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Algebraic methods in Theoretical Mechanics

Semjon Adlaj

By the beginning of the third millennium AD, the fundamental problems of classical mechanics remained unsettled. In accordance with the principle "nothing has been done if something remains to be done", Carl Gauss (regarded as the first mathematician of Germany) delegated the solution to the problem of a (nonlinear) pendulum and the solution to the problem of free rigid body motion to Carl Jacobi (regarded as the second mathematician of Germany). Jacobi's works remained quintessential to all (without a single exception) treatments of these problems in "authoritative" sources on mechanics (including Paul Appell's volumes on Theoretical Mechanics) until the end of the twentieth and the beginning of the twenty-first century. However, the completion of Jacobi's superb works remained impossible due to the traditional lag of Theoretical Mechanics in implementing the (revolutionary) Theory of Evariste Galois (set forth in an immortal manuscript, dated May 29, 1832) and the widespread misconception, concerning the perceived detachment of Galois Theory from the attainment of constructive solutions to the fundamental problems of Classical Mechanics. A precise observation by V.F. Zhuravlev, concerning an "incompleteness of Poinsot geometrical construction" was stubbornly ignored, just as the difficulties (specified by him) in solving the problem of free rigid body motion were "passed with silence". Now comes the time for a broad identification of such difficulties, because (as Galois noted) "an author never does more harm than hide a difficulty", and to demonstrate the ripened need for an algebraic approach to achieve an exhaustive and necessarily exact solution (which has nothing to do with the so-called "approximate solutions", which, strictly speaking, are not at all solutions) of the problems, being discussed.

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On the topology and singularity theory of functions on two-dimensional sphere

Alekseev Mark

Introduction

Let be a locally trivial bundle of smooth compact manifolds $F \hookrightarrow W \to M$ and smooth general function $f : W \to \mathbb{R}$ in total space. The purpose of this work is to relate the topological characteristics of the bundle, which serve as an obstruction to its triviality, with the types of degeneracies of the critical points of the restriction of the function to fibers. If the bundle is trivial, $W = M \times F$, then there is such a function f, whose restriction on each fiber is Morse, for example, we can take fixed Morse function $g\colon F \to \mathbb{R}$ and set $f = g \circ p_2$ where $p_2 : W \to F$ is a natural projection to the second factor. If the bundle $W \to M$ is nontrivial then we can expect the existence of fibers where restriction of f will have degeneracy. This connection may be used in two directions: on the one hand, by studying the singularities for a given function, one can try to build topological obstructions to the triviality of the bundle (i.e. characteristic classes). On the other hand, the nontriviality of the bundle should imply the necessity of the existence of singularities.

In the case when the bundle layer is a circle, the theory describing the correspondence between the characteristic classes of the bundle and the singularities of the function on its layers is constructed in [1]. In this paper, we take the first step in extending this theory to the case when the bundle fiber is a two-dimensional oriented sphere, and the singularities of the global minimum are considered as singularities. We give a construction of the classifying space for this case. It has homotopy type $BSO(3)$, and hence its cohomology groups are known. On the other hand, the space itself is divided into strata that correspond to one-to-one types of degenerations of the global minimum of functions on the sphere. We give a classification of the singularities of the global minimum of functions on a sphere of small codimension, and give a description of the homotopy type of most strata of small codimension as well as two of infinite series of strata. This information is used

to construct a spectral sequence converging to the cohomology of the classifying space.

The main result is description of strata of small codimensions (up to codimension 4) and two infinite series of strata (1)$[1^n]$, 2) $[2k-1]$; 3) $[2k-1,1]$ which means respectively 1) strata of functions with n points of global minima each of them is a singularity of A_1 type; 2) strata of functions with one point of global minima that has a A_{2k-1} type, 3) strata of functions with two points of global minima one of them of type A_{2k-1} and the second of type A_1).

Список литературы

[1] М. Э. Казарян, Относительная теория Морса одномерных расслоений и циклические гомологии, Функц. анализ и его прил., 1997, том 31, выпуск 1, 20–31

[2] В.А. Васильев, Лагранжевы и лежандровы характеристические классы.— М.: МЦНМО, 2000. — 312 с.

[3] М. Э. Казарян, Характеристические классы лагранжевых и лежандровых особенностей, УМН, 1995, том 50, выпуск 4(304), 45–70

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Painleve Test and Integrability of Polynomial ODEs

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Abstract. We calculate Painleve Test of integrability for some integrable polynomial ODEs. Some of them don't pass Painleve Test.

Introduction

The essence of Painleve Test of integrability polynomial ODEs is solving this equations in form of Puiseux series with finite nonzero principal part [1]. We calculate that series for solutions of following integrable ODEs: Lotka-Volterra system [2], Chazy equation [3], Euler-Poisson system ODEs described rigid body motion around a fixed point [4].

Results

For calculations of that Puiseux series we used algorithms and programs described in [5, 6]. Solutions of Lotka-Volterra system and Chazy equation don't have that Puiseux expansions and don't pass Painleve Test. For solutions of Euler-Poisson system that Puiseux expansions exist under certain conditions on the parameters of the system. The set of calculated conditions include all known contitions of Euler-Poisson system integrability. Other calculated conditions point to new integrability case possibility.

Conclusion

If polynomial ODEs dont't pass Painleve Test, then first integrals are complicated than algebraic integrals.

References

[1] M. Tabor, *Chaos and Integrability in Nonlinear Dynamics: An Introduction*, Wiley, 1989.

[2] V.I. Arnold, *Ordinary Differential Equations*, Springer-Verlag, 1992.

[3] V.P. Varin, *Special solutions to Chazy equation*, Keldysh Institute Preprints. 2015. No. 43. 39 p., available at `http://library.keldysh.ru/preprint.asp?id=2015-43&lg=e`.

[4] V.V. Golubev, *Lectures on Integration of Equations Motion of a Heavy Rigid Body near Fixed Point*, Moscow: GITTL, 1953.

[5] A.B. Aranson, *Computation of Collections of Correlate Faces for Several Polyhedrons*, in Computer Algebra in Scientific Computing, Munchen: Techn, Univ. Munchen, 2003, pp. 13–17.

[6] A.B. Aranson, *Calculation of power expansion solutions of N. Kowalewski modified ODE system by power geometry algorithms*, Programming and Computer Software 37, 87–98 (2011).

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On Jordan structure of nilpotent $so(N, C)$-matrices.

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Abstract. I consider the nilpotent case now, it is most interesting and complicated case. A nilpotent matrix from the Lie algebra of orthogonal group can be considered as a matrix of a nilpotent linear transformation of some auxiliary linear space. Any matrix has Jordan structure, that means that there is a basis collected by cyclic vectors. This structure does not correlate with the Euclidean structure, generally speaking.

I will demonstrate how to construct a basis from the cyclic vectors that is in according with the Euclidean structure. It gives a splitting of the linear space on the orthogonal sum of Euclidean subspaces with the Euclidean structure inherited from the ambient space. Each such subspace equipped with the standard ("hyperbolic") basis collected from the cyclic vectors.

Each Jordan chain consisting of odd number of elements belongs to each own subspace, and Jordan chains consisting of even number of elements come in pairs. Each such pair forms a standard basis of the corresponding subspace.

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Asymptotic forms of solutions to system of nonlinear partial differential equations

Alexander Batkhin and Alexander Bruno

Abstract. In [1, 2] we considerably develop the methods of power geometry for a system of partial differential equations and apply them to computing the asymptotic forms of solutions to the problem of evolution of the turbulent flow. For each equation of the system, its Newton polyhedron and its hyperfaces with their normals and truncated equations are calculated. To simplify the truncated systems, power-logarithmic transformations are used and the truncated systems are further extracted. Results: (1) the boundary layer on the needle is absent in liquid, while in gas it is described in the first approximation; (2) one-dimensional model of evolution of turbulent bursts have eight asymptotic forms, presented explicitly.

1. Introduction

A universal asymptotic nonlinear analysis is formed, whose unified methods allow finding asymptotic forms and expansions of solutions to nonlinear equations and systems of different types: Algebraic; Ordinary differential equations (ODEs); Partial differential equations (PDEs).

This calculus contains two main methods: 1) *Transformation of coordinates, bringing equations to normal form*; 2) *Separating truncated equations.*

Two kinds of coordinate changes can be used to analyze the resulting equations: A) *Power*; B) *Logarithmic.*

Here, we consider systems of nonlinear partial differential equations in two variants:

a) *with solvable truncated system*; b) *without solvable truncated system.* We show how to find asymptotic forms of their solutions using algorithms of power geometry. In this case, by asymptotic form of solution, we mean a simple expression in which each of the independent or dependent variables tends to zero or infinity.

Here, we consider two fluids problems: (a) boundary layer and (b) turbulence flow by methods of power geometry.

For problem (a), it was firstly given in [3, Chapter 6, Section 6]; see also [4, 5]. A boundary layer on a needle has a stronger singularity than on a plane, and it was first considered in [5].

For problem (b), we firstly make it in [1, 2] and we are not sure that it can be solved with the usual analysis.

The structure of the paper is as follows. Section 2 outlines the basics of power geometry for partial differential equations. In Section 3, the theory and algorithms are further developed to apply to variant (b) problems. In Section 4, they are used to compute asymptotic forms of evolution of turbulent flow.

