 sec