2. Basics of Power Geometry [3, Chapters VI–VIII]

Let $X = (x_1, \ldots, x_m) \in \mathbb{C}^m$ be independent and $Y = (y_1, \ldots, y_n) \in \mathbb{C}^n$ be dependent variables. Place $Z = (X, Y) \in \mathbb{C}^{n+m}$. *Differential monomial* $a(Z)$ is a product of an ordinary monomial $cZ^R = cz_1^{r_1} \cdots z_{m+n}^{r_{m+n}}$, where $c = \text{const} \in \mathbb{C}$, and a finite number of derivatives of the form

$$\frac{\partial^l y_j}{\partial x_1^{l_1} \cdots \partial^{l_m} x_m} \equiv \frac{\partial^l y_j}{\partial X^L}, \quad l_j \geq 0, \sum_{j=1}^{m} l_j = l, \quad L = (l_1, \ldots, l_m)\,. \tag{1}$$

The differential monomial $a(Z)$ corresponds to its vector exponent of degree $Q(a) \in \mathbb{R}^{m+n}$, formed by the following rules:

$$Q\left(Z^R\right) = R, \quad Q\left(\partial^l y_j / \partial X^L\right) = (-L, E_j), \tag{2}$$

where E_j is the unit vector. The product of monomials corresponds to the sum of their vector exponents of degree: $Q(ab) = Q(a) + Q(b)$. *Differential sum* is the sum of differential monomials:

$$f(Z) = \sum a_k(Z). \tag{3}$$

The set $\mathbf{S}(f)$ of vector exponents $Q(a_k)$ is called *support of sum* $f(Z)$. The closure of the convex hull

$$\mathbf{\Gamma}(f) = \left\{Q = \sum \lambda_j Q_j, Q_j \in \mathbf{S}, \lambda_j \geq 0, \sum \lambda_j = 1\right\}$$

of the support $\mathbf{S}(f)$ is called the *polyhedron of the sum* $f(Z)$. The boundary $\partial\mathbf{\Gamma}$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of generalized faces $\mathbf{\Gamma}_j^{(d)}$, where $d = \dim \mathbf{\Gamma}_j^{(d)}$, $0 \leq d \leq m + n - 1$. Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to:

- *Normal cone:* $\mathbf{U}_j^{(d)} = \{P \in \mathbb{R}_*^{m+n} : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle\,,\}$, where $Q', Q'' \in \mathbf{\Gamma}_j^{(d)}, Q''' \in \mathbf{\Gamma} \backslash \mathbf{\Gamma}_j^{(d)}$, and the space $\mathbb{R}_*^{m+n}$ is conjugate to the space $\mathbb{R}^{m+n}$ and $\langle \cdot, \cdot \rangle$ is a scalar product;
- *Truncated sum:* $\hat{f}_j^{(d)}(Z) = \sum a_k(Z)$ over $Q(a_k) \in \mathbf{\Gamma}_j^{(d)} \bigcap \mathbf{S}$.

Consider a system of equations:

$$f_i(X, Y) = 0, \quad i = 1, \ldots, n, \tag{4}$$

where f_i are differential sums. Each equation $f_i = 0$ corresponds to: its support $\mathbf{S}(f_i)$; its polyhedron $\mathbf{\Gamma}(f_i)$ with a set of faces $\mathbf{\Gamma}_{ij}^{(d_i)}$ in the main space $\mathbb{R}^{m+n}$; set of their normal cones $\mathbf{U}_{ij}^{(d_i)}$ in the dual space $\mathbb{R}_*^{m+n}$; set of truncated equations $\hat{f}_{ij}^{(d_i)}(X,Y) = 0$.

The set of truncated equations

$$\hat{f}_{ij_i}^{(d_i)}(X,Y) = 0, \quad i = 1,\dots,n, \tag{5}$$

is a *truncated system* if the intersection

$$\mathbf{U}_{1j_i}^{(d_1)} \cap \dots \cap \mathbf{U}_{nj_n}^{(d_n)}. \tag{6}$$

is not empty. A truncated system is always a quasi-homogeneous system.

In the solution of the system (4),

$$y_i = \varphi_i(X), \quad i = 1,\dots,n, \tag{7}$$

where φ_i are series in powers of x_k and their logarithms, each φ_i corresponds to its support, polyhedron, normal cones $\mathbf{u}_i$, and truncations. Here, the logarithm $\ln x_i$ has a zero exponent of degree on x_i. The set of truncated solutions $y_i = \hat{\varphi}_i$, $i = 1,\dots,n$, corresponds to the intersection of their normal cones: $\mathbf{u} = \bigcap_{i=1}^{n} \mathbf{u}_i \subset \mathbb{R}_*^{m+n}$. If it is not empty, it corresponds to *truncated solution*: $y_i = \hat{\varphi}_i$, $i = 1,\dots,n$.

Theorem 1. *If the normal cone* $\mathbf{u}$ *intersects the normal cone* (6)*, then the truncation* $y_i = \hat{\varphi}_i(X)$*,* $i = 1,\dots,n$*, of this solution satisfies the truncated system* (5)*.*

Multiplying the differential sum (5) with the support $\mathbf{S}(f)$ by the monomial Z^R gives the differential sum, $g(Z) = Z^R f(Z)$, with the support $\mathbf{S}(g) = R + \mathbf{S}(f)$. Thus, the multiplication leads to a shift of supports. Multiplications by monomials form a group of linear transformations of supports, and they can be used to simplify supports, differential sums, and systems of equations.

3. Algorithms of power geometry and their implementation

A matrix α is called *unimodular* if all its elements are integer and $\det \alpha = \pm 1$.

Problem 1. *Let* n*-dimensional integer vector* $A = (a_1, a_2, \dots, a_n)$ *be given. Find an* n*-dimensional unimodular matrix* α *such that the vector* $A\alpha = C = (c_1, \dots, c_n)$ *contains only one coordinate* c_n *different from zero.*

Its solution was given in [6, 7, 8].

Transformation of the variables

$$\ln W = (\ln Z)\,\alpha, \text{ where } \alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix}, \tag{8}$$

is called *power transformation*, where α_{11}, α_{22} are square matrices of sizes m and n, respectively and $\ln Z = (\ln z_1, \dots, \ln z_{m+n})$.

Theorem 2 ([3]). *The power transformation* (8) *changes a differential monomial* $a(Z)$ *with exponent of degree* $Q(a)$ *into a differential sum* $b(W)$ *with exponent of degree* $Q(b)$*:*

$$R = Q(b) = Q(a)\alpha^{-1*}, \tag{9}$$

where $*$ *denotes transposition.*

Theorem 3 ([3]). *If the system* (4) *is a quasi-homogeneous system and* $d = \dim \tilde{\mathbf{\Gamma}}$*, then there exist a power transformation* (8) *and monomials* Z^{T_i}*,* $i = 1, \ldots, n$ *which change the system* (4) *into the system* $g_i(W) \equiv Z^{T_i} f_i(Z) = 0$*,* $i = 1, \ldots, n$*, where all* $g_i(W)$ *are differential sums, and all their supports* $\mathbf{S}(g_i)$ *have* $m+n-d$ *identical coordinates* q_j *equal to zero.*

Transformation

$$\zeta_j = \ln z_j \tag{10}$$

is called *logarithmic transformation.*

Theorem 4 ([9]). *Let* $f(Z)$ *be such a differential sum that for all its monomials,* j*th component of* q_j *vector degree exponent* $Q = (q_1, \ldots, q_{m+n})$ *is zero, then as a result of the logarithmic transformation* (10)*, a differential sum* $f(Z)$ *transforms into a differential sum from* $z_1, \ldots, \zeta_j, \ldots, z_n$*.*

For $z_j \to 0$ or ∞, the coordinate $\zeta_j = \ln w_j$ always tends to $\pm\infty$. If we are interested only in those solutions (7) which have a normal cone $\mathbf{u}$ intersecting a given cone K, then the cone K is called the *cone of problem.* Thus, after the logarithmic transformation (10) for the coordinate ζ_j in the cone of the problem, we have $p_j \geq 0$.

In the following, we will not consider all possible truncated systems (5), but only those in which one of the equations has dimension $d_i = m + n - 1$. The calculations show that in this case the above procedure will cover all the truncated systems. Finally, it is convenient to combine the power and logarithmic transformations.

The CAS `Maple 2021` was used for calculations in this work. A library of procedures based on the `PolyhedralSets` CAS `Maple` package was developed to implement the algorithms of power geometry. The library includes calculation procedures:

- vector power exponent Q of the differential monomial $a(Z)$ for a given order of independent and dependent variables;
- support $\mathbf{S}$ of a partial differential equation written as a sum of differential monomials;
- Newton's polyhedron $\mathbf{\Gamma}$ in the form of a graph of generalized faces $\mathbf{\Gamma}_j^{(d)}$ of all dimensions d for the given support of the equation ; the number j is given by the program; each generalized face has its own number j; each line of the graph contains all generalized faces $\mathbf{\Gamma}_j^{(d)}$ of the same dimension d, the first line contains the Newton's polyhedron $\mathbf{\Gamma}$, the next line contains all faces $\mathbf{\Gamma}_j^{(m+n-1)}$ of dimension $m+n-1$ and so on; the last line contains the empty

set; if $\mathbf{\Gamma}_j^{(d)} \subset \mathbf{\Gamma}_k^{(d+1)}$, then they are connected by an arrow; in [3, Ch. 1, Section 1], "the structural diagram" was used that is similar to the graph and differs from it in two properties: numeration of faces $\mathbf{\Gamma}_j^{(d)}$ is independent for each dimension d and arrows are replaced by segments (see also [10]);

- normal vector N_j for the each generalized face $\mathbf{\Gamma}_j^{(m+n-1)}$ for the second line of the graph;
- truncated equation $\hat{f}_j^{(d)} = 0$ by the given number j of the generalized face or by a given normal vector N_j;
- normal cone of the corresponding generalized face: if the face

$$\mathbf{\Gamma}_j^{(d)} = \mathbf{\Gamma}_i^{(m+n-1)} \cap \mathbf{\Gamma}_k^{(m+n-1)} \cap \cdots \cap \mathbf{\Gamma}_l^{(m+n-1)},$$

then the normal cone $\mathbf{U}_j^{(d)}$ is the conic hull of the normals $N_i, N_k, \ldots, N_l$;

- power or logarithmic transformation of the original variables by a given normal N of the hyperface. For this purpose, the algorithms for constructing the unimodular matrix described in [6, 7, 8] are used.

4. The k–ε Model of Evolution of Turbulent Bursts

According to [11, 12, 13], the model is described by the system

$$\begin{aligned} k_t &= \left(\frac{k^2}{\varepsilon} k_x\right)_x - \varepsilon, \\ \varepsilon_t &= \left(\frac{k^2}{\varepsilon} \varepsilon_x\right)_x - \gamma \frac{\varepsilon^2}{k}. \end{aligned} \tag{11}$$

Here, time t and coordinate x are independent variables, the turbulent density k and the dissipation rate ε are dependent variables, and γ is a real parameter. Here, $m = n = 2$, $m + n = 4$ and $x_1 = t$, $x_2 = x$, $y_1 = k$, $y_2 = \varepsilon$.

In [1, 2] equations (11) are written as differential sums, such truncated systems are selected, which have one 3-dimensional equation, power and logarithmic transformations are applied and more simple systems are obtained. If they are not solvable, the computations are repeated till solvable systems are obtained, Their solutions, written in initial coordinates, are asymptotic forms of solutions to initial system.

References

[1] A. D. Bruno and A. B. Batkhin. Computation of asymptotic forms of solutions to system of nonlinear partial differential equations. *Preprints of KIAM*, (48):36, 2022. (in Russian). https://doi.org/10.20948/prepr-2022-48.

[2] A. D. Bruno and A. B. Batkhin. Asymptotic forms of solutions to system of nonlinear partial differential equations. *Universe*, 9(1):35, 2023. https://doi.org/10.3390/universe9010035.

[3] A. D. Bruno. *Power Geometry in Algebraic and Differential Equations*. Elsevier Science, Amsterdam, 2000.

[4] H. Blasius. Grenzschichten in flüssigkeiten mit kleiner rebung. *Zeit. für Math. und Phys.*, 56:1–37, 1908.

[5] A. D. Bruno and T. V. Shadrina. Axisymmetric boundary layer on a needle. *Transactions of Moscow Math. Soc.*, 68:201–259, 2007.

[6] A. D. Bruno and A. A. Azimov. Computation of unimodular matrices. *KIAM Preprints*, (36):1–20, 2022 (in Russian). `https://doi.org/10.20948/prepr-2022-46`.

[7] A. D. Bruno and A. A. Azimov. Computation of unimodular matrices of power transformations. *Programming and Computer Software*, 49(1):32–41, 2023. `https://doi.org/10.1134/S0361768823010036`.

[8] A. D. Bruno and A. A. Azimov. Computation of unimodular matrices of power transformations (this book).

[9] A. D. Bruno. Algorithms of the nonlinear analysis. *Russian Math. Surveys*, 51(5):956, 1996.

[10] C. Conradi, E. Feliu, M. Mincheva, and C. Wiuf. Identifying parameter regions for multistationarity. *PLOS Computational Biology*, 13(10):e1005751, oct 2017. `https://doi.org/10.1371/journal.pcbi.1005751`.

[11] A. N. Kolmogorov. *The equation of turbulent motion of an incompressible fluid*, volume I. Mathematics and Mechanics, pages 328–330. Kluwer Academic Publisher, Dordrecht, 1991.

[12] M. Bertsch, R. Dal Pazo, and R. Kersner. The evolution of turbulent bursts: the b-epsilon model. *Europ. J. Appl. Math.*, 5:537–557, 1994.

[13] V. A. Galaktionov. Invariant solutions of two models of evolution of turbulent bursts. *Europ. J. Appl. Math.*, 10:237–249, 1999.

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First differential approximation for ODE systems with parameters

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Abstract. Using the example of Jacobi oscillator, the first differential approximation for various numerical methods for systems of ordinary differential equations is constructed. Computational experiments show a good coincidence of the residuals on the first integrals of the system and their first differential approximations. The presented methods make it possible to carry out effective calculations by means of computer algebra systems.

In the 60s of the last century, N. N. Yanenko [1] formulated a method for differential approximation of a difference scheme. The main idea of this method is to replace the study of the properties of a difference scheme with the study of a problem with differential equations occupying an intermediate position between the original differential problem and its difference scheme. In the works of N. N. Yanenko and his students, as a result, concepts such as the approximation viscosity of the difference scheme and the first differential approximation (FDA) of the difference scheme were formulated.

The paper considers systems of ordinary differential equations (ODE) depending on the parameters. Examples of such systems can be both systems with parameters and systems with the first integrals. The first integrals retain their values on the solutions of the original system and these values can be considered as parameters of the original system.

There are a large number of numerical methods for solving ODE. The FDA allows to obtain information about the quality of the selected numerical method for a specific system using only symbolic calculations.

We consider a first-order ODE system resolved with respect to the first derivatives and present an algorithm to calculate the FDA. The algorithm is a set of simple operations with formal power series.

Consider Jacobi oscillator as an example. By definition, the Jacobi functions $p = \operatorname{sn} t$, $q = \operatorname{cn} t$, $r = \operatorname{dn} t$ are a particular solution of a nonlinear autonomous system with the initial conditions are $p = 0$, $q = r = 1$, $k = 1/2$ for $t = 0$. This

autonomous system has two quadratic integrals of motion:

$$\begin{cases} p_t - qr = 0, \\ q_t + pr = 0, \\ r_t + k^2 pq = 0, \\ p^2 + q^2 = C_1, \\ k^2 p^2 + r^2 = C_2. \end{cases} \tag{1}$$

Consider the original Runge–Kutta method of 4 orders for Jacobi oscillator (1)

$$\begin{cases} \frac{\widehat{p}-p}{h} - qr + h\big(\frac{k^2pq^2}{2} + \frac{pr^2}{2}\big) + h^2\big(-\frac{k^2qr(2p-q)(2p+q)}{6} + \frac{qr^3}{6}\big) + \ldots = 0, \\ \frac{\widehat{q}-q}{h} + pr + h\big(-\frac{k^2p^2q}{2} + \frac{qr^2}{2}\big) + h^2\big(\frac{k^2pr(p-2q)(p+2q)}{6} - \frac{pr^3}{6}\big) + \ldots = 0, \\ \frac{\widehat{r}-r}{h} + k^2pq + h\big(\frac{k^2r(q-p)(p+q)}{2}\big) + h^2\big(\frac{k^4pq(p-q)(p+q)}{6} - \frac{2k^2pqr^2}{3}\big) + \ldots = 0. \end{cases} \tag{2}$$

Here, for the compactness of formulas, p, q, r are denoted by $p(t), q(t), r(t)$, $\widehat{p}, \widehat{q}, \widehat{r}$ are denoted by $p(t+h), q(t+h), r(t+h)$, and according to the degree of h, the first two terms are given. Let's add the first two integrals to the system (2).

$$\begin{cases} \dfrac{(\widehat{p}^2 + \widehat{q}^2) - (p^2 + q^2)}{h} = 0, \\ \dfrac{(k^2\widehat{p}^2 + \widehat{r}^2) - (k^2p^2 + r^2)}{h} = 0, \end{cases} \tag{3}$$

Substitute Taylor series expansion at the point t for the functions $p(t), q(t), r(t), p(t+h), q(t+h), r(t+h)$ in the system (2), (3)

$$\begin{cases} p_t - qr + h\left(\frac{k^2pq^2}{2} + \frac{pr^2+p_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ q_t + pr + h\left(-\frac{k^2p^2q}{2} + \frac{qr^2+q_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ r_t + k^2pq + h\left(-\frac{k^2r(p-q)(p+q)}{2} + \frac{r_{tt}}{2}\right) + \mathcal{O}(h^2) = 0, \\ 2\left(pp_t + qq_t\right) + h\big(pp_{tt} + p_t^2 + qq_{tt} + q_t^2\big) + \mathcal{O}(h^2) = 0, \\ 2k^2pp_t + 2rr_t + h\big(k^2\left(pp_{tt} + p_t^2\right) + rr_{tt} + r_t^2\big) + \mathcal{O}(h^2) = 0. \end{cases} \tag{4}$$

To obtain the correct order of the Runge-Kutta method, it is necessary to construct the FDA. The system (4) forms a differential ideal [2] with respect to the operations of addition, multiplication and differentiation of series equal to zero. The form of the series(4), resolved with respect to the first derivatives, allows you to replace the derivatives of functions in coefficients at h through the functions themselves.

It is possible to formulate the following FDA construction algorithm for Runge–Kutta methods, which ends when the first nonzero term in powers of h is expressed only through the functions themselves. As a consequence, the FDA view does not depend on the decomposition point in the Taylor series and is a canonical form.

In our talk, a program to construct the FDA for given ODE system will be presented. This program written in the SymPy computer algebra system and

available at `https://github.com/blinkovua/sharing-blinkov/tree/master/FDA_ODE`. For presented example, it give us FDA in the form

$$\begin{cases} p_t - qr + h^4\left(-\frac{k^4qr\left(3p^4+3p^2q^2-2q^4\right)}{240} - \frac{k^2qr^3(p-q)(p+q)}{80} + \frac{qr^5}{120}\right) + \mathcal{O}(h^5) = 0, \\ q_t + pr + h^4\left(-\frac{k^4pr\left(2p^4-3p^2q^2-3q^4\right)}{240} + \frac{k^2pr^3(p-q)(p+q)}{80} - \frac{pr^5}{120}\right) + \mathcal{O}(h^5) = 0, \\ r_t + k^2pq + h^4\left(-\frac{k^6pq\left(2p^4-3p^2q^2+2q^4\right)}{240} + \frac{k^4pqr^2(p-q)(p+q)}{80} + \frac{k^2pqr^4}{80}\right) + \mathcal{O}(h^5) = 0, \\ h^4\left(\frac{k^4pqr(p-q)(p+q)\left(p^2+q^2\right)}{24}\right) + \mathcal{O}(h^5) = 0, \\ h^4\left(\frac{k^2pqr(kp-r)(kp+r)\left(k^2p^2+r^2\right)}{24}\right) + \mathcal{O}(h^5) = 0. \end{cases} \tag{5}$$

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References

[1] Yanenko N. N., Shokin Yu. I. The first differential approximation of difference schemes for hyperbolic systems of equations. Siberian Mathematical Journal, 1969, vol. 10, no. 5, pp. 868–880. https://doi.org/10.1007/BF00971662

[2] Kolchin E. R. Differential Algebra and Algebraic. Academic Press, New York, 1973. 446 P.

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Parametric expansions of an algebraic variety near its singularities

Alexander Bruno and Alijon Azimov

Abstract. Now there is a method, based on Power Geometry, that allows to find asymptotic forms and asymptotic expansions of solutions to different kinds of non-linear equations near their singularities. The method contains three algorithms: (1) Reducing equation to its normal form, (2) Separating truncated equations, (3) Power transformations of coordinates. Here we describe the method for the simplest case: a single algebraic equation, and apply it to an algebraic variety, described by an algebraic equation of order 12 in three variables. The variety was considered in study of Einstein's metrics and has several singular points and singular curves. Near some of them we compute a local parametric expansion of the variety.

1. Introduction

Here we propose a new method for solution of a polynomial equation

$$f(x_1, \dots, x_n) = 0$$

near its singular point. In the example we show computations of the method for a certain polynomial f and $n = 3$. The method is used:

I: The *Newton polyhedron* for separation of truncated equations and

II: *Power transformations* for simplification of these equations.

Here the basic ideas of this method are explained for the simplest case: a single algebraic equation. In Section 2 we give a generalization of Implicit Function Theorem. In Sections 3 and 4 we remind some constructions of Power Geometry [1]. In Section 5 we explain a way of computation of asymptotic parametric expansions of solutions. In Section 6 we show a variety Ω and some its singularities.

2. The implicit function theorem

Let $X = (x_1, \dots, x_n)$, $Q = (q_1, \dots, q_n)$, then $X^Q = x_1^{q_1}, \dots, x_n^{q_n}$, $\|Q\| = \sum_{j=1}^n q_j$.

Theorem 1. *Let*

$$f(X,\varepsilon,T)=\Sigma a_{Q,r}(T)X^Q\varepsilon^r,$$

where $0\le Q\in\mathbb{Z}^n$, $0\le r\in\mathbb{Z}$, *the sum is finite and* $a_{Q,r}(T)$ *are some functions of* $T=(t_1,\dots,t_m)$, *besides* $a_{00}(T)\equiv 0$, $a_{01}(T)\not\equiv 0$. *Then the solution to the equation* $f(X,\varepsilon,T)=0$ *has the form*

$$\varepsilon=\Sigma b_R(T)X^R, \tag{1}$$

where $0\le R\in\mathbb{Z}^n$, $0<\|R\|$, *the coefficients* $b_R(T)$ *are functions on* T *that are polynomials from* $a_{Q,r}(T)$ *with* $\|Q\|+r\le\|R\|$ *divided by* $a_{01}^{2\|R\|-1}$. *The expansion* (1) *is unique.*

This is a generalization of Theorem 1.1 of [1, Ch. II] on the implicit function and simultaneously a theorem on reducing the algebraic equation $f=0$ to its normal form (1) when the linear part $a_{01}(T)\not\equiv 0$ is non degenerate. In it, we must exclude the values of T near the zeros of the function $a_{01}(T)$.

Let $X=(x_1,\dots,x_n)\in\mathbb{R}^n$ or $\mathbb{C}^n$, and $f(X)$ be a polynomial. A point $X=X^0$, $f(X^0)=0$ is called *simple* if in it vector $(\partial f/\partial x_1,\dots,\partial f/\partial x_n)\ne 0$.

Definition 1. Let $\varphi(X)$ be some polynomial, $X=(x_1,\dots,x_n)$. A point $X=X^0$ of the set $\varphi(X)=0$ is called **singular point of the k-order**, if all partial derivatives of the polynomial $\varphi(X)$ for the $x_1,\dots,x_n$ turn into zero at this point, up to and including k-th order derivatives, and at least one partial derivative of order $k+1$ is nonzero.

3. The Newton polyhedron

Let the point $X^0=0$ be singular. Write the polynomial in the form $f(X)=\Sigma a_Q X^Q$, where $a_Q=\text{const}\in\mathbb{R}$, or $\mathbb{C}$. Let $\mathbf{S}(f)=\{Q:a_Q\ne 0\}\subset\mathbb{R}^n$.

The set $\mathbf{S}$ is called the *support* of the polynomial $f(X)$. Let it consist of points $Q_1,\dots,Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(f)=\left\{Q=\sum_{j=1}^k\mu_jQ_j,\quad \mu_j\ge 0,\quad \sum_{j=1}^k\mu_j=1\right\},$$

which is called the *Newton polyhedron.*

Its boundary $\partial\Gamma(f)$ consists of generalized faces $\Gamma_j^{(d)}$, where d is its dimension of $0\le d\le n-1$ and j is its number. Numbering is unique for all dimensions d.

Each (generalized) face $\Gamma_j^{(d)}$ corresponds to its:

- *boundary subset* $\mathbf{S}_j^{(d)}=\mathbf{S}\cap\Gamma_j^{(d)}$,
- *truncated polynomial* $\hat{f}_j^{(d)}(X)=\Sigma a_QX^Q$ over $Q\in\mathbf{S}_j^{(d)}$, and
- *normal cone*
  $\mathbf{U}_j^{(d)}=\left\{P:\langle P,Q'\rangle=\langle P,Q''\rangle>\langle P,Q'''\rangle,Q',Q''\in\mathbf{S}_j^{(d)},Q'''\in\mathbf{S}\backslash\mathbf{S}_j^{(d)}\right\}$,
  where $P=(p_1,\dots,p_n)\in\mathbb{R}_*^n$, the space $\mathbb{R}_*^n$ is conjugate (dual) to the space $\mathbb{R}^n$ and $\langle P,Q\rangle=p_1q_1+\ldots+p_nq_n$ is the scalar product.

At $X \to 0$ solutions to the full equation $f(X) = 0$ tend to non-trivial solutions of those truncated equations $\hat{f}_j^{(d)}(X) = 0$ whose normal cone $\mathbf{U}_j^{(d)}$ intersects with the negative orthant $P \le 0$ in $\mathbb{R}_*^n$.

4. Power transformations

Let $\ln X = (\ln x_1, \dots, \ln x_n)$. The linear transformation of the logarithms of the coordinates

$$(\ln y_1, \dots, \ln y_n) \stackrel{\text{def}}{=} \ln Y = (\ln X)\alpha, \tag{2}$$

where α is a nondegenerate square n-matrix, is called *power transformation*.

By the power transformation (2), the monomial X^Q tranforms into the monomial Y^R, where $R = Q(\alpha^*)^{-1}$ and the asterisk indicates a transposition.

A matrix α is called *unimodular* if all its elements are integers and $\det \alpha = \pm 1$. For an unimodular matrix α, its inverse α^{-1} and transpose α^* are also unimodular.

Theorem 2. *For the face $\Gamma_j^{(d)}$ there exists a power transformation (2) with the unimodular matrix α which reduces the truncated sum $\hat{f}_j^{(d)}(X)$ to the sum from d coordinates, that is, $\hat{f}_j^{(d)}(X) = Y^S \hat{g}_j^{(d)}(Y)$ where $\hat{g}_j^{(d)}(Y) \equiv \hat{g}_j^{(d)}(y_1, \dots, y_d)$ is a polynomial. Here $S \in \mathbb{Z}^n$. The additional coordinates $y_{d+1}, \dots, y_n$ are local (small).*

The article [2] specifies an algorithm for computing the unimodular matrix α of Theorem 2.

5. Parametric expansion of solutions

Let $\Gamma_j^{(d)}$ be a face of the Newton polyhedron $\Gamma(f)$. Let the full equation $f(X) = 0$ is changed into the equation $g(Y) = 0$ after the power transformation of Theorem 2. Thus $\hat{g}_j^{(d)}(y_1, \dots, y_d) = g(y_1, \dots, y_d, 0, \dots, 0)$.

Let the polynomial $\hat{g}_j^{(d)}$ be the product of several irreducible polynomials

$$\hat{g}_j^{(d)} = \prod_{k=1}^{m} h_k^{l_k}(y_1, \dots, y_d), \tag{3}$$

where $0 < l_k \in \mathbb{Z}$. Let the polynomial h_k be one of them. Three cases are possible:
Case 1. The equation $h_k = 0$ has a polynomial solution $y_d = \varphi(y_1, \dots, y_{d-1})$. Then in the full polynomial $g(Y)$ let us substitute the coordinates $y_d = \varphi + z_d$, for the resulting polynomial $h(y_1, \dots, y_{d-1}, z_d, y_{d+1} \dots, y_n)$ again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were made in [3] and were shown in Introduction to [1].

Case 2. The equation $h_k = 0$ has no polynomial solution, but has a parametrization of solutions $y_j = \varphi_j(T), j = 1, \dots, d, \quad T = (t_1, \dots, t_{d-1})$.

Then in the full polynomial $g(Y)$ we substitute the coordinates

$$y_j = \varphi_i(T) + \beta_j\varepsilon, \quad j = 1, \dots, d, \tag{4}$$

where $\beta_j = \text{const}$, $\Sigma\,|\beta_j| \neq 0$, and from the full polynomial $g(Y)$ we get the polynomial

$$h = \Sigma a_{Q'',r}(T) {Y''}^{Q''} \varepsilon^r, \tag{5}$$

where $Y'' = (y_{d+1}, \dots, y_n)$, $0 \leq Q'' = (q_{d+1}, \dots, q_n) \in \mathbb{Z}^{n-d}$, $0 \leq r \in \mathbb{Z}$. Thus $a_{00}(T) \equiv 0$, $a_{01}(T) = \sum_{j=1}^{d} \beta_j \partial \hat{g}_j^{(d)} / \partial y_j(T)$.

If in the expansion (3) $l_k = 1$, then $a_{01} \not\equiv 0$. By Theorem 1, all solutions to the equation $h = 0$ have the form $\varepsilon = \Sigma b_{Q''}(T) {Y''}^{Q''}$, i.e., according to (4) the solutions to the equation $g = 0$ have the form $y_j = \varphi_j(T) + \beta_j \Sigma b_{Q''}(T) {Y''}^{Q''}$, $j = 1, \dots, d$. Such calculations were proposed in [4].

If in (3) $l_k > 1$, then in (5) $a_{01}(T) \equiv 0$ and for the polynomial (5) from Y'', ε we construct the Newton polyhedron by support $\mathbf{S}(h) = \{Q'', r : a_{Q'',r}(T) \not\equiv 0\}$, separate the truncations and so on.

Case 3. The equation $h_k = 0$ has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [4], one can compute a piecewise approximate parametric solution to the equation $h_k = 0$ and look for an approximate parametric expansion.

Similarly, one can study the position of an algebraic manifold in infinity.

6. Variety Ω and its singularities

In [5], the investigation of the three-parametric family of special homogeneous spaces from the viewpoint of the normalized Ricci flow was started. The Ricci flows describe the evolution of Einstein's metrics on a variety. The equations of the normalized Ricci flow are reduced to a system of two differential equations with three parameters a_1, a_2 and a_3:

$$dx_j/dt = \tilde{f}_1(x_1, x_2, a_1, a_2, a_3), \quad j = 1, 2, \tag{6}$$

here $\tilde{f}_1$ and $\tilde{f}_2$ are certain functions. The singular point of this system are associated with invariant Einstein's metrics. At the singular (stationary) point x_1^0, x_2^0, system (6) has two eigenvalues λ_1 and λ_2. If at least one of them is equal to zero, then the singular (fixed) point x_1^0, x_2^0 is said to be *degenerate*. It was proved in [5] that the set Ω of the values of the parameters a_1, a_2, a_3 at which system (6) has at least one degenerate singular point is described by the equation

$$\begin{aligned}
&Q(s_1, s_2, s_3) \equiv (2s_1 + 4s_3 - 1)\left(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - \right.\\
&\left. -4096 s_3^3 + 12 s_1^2 - 240 s_1 s_3 + 768 s_3^2 - 6 s_1 + 60 s_3 + 1\right) - 8 s_1 s_2 (2s_1 + 4s_3 - 1) \times\\
&\times (2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) - 16 s_1^2 s_2^2 \left(52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - \right.\\
&\left. -320 s_3 + 13\right) + 64 (2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4 = 0,
\end{aligned}$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal respectively to $s_1 = a_1 + a_2 + a_3$, $s_2 = a_1a_2 + a_1a_3 + a_2a_3$, $s_3 = a_1a_2a_3$.

In [6], for symmetry reasons, the coordinates $\mathbf{a} = (a_1, a_2, a_3)$ were changed to the coordinates $\mathbf{A} = (A_1, A_2, A_3)$ by a linear transformation $\mathbf{a} = M\mathbf{A}$.

In [6] all singular points of the variety Ω in coordinates $\mathbf{A} = (A_1, A_2, A_3)$ were found. There are five points of the third order. Among them $P_1^{(3)} = (0, 0, 3/4)$. There are three second-order points and three algebraic curves of singular points of the first order. Among them is $\mathcal{I} = \{A_1 + A_2 + 1 = 0, A_3 = 1/2\}$.

In the talk we will consider the variety Ω in the neighborhood of point $P_1^{(3)}$ and curve $\mathcal{I}$. The methods proposed in [4] and described in Sections 2-5 are implemented.

References

[1] A. D. Bruno. *Power Geometry in Algebraic and Differential Equations*. Elsevier Science, Amsterdam, 2000.

[2] A. D. Bruno and A. A. Azimov. Computation of unimodular matrices of power transformations. *Programming and Computer Software*, 49(1):32–41, 2023. https://doi.org/10.1134/S0361768823010036

[3] A. D. Bruno and A. B. Batkhin. Resolution of an algebraic singularity by power geometry algorithms. *Programming and Computer Software*, 38(2):57–72, 2012. https://doi.org/10.1134/S036176881202003X

[4] A. D. Bruno. Algorithms for solving an algebraic equation. *Programming and Computer Software*, 44(6):533–545, 2018. https://doi.org/10.1134/S0361768819100013

[5] Yu. G. Nikonorov. Classification of generalized wallach spaces. *Geometriae Dedicata*, 111(1):193–212, 2016. https://doi.org/10.1007/s10711-015-0119-z

[6] A. D. Bruno and A. B. Batkhin. Investigation of a real algebraic surface. *Programming and Computer Software*, 41(2):74–82, 2015.

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Computing unimodular matrices of power transformations

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Abstract. An algorithm for solving the following problem is described. Let $m < n$ integer vectors in the n-dimensional real space be given. Their linear span forms a linear subspace L in $\mathbb{R}^n$. It is required to find a unimodular matrix such that the linear transformation defined by it takes the subspace L into a coordinate subspace. Computer programs implementing the proposed algorithms and the power transforms for which they are designed are described.

1. Introduction

Recall that a square matrix is said to be unimodular if all its elements are integers and its determinant equals ± 1. Its inverse is also unimodular.

We will write vectors as row vectors $A = (a_1, \ldots, a_n)$,and $[a]$ is the integer part of the real number a.

Problem 1. Let m, $(m < n)$ integer vectors $A_1, \ldots, A_m$ be given in the n-dimensional real space $\mathbb{R}^n$. Their linear span

$$L = \left\{ X = \sum_{j=1}^{m} \lambda_j A_j,\ \lambda_j \in \mathbb{R}, \quad j = 1, \ldots, m \right\} \tag{1}$$

forms a linear subspace in $\mathbb{R}^n$. It is required to find a unimodular matrix α such that the transformation $X\alpha = Y$ takes L to the coordinate subspace

$$M = \{Y : y_{n-l+1} = \cdots = y_n = 0\},$$

where $l = \dim L$.

In this talk, we give an algorithm for solving this problem and provide its implementations in computer algebra systems [1]. If $n = 2$ and $m = 1$, then Problem 1 is solved by Eucledean algorithm or by continued fraction [2]. In Section 2, we describe the Euler algorithm [3], which generalizes the Euclidean algorithm (i.e.,

the continued fraction algorithm) to the n-dimensional integer vector. In Section 3 we describe a solution of Problem 1. In Section 4 we consider power transformations, for the calculation of the unimodular matrices of which, all these algorithms are developed.

2. Euler's algorithm and a generalization of continued fraction

Problem 2. Let an n-dimensional integer vector $A = (a_1, a_2, \dots, a_n)$ be given. Find an n-dimensional unimodular matrix α such that the vector $A\alpha = C = (c_1, \dots, c_n)$ contains only one nonzero component c_n.

Euler proposed the following algorithm for solving this problem [3]. Suppose for the time being that all components of vector A are nonzero. Using the permutation $A\alpha_0 = (\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_n)$ arrange its components in nondecreasing order $\widetilde{a}_j \le \widetilde{a}_{j+1}$, $j = 1, \dots, n-1$. Hereα_0 is the unimodular matrix of the permutation. Let $\widetilde{a}_k$ be the least number among $\widetilde{a}_j$ that is distinct from zero.

Let $b_j = [\widetilde{a}_j/\widetilde{a}_k]$, $j = 1, \dots, n$. Here $b_1 = \dots = b_{k-1} = 0$, $b_k = 1$. Make the transformation

$$d_j = \widetilde{a}_j - b_j\widetilde{a}_k,\ 1 \le j \le n,\ j \ne k, d_k = \widetilde{\alpha}_k. \tag{2}$$

It is associated with the unimodular matrix α_1 the diagonal of which consists of ones, and the k-th row is

$$0, 0, \dots, 0, 1, -b_{k+1}, \dots, -b_n, \text{ i.e. } \tilde{A}\alpha_1 = D = (d_1, \dots, d_n).$$

Now arrange the components of the vector D in non-decreasing order using the unimodular permutation matrix β_0 so that $D\beta_0 = \tilde{D} = \left(0, \dots, 0, \widetilde{d}_k, \dots, \widetilde{d}_n\right)$,where $\widetilde{d}_j \le \widetilde{d}_{j+1}$.

Let $\widetilde{d}_l$ be the least of $\widetilde{d}_j$, distinct from zero, and let $e_j = \left[\widetilde{d}_j/\widetilde{d}_l\right]$, $j = 1, \dots, l$. Make the transformation

$$f_j = \widetilde{d}_j - e_j\widetilde{d}_l, \quad 1 \le j \le n, \quad j \ne l, \quad f_l = \widetilde{d}_l,$$

and soon. At each step, the maximum of the components of the vector decreases and it is the n-th component. Therefore, in a finite number of steps we obtain a vector with the only (last) nonzero component. This component equals the GCD of all original components $a_1, \dots, a_n$. Each step involves a permutation matrix and a triangular matrix with the unit diagonal:

$$A\alpha_0\alpha_1\beta_0\beta_1\gamma_0\gamma_1 \dots \omega_0\omega_1 = A\alpha = C = (0, \dots, 0, c_n).$$

The matrix

$$\alpha = \alpha_0\alpha_1\beta_0\beta_1\gamma_0\gamma_1 \cdots \omega_0\omega_1 \tag{3}$$

is a solution of Problem 2.

If not all components a_j of the original vector A have the same sign, then we first arrange them in non-decreasing order of their moduli $|\widetilde{a}_j| \le |\widetilde{a}_{j+1}|$ and set $b_j = [|\widetilde{a}_j| \,/\, |\widetilde{a}_k|] \operatorname{sign} \widetilde{a}_j \operatorname{sign} \widetilde{a_k}$.

Let the given vector A be perpendicular to a linear variety. Then, after the transformation using the matrix α, we obtain the vector in which all first $n-1$ components are zero. Therefore, the last component of all vectors of the original variety will be zero after this transformation.

Euler's algorithm generalizes the continued fraction algorithm only for integer vectors. Such a generalization for arbitrary real vectors was sought by all major mathematicians of the 19th century, but without success. Such a generalization of the continued fraction algorithm for the n-dimensional vector was proposed in [4]. It gives a sequence of best approximations, and it is periodic if all the components of the original vector are roots of a polynomial of degree n with integer coefficients.

3. Solution to Problem 1

Let integer vectors

$$\begin{aligned} A_1 &= (a_{11}, a_{12}, \ldots, a_{1n}), \\ A_2 &= (a_{21}, a_{22}, \ldots, a_{2n}), \\ &\ldots \\ A_m &= (a_{m1}, a_{m2}, \ldots, a_{mn}) \end{aligned} \tag{4}$$

($m < n$) and a linear space (1) be given.

First, we check if there are identical vectors among them. If there are any, we discard duplicates and leave only one of them. Now, we are sure that all vectors (4) are different. Apply Euler's algorithm to the vector A, i.e., calculate the matrix α such that $A_1\alpha_0 = C_1 = c_nE_n$, where c_n is an integer and E_k is the k-th unit vector.

Let $A_j\alpha_0 = C_j = (c_{j1}, \ldots, c_{jn})$, $j = 2, \ldots, m$. Set $A_j^1 = (c_{j1}, \ldots, c_{jn-1})$, $j = 2, \ldots, m$. Apply Euler's algorithm to the $(n-1)$-dimensional vector A_2^1 to obtain $A_2^1\alpha_1 = C_2^1 = (0, 0, \ldots, c_{n-1}^1)$, where α_1 is an $(n-1)$-dimensional square matrix. Let

$$A_j^1\alpha_1 = C_j^1 = (c_{j1}^1, \ldots, c_{jn-1}^1), \quad j = 3, \ldots, m.$$

Apply Euler's algorithm to the $(n-2)$-dimensional vector C_3^1, and so on. Finally, we obtain the sequence of matrices $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ of decreasing size $n, n-1, \ldots, n-m+1$. Form the block matrices

$$\beta_j = \begin{pmatrix} \alpha_j & 0 \\ 0 & I_{j+1} \end{pmatrix}, \quad j = 0, \ldots, n-m,$$

of size n, where I_{j+1} are the identity matrices of size $j+1$. Set $\gamma = \beta_0\beta_1 \cdots \beta_{m-1}$. Then

$$A_j\gamma = (0, 0, \ldots, 0, w_{j,n-j+1}, \ldots, w_{j,n}) = W_j, \quad j = 1, \ldots, m.$$

The matrix γ is a solution to Problem 1.

4. Power transformations

Let the polynomial

$$f(X) = \sum f_Q X^Q, Q \in \mathbf{S}, \tag{5}$$

where $X = (x_1, \dots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, $Q = (q_1, \dots, q_n) \in \mathbb{Z}^n, Q \geq 0, f_Q$ are constant coefficients from $\mathbb{R}$ or $\mathbb{C}$, $\mathbf{S} = \mathbf{S}(f)$ is the support of f, be given. Let $\mathcal{F}$ be the algebraic variety $f(X) = 0$ and the point $X = X^0 \in \mathcal{F}$.

If X^0 is a simple point, i.e., if at least one derivative $\partial f/\partial x_j$ is nonzero at X^0 then the implicit function theorem implies that the variety $\mathcal{F}$ in the neighborhood of X^0 is described by the equation

$$\Delta x_j = \varphi(\Delta x_1, \dots, \Delta x_{j-1}, \Delta x_{j+1}, \dots, \Delta x_n), \tag{6}$$

where $\Delta x_k = x_k - x_k^0$ and φ is a convergent series of its arguments.

If X^0 is not a simple point, then, according to [5, 6] we can seek the branches of the variety $\mathcal{F}$, passing through X^0 in the form of parametric expansions

$$\Delta x_j = \varphi_j(\xi_1, \dots, \xi_{n-1}), i = 1, \dots, n, \tag{7}$$

where ξ_k are small parameters and φ_j — are converging power series. To this end the convex hull Γ of the support $\mathbf{S}$ in the space is constructed. Then, Γ is the polyhedron the boundary $\partial\Gamma$ of which consists of (generalized) faces $\Gamma_j^{(d)}$ of dimension d, $0 \leq d < n$. Here j is the face index. Since all vertices $\Gamma_j^{(0)}$ of Γ are integer, each face $\Gamma_j^{(d)}$ has $n-d$ integer linearly independent normals $N_{j1}^{(d)}, \dots, N_{jn-d}^{(d)} \in \mathbb{R}_*^n$ i.e., normals belonging to the space $\mathbb{R}_*^n$, which is dual of the space $\mathbb{R}^n$.

In addition, each face $\Gamma_j^{(d)}$ is associated with the boundary set

$$D_j^{(d)} = \left\{ Q \in \mathbf{S} \cap \Gamma_j^{(d)} \right\},$$

and the truncated sum is

$$\hat{f}_j^{(d)}(X) = \sum f_Q X^Q \text{ over } Q \in D_j^{(d)}. \tag{8}$$

Theorem 1 ([5, Corollary in Chapter II, § 3]**,** [6, Theorem 3.1]**).** *For the face $\Gamma_j^{(d)}$ there exists a power transformation*

$$\ln Y = \ln X \cdot \alpha,$$

where $\ln Y = (\ln y_1, \dots, \ln y_n)$ and $\ln X = (\ln x_1, \dots, \ln x_n)$ with a unimodular matrix α, that takes the truncated sum (8) *to a polynomial g of d variables, i.e.,*

$$\hat{f}_j^{(d)}(X) = Y^T g(y_1, \dots, y_d), \tag{9}$$

where $T = (t_1, \dots, t_n) \in \mathbb{Z}^n$.

However in [5, 6], it was not pointed out how the unimodular matrix α can be calculated. This is done in the current paper. In [7, Part I, Ch. I, Section 1.9] it was made for $n = 2$. In [1, 8] we describe software of these algorithms. It will be considered in our talk.

References

[1] *Bruno, A.D., Azimov, A.A.* Computing unimodular matrices of power transformations // Programming and Computer Software, 2023, Vol. 49, No. 1, pp. 32–41. DOI: 10.1134/S0361768823010036

[2] *Khinchin, A. Ya.* Continued Fractions, Moscow: Fizmatgiz, 1961 (in Russian); Mineola, NY: Dover, 1997.

[3] *Euler L.* De relatione inter ternas pluresve quantitates instituenda // 1785, All Works 591.

[4] *Bruno, A.D.* Computation of the fundamental units of number rings using a generalized continued fraction, Program. Comput. Software, 2019, vol. 45, no. 2, pp. 37–50. DOI: 10.1134/S036176881902004X.

[5] *Bruno, A.D.* Power Geometry in Algebraic and Differential Equations, Moscow: Nauka, 1998 (in Russian); Amsterdam: Elsevier, 2000.

[6] *Bruno, A.D., Batkhin, A.B.* Resolution of an algebraic singularity by power geometry algorithms, Program. Comput. Software, 2012, vol. 38, no. 2, pp. 57–72.

[7] *Bruno, A.D.* Local Methods in Nonlinear Differential Equations, Berlin: Springer, 1989.

[8] *Bruno, A.D., Azimov, A.A.* Computation of Unimodular Matrices // Preprints of KIAM, Moscow, No 46, 2022 (in Russian). DOI: `https://doi.org/10.20948/prepr-2022-46`

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Effective algorithm for factoring polynomials in the ring of multivariable formal power series in zero–characteristic

Alexander L. Chistov

Let k be a ground field of zero–characteristic with algebraic closure $\overline{k}$. We assume that k is finitely generated over its primitive subfield. Let $k[[X_1, \ldots, X_n]]$ (respectively $\overline{k}[[X_1, \ldots, X_n]]$) the ring of formal power series in the variables $X_1, \ldots, X_n$ with coefficients from the field k (respectively $\overline{k}$). By definition an algorithm constructs a polynomial with coefficients in the ring of formal power series if and only if it can construct arbitrary approximations of all the coefficients of this polynomial.

Let $f \in k[X_1, \ldots, X_n, Z]$ be a polynomial of degree $\deg_{Z,X_1,\ldots,X_n} \leqslant d$, $d \geqslant 2$, and the leading coefficient with respect to Z of f is equal to 1. We suggest algorithms for factorization such a polynomial f in the rings $k[[X_1, \ldots, X_n]][Z]$ and $\overline{k}[[X_1, \ldots, X_n]][Z]$. To our knowledge so far nobody has described such algorithms for the case $n \geqslant 2$ (may be only particular cases has been considered). As a direct consequence of the suggested algorithms we get algorithms for factorization of polynomials from $k[X_1, \ldots, X_n]$ in the rings of formal power series $k[[X_1, \ldots, X_n]]$ and $\overline{k}[[X_1, \ldots, X_n]]$. Again as far as we know no such algorithms have been obtained for $n \geqslant 3$ (the case $n = 1$ is trivial and the case $n = 2$ can be treated using the method of Newton's broken lines, cf. [6]).

For any $j \geqslant 1$ the suggested algorithms can construct the j-th approximation of all the objects at their output. We give explicit complexity bounds for the running time of the described algorithms. These complexity bounds are polynomial in j and the size of the input data if the number n of variables is fixed, say $n = 2, 3, 4, \ldots$.

There is no easy solution of the considered problem of factorization of a polynomial $f \in k[X_1, \ldots, X_n, Z]$ using only Newton polygons or polyhedrons for $n \geqslant 2$. Of course the roots of the polynomial f belong to the field of multiple formal fractional power series in $X_1, \ldots, X_n$, i.e. to the union by all integers $\nu_1, \ldots, \nu_n \geqslant 1$ of the fields of multiple formal power series

$$\overline{k}((X_1^{1/\nu_1}))((X_2^{1/\nu_2})) \ldots ((X_n^{1/\nu_2})).$$

For example, it is difficult to decide whether a root z of the polynomial f from this field actually belongs to $k[[X_1, \ldots, X_n]]$.

Our method is based on the results on normalization of algebraic varieties and completions of their local rings. First of all it is an effective normalization of algebraic varieties in zero–characteristic with the explicit complexity bound. It was described by the author erlier, see [3], [4], [5]. Secondly we use the theorems related to analytical irreducibility and analytical normality of normal algebraic varieties, see [9] v.II, Chapter 8 §13 Theorems 31–33. Of course we use also the results from [2].

We don't consider the case of nonzero characteristic mainly since no results similar to [3] have been obtained so far in this case. But, of course, one can use another algorithms for normalization of algebraic varieties in nonzero characteristic (there are no explicit estimates of complexity for these algorithms in literature) and get an analog of our result in nonzero characteristic but without a bound for the complexity of algorithms.

For more details, see Theorem 1 [7]. Actually the complexity of the algorithm from this theorem is polynomial in $d^{2^{n^c}}$ and j^n for a constant $c > 0$. At present we have analysed the construction of this algorithm thoroughly. We hope to improve it using the result of [8]. The complexity bound of the new version of this algorithm will be polynomial in d^{n^c} and j^n (the constant c will be specified).

Note also that in [7] we refer to Theorem 1 §3 Chapter IV [1] about factroring polynomials over a field complete with respect to a discrete valuation (although factually one can manage without this theorem in [7]). Recently we have found that it is not quite obvious that the construction from the proof of this theorem in [1] gives a polynomial time algorithm in our situation. Still it is true. Only minor modifications are required in this construction. We are going to clarify this question in the next paper.

References

[1] Z. I. Borevich, I. R. Shafarevich *Number theory*, New York Academic Press 1966.

[2] A.L. Chistov, *Polynomial complexity algorithm for factoring polynomials and constructing components of a variety in subexponential time*, Zap. Nauchn. Semin. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 137 (1984), p. 124–188 (in Russian) [English transl.: J. Sov. Math. 34 (4) (1986)].

[3] A.L. Chistov, *Effective Construction of a Nonsingular in Codimension One Algebraic Variety over a Zero-Characteristic Ground Field*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 387 (2011), p. 167–188 [English transl.: Journal of Mathematical Sciences v. 179 (2011), p. 729–740].

[4] A.L. Chistov, *An overview of effective normalization of a nonsingular in codimension one projective algebraic variety'*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 373 (2009), p. 295–317 [English transl.: Journal of Mathematical Sciences v. 168 (2010), p. 478–490].

[5] A.L. Chistov, *Effective normalization of a nonsingular in codimension one projective algebraic variety*, Doklady Academii Nauk 427, no. 5 (2009), p. 605–608 (in Russian) [English transl.: Doklady Mathematics, 80:1 2009, p. 577–580].

[6] A.L. Chistov, *Polynomial complexity of the Newton–Puiseux algorithm*, In: International Symposium on Mathematical Foundations of Computer Science 1986. Lecture Notes in Computer Science Vol. 233 Springer (1986) p. 247–255.

[7] A.L. Chistov *An algorithm for factoring polynomials in the ring of multivariable formal power series in zero–characteristic*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 517 (2022), p. 268–290 (in Russian) [English transl.: to appear in Journal of Mathematical Sciences].

[8] W.-L. Chow *On the theorem of Bertini for local domains*, Proceedings of the National Academy of Sciences, (1958) v.44, #6 p. 580–584.

[9] O. Zariski, P. Samuel P. *Commutative algebra*, v.I-II, Berlin New York Springer–Verlag, 1958–1960.

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Modeling of bumping routes in the RSK algorithm

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The Robinson-Schensted-Knuth algorithm defines a bijection between sets of sequences of a linearly ordered set and pairs of Young tableaux of the same shape: a semi-standard tableau (SSYT) P and a standard tableau (SYT) Q. At the same time, it is known [1] that RSK establishes the correspondence between a uniform measure on sequences and a Plancherel measure on the shapes of Young tableaux. A sequence of values bumped during a single iteration of the algorithm in the P tableau forms the so-called "bumping route". In Fig. 1 is an example of a bumping route in a tableau of 50 boxes when processing the number 18.

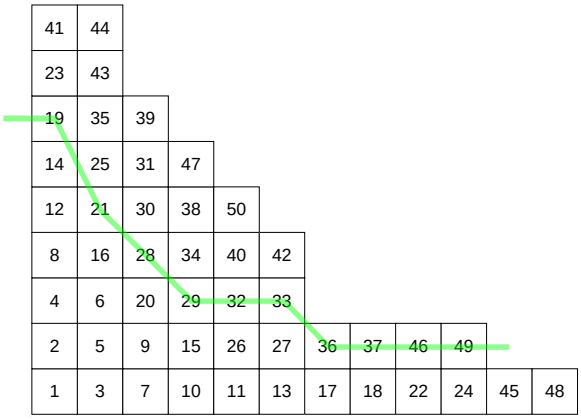

FIGURE 1. An example of a bumping route

The explicit formulae for the limit bumping routes in tableau P which correspond to uniformly-distributed random sequences were obtained in [2]. However, the question remains open about how the bumping routes converge to their limit curves with increasing size of Young tableaux.

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A series of computer experiments [3, 4] were conducted in order to investigate this problem. It was studied how the distance between the bumping routes and their limit shapes changes with the size of the tableaux. We considered SSYT filled with real numbers from the range [0, 1]. Tableaux sizes were taken from the range of $n \in [10^5, ..., \cdot 10^7]$ boxes with a step of 10^5. A fixed number of Young tableaux P of each of the considered sizes was generated. Then, we inserted various input values of α in the resulting tableaux and calculated the averages and variances of the deviations of the discrete bumping routes from the corresponding limit curves. The computer experiments show that the distance between the bumping routes and the limit shapes is well approximated by the formula

$$f(n) = a \cdot n^{-\frac{1}{4}} + b \cdot n^{-\frac{1}{2}}.$$

Note that the first term of this equation shows the extremely slow convergence of the bumping routes to their limit shapes. Fig. 2 shows the average distances of the bumping routes from the limit shapes for various input values α.

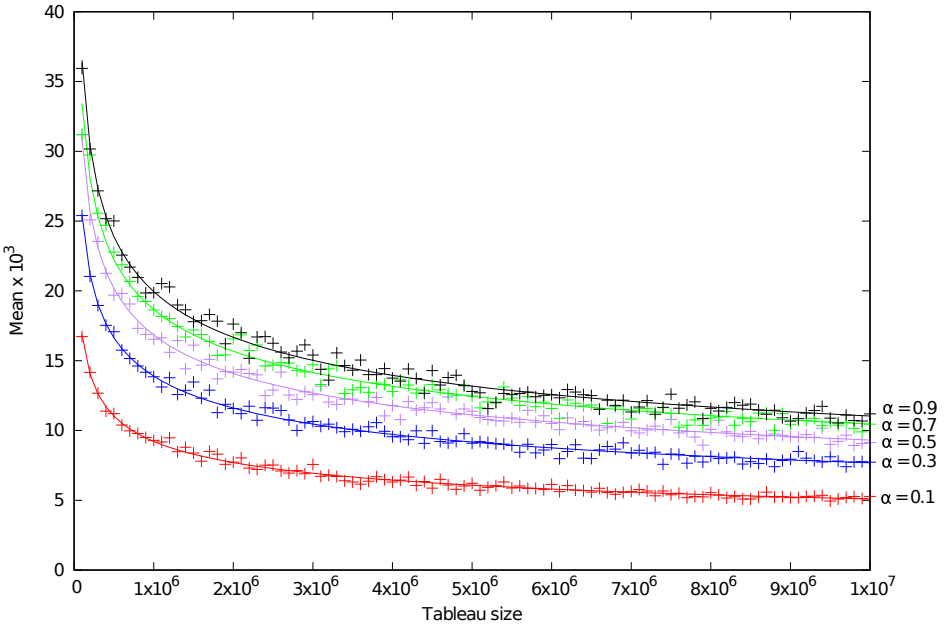

FIGURE 2. Mean values of deviations of bumping routes from their limit shapes and corresponding approximating curves

Also, as a result of experiments, it was found out that the distribution of the ends of the bumping routes at the profile of Young tableaux is quite close to Gaussian [4]. We estimated the parameters of the Gaussian distribution depending on the values $\alpha = 0.1, 0.3, 0.5, 0.7, 0.9$ fed to the input of the RSK algorithm.

References

[1] Donald E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.* Volume 34, Number 3 (1970), pp. 709-727.

[2] Dan Romik, Piotr Śniady, "Limit shapes of bumping routes in the Robinson-Schensted correspondence", *Random Struct. Algorithms*, 48 (1), pp. 171-182 (2016)

[3] Vassiliev N. N., Duzhin V. S., Kuzmin A. D. On the convergence of bumping routes to their limit shapes in the RSK algorithm. Numerical experiments. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2021, no. 6, pp. 2–9. doi:10.31799/1684-8853-2021-6-2-9

[4] Vassiliev N. N., Duzhin V. S., Kuzmin A. D. Modeling of bumping routes in the RSK algorithm and analysis of their approach to limit shapes. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2022, no. 6, pp. 2–9. doi:10.31799/1684-8853-2022-6-2–9, EDN: WRCOSH

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On integrability of the resonant cases of the generalized Lotka–Volterra system

Victor F. Edneral and Alexander B. Aranson

Abstract. The paper discusses a possible connection between local integrability near stationary points and global integrability of an autonomous two-dimensional polynomial ODE system. As an example, we use the resonance case of the generalized Lotka–Volterra system. We parametrized its right-hand sides as quadratic polynomials with resonance linear part. The conditions of local integrability near stationary points are written as systems of algebraic equations in the parameters of the system. We solve these systems. It is established that for the values of the parameters obtained in this way, the system of ODEs under consideration turns out to be integrable. Thus, we can speak of a heuristic approach that allows us to determine cases of ODE integrability a priori.

Introduction

We use an approach based on local analysis. It uses the resonant normal form computed near stationary points [1]. In the paper [2] we proposed a method for searching for integrable cases based on determining the parameter values for which the dynamical system is locally integrable at all stationary points simultaneously. Because at regular points, local integrability always holds, so such a requirement is equivalent to the requirement of local integrability at every point of the domain under consideration.

Note that the integrability of an autonomous planar system implies the solvability of the system in quadratures.

Problem

We will check our method on the example of a resonance case of the generalized Lotka–Volterra system

$$\begin{aligned} \dot{x} = &\quad Mx + a_1x^2 + a_2x\,y + a_3y^2, \\ \dot{y} = &\quad -y + b_1x^2 + b_2x\,y + b_3y^2, \end{aligned} \tag{1}$$

here x and y are functions in time and parameters $a_1, a_2, a_3, b_1, b_2, b_3$ are real. M is non-negative integer. Each value of M corresponds to the resonance $M:1$.

The problem is to construct the first integrals of system (1).

Method

The main task of the method under discussion is to find conditions on the parameters of the system under which the system is locally integrable near its stationary points. Local integrability means the presence of a sufficient number (one for an autonomous flat system) of local integrals. Local integrals may be different for different points of this region of the phase space, but in our opinion for the existence of a global integral, the enough number of local integrals (one in our case) must exist for in each point simultaneously. This condition is not satisfied for arbitrary parameter values. In the book[1] the algebraic condition of local integrability is written out. This is the so-called **A** condition. It is satisfied at all regular points, but it is nontrivial in resonance cases at stationary points.

Firstly we look for sets of parameters under which the condition **A** is satisfied at the stationary point of the system (1) at the origin. We solve the corresponding systems of algebraic equations with respect to the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ and check the integrability at other stationary points for each found set of parameters. Received parameter sets are good candidates for the existence of a single function for all points - the first integral. These integrals are sought by one method or another. We did this procedure for 3 resonances $M:1, M=1,2,3$.

Condition of Local Integrability

In our case the condition of local integrability **A** is some infinite sequence of polynomial equations with respect to the coefficients of the system. Each of the stationary points has its own system of equations. The joined system should be solved. Another tactic involves solving system at stationary point in the origin and checking the corresponding solutions with the condition **A** at other points of the phase plane. Recall that the normal form has a non-trivial form in the resonant case only.

Calculation of the normal form is an iterative process, we do it step by step. Computing each $M+1$ normal form order adds one equation to the condition. Condition **A** is an infinite system of equations, we have to work with a finite (truncated) condition. But as a result of many calculations in various systems of

ODEs, we noticed a remarkable fact. After a certain order, adding new equations to condition **A** ceases to affect the solutions of the system, i.e. several of its lower polynomials form the basis of the entire infinite ideal. For the system under consideration, it suffices to consider the first 3 equations. We generated them by the package [3].

For a 1:1 resonance, this is the truncated **A** system at the origin. It has been experimentally established that adding further equations does not change its solution

$$
\begin{aligned}
&a_1a_2 - b_2b_3 = 0,\\
&-a_3b_2(-6a_1^2 + 9a_1b_2 + 14b_1b_3 + 6b_2^2) + 9a_2^2(a_1b_2 + b_1b_3) + a_2(14a_1a_3b_1 -\\
&\quad 3b_3(2b_1b_3 + 3b_2^2)) + 6a_2^3b_1 = 0,\\
&432a_1^4a_2a_3 + 36a_1^3(54a_2^3 + 18a_2^2b_3 - 61a_2a_3b_2 - 18a_3b_2b_3) -\\
&\quad 6a_1^2(162a_2^3b_2 + a_2^2(131a_3b_1 - 162b_2b_3) + 3a_2a_3(106b_1b_3 + 75b_2^2) +\\
&\quad 2a_3b_2(194a_3b_1 - 381b_2b_3)) + a_1(3708a_2^4b_1 - 108a_2^3(33b_2^2 - 38b_1b_3) -\\
&\quad 3a_2^2b_1(5299a_3b_2 + 1524b_3^2) - 4a_2(868a_3^2b_1^2 - 981a_3b_2^3 + 81b_3^2(3b_2^2 - 2b_1b_3)) +\\
&\quad 36b_2(142a_3^2b_1b_2 + a_3b_3(53b_1b_3 - 114b_2^2) - 18b_2b_3^2)) - 1782a_2^4b_1b_2\\
&\quad -6a_2^3b_1(523a_3b_1 + 654b_2b_3) + 18a_2^2b_3(-284a_3b_1^2 + 75b_1b_2b_3 + 198b_2^3) +\\
&\quad 3a_2(a_3(776b_1^2b_3^2 + 5299b_1b_2^2b_3 + 594b_2^4) + 12b_2b_3^2(61b_1b_3 + 27b_2^2)) +\\
&\quad 2b_2(a_3^2b_1(1736b_1b_3 + 1569b_2^2) + 3a_3b_2b_3(131b_1b_3 - 618b_2^2) -\\
&\quad 108b_3^3(2b_1b_3 + 9b_2^2)) = 0.
\end{aligned}
\tag{2}
$$

Equations of a similar form were obtained for resonances $1:2$ and $1:3$ also.

Results

The MATHEMATICA-11 system received 11 rational solutions of system (2). Some of them are a consequence of others. 7 solutions turned out to be independent:

$$
\begin{aligned}
&1)\{a_1 \to -\tfrac{b_2}{2}, b_3 \to -\tfrac{a_2}{2}\};\\
&2)\{a_3 \to \tfrac{a_2^3b_1}{b_2^3}, b_3 \to \tfrac{a_1a_2}{b_2}\};\\
&3)\{a_1 \to 2b_2, a_3 \to \tfrac{a_2b_2}{b_1}, b_3 \to 2a_2\};\\
&4)\{a_1 \to 2b_2, a_3 \to 0, b_1 \to 0, b_3 \to 2a_2\};\\
&5)\{a_2 \to 0, b_2 \to 0\};\\
&6)\{a_1 \to 0, b_1 \to 0, b_2 \to 0\};\\
&7)\{a_1 \to 2b_2, a2 \to 0, b_1 \to 0, b_3 \to 0\}.
\end{aligned}
\tag{3}
$$

At these sets of parameters we checked the integrability condition at other stationary points of system (2).

The corresponding cases of system (1) look like this:

$$\begin{array}{lll}
1)\dot{x}= & x-\frac{1}{2}b_2x^2+a_2xy+a_3y^2, & \dot{y}=-y+b_1x^2+b_2xy-\frac{1}{2}a_2y^2;\\
2)\dot{x}= & x+a_1x^2+a_2xy+\frac{a_2^3b_1}{b_2^3}y^2, & \dot{y}=-y+b_1x^2+b_2xy+\frac{a_1a_2}{b_2}y^2;\\
3)\dot{x}= & x+2b_2x^2+a_2xy+\frac{a_2b_2}{b_1}y^2, & \dot{y}=-y+b_1x^2+b_2xy+2a_2y^2;\\
4)\dot{x}= & x+2b_2x^2+a_2xy, & \dot{y}=-y+b_2xy+2a_2y^2;\\
5)\dot{x}= & x+a_1x^2+a_3y^2, & \dot{y}=-y+b_1x^2+b_3y^2;\\
6)\dot{x}= & x+a_2xy+a_3y^2, & \dot{y}=-y+b_3y^2;\\
7)\dot{x}= & x+2b_2x^2+a_3y^2, & \dot{y}=-y+b_2xy.
\end{array} \tag{4}$$

Cases 1), 4),6) and 7) have been integrated by the MATHEMATICA-11 solver. 2) and 5) were integrated by the Darboux method. At this moment we could not integrate case 3). For the $1:2$ and $1:3$ resonances, we also managed to calculate the integrals for almost all predicted cases.

We then combined the conditions for the three resonances and tried to integrate the corresponding cases of the general (non-resonant) form. Systems with coefficients thus obtained have the form

$$\begin{array}{lll}
1) & \dot{x}=\alpha x+a_1x^2, & \dot{y}=-y+b_1x^2+b_3y^2;\\
2) & \dot{x}=\alpha x+a_1x^2, & \dot{y}=-y+b_1x^2+b_2xy;\\
3) & \dot{x}=\alpha x+a_2xy+a_3y^2, & \dot{y}=-y+b_3y^2;\\
4) & \dot{x}=\alpha x+2b_2x^2+a_3y^2, & \dot{y}=-y+b_2xy;\\
5) & \dot{x}=\alpha x+a_2xy, & \dot{y}=-y+b_2xy;\\
6) & \dot{x}=\alpha x+a_2xy+a_3y^2, & \dot{y}=-y;\\
7) & \dot{x}=\alpha x+a_2xy+a_3y^2, & \dot{y}=-y-a_2y^2/2;\\
8) & \dot{x}=\alpha x+b_2x^2+a_2xy, & \dot{y}=-y+b_2xy+a_2y^2;\\
9) & \dot{x}=\alpha x+a_2xy+a_3y^2, & \dot{y}=-y+a_2y^2;\\
10) & \dot{x}=\alpha x+a_2xy, & \dot{y}=-y+b_1x^2+2a_2y^2;\\
11) & \dot{x}=\alpha x+a_2xy+a_3y^2, & \dot{y}=-y+2a_2y^2.
\end{array} \tag{5}$$

α here is voluntary parameter. That is, we have moved away from the resonant limitation.

All systems (5) are integrable. So we have an algorithm that can predict integrable cases.

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References

[1] A.D. Bruno, *Analytical form of differential equations (I, II).* Trudy Moskov. Mat. Obsc. **25**, 119–262 (1971), **26**, 199–239 (1972) (in Russian) = Trans. Moscow Math. Soc. **25**, 131–288 (1971), **26**, 199–239 (1972) (in English) A.D. Bruno, *Local Methods*

in Nonlinear Differential Equations. Nauka, Moscow 1979 (in Russian) = Springer-Verlag, Berlin (1989) P.348.

[2] A.D. Bruno, V.F. Edneral, V.G. Romanovski, On new integrals of the Algaba-Gamero-Garcia system, Proceedings of the CASC 2017, Springer-Verlag series: LNCS **10490** (2017) 40–50

[3] V.F. Edneral, R. Khanin, *Application of the resonance normal form to high-order nonlinear ODEs using Mathematica*. Nuclear Instruments and Methods in Physics Research, Section A: Accelerators, Spectrometers, Detectors and Associated Equipment **502**(2–3) (2003) 643–645.

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Bernstein polynomials and MacWilliams transform

Nikita Gogin and Vladislav Shubin

Abstract. In this report we show that the vector of coefficients of the Bernstein polynomial (in monomial basis) for a function given on the interval $[-1, 1]$ is (up to a rational multiplier) the MacWilliams transform of the vector of selected samples of this function taken with binomial weights.

Keywords. Bernstein polynomials, Krawtchouk polynomials, MacWilliams matrices, Pascal-MacWilliams pyramid, Cellular automata.

Introdution

Bernstein polynomials are apparently the first historical example of a constructive proof of Weierstrass approximation theorem. These polynomials are widely used for approximation problems alongside with other methods (such as the least-squares method) and play an important role in computer graphics, as one of the forms of analytical representation of Bézier curves. [2, p. 41]

1. Bernstein polynomials and Krawtchouk polynomials

The classical definition of Bernstein polynomials is as follows:

Definition 1.1. *Let $f(x) \in C[0, 1]$. The Bernstein polynomial $B_n(f; x)$ of degree n for the sampling vector $f_r = f(x_r) = (f(0), f(1/n), \ldots, f(1))$ on the uniform grid $x_r = r/n$, $r = 0, 1, \ldots, n$ is defined as the polynomial*

$$B_n(f; x) = \sum_{r=0}^{n} \binom{n}{r} f_r x^r (1-x)^{n-r}, \tag{1}$$

where the products $\binom{n}{r} x^r (1-x)^{n-r}$ are called Bernstein basis polynomials or Bézier polynomials.

The approximation property of these polynomials is expressed by the following theorem:

Theorem 1.1. *If $f(x)$ is a continuous function on the interval $[0,\,1]$, then as $n \to \infty$, the sequence of polynomials $B_n(f;\,x)$ converges uniformly on the interval $[0,\,1]$ to the function $f(x)$.*

For other intervals, it is necessary to change the variable. For the purposes of this paper we introduce a new variable

$$t = 2x - 1 \quad \text{i.e.} \quad x = \frac{t+1}{2}. \tag{2}$$

It is easy to see that with this new variable, the domain of fucntion f is the interval $[-1,\,1]$.

Formula (1), in accordance with (2), can be rewritten as follows

$$B_n(f;\,t) = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} f_r (1+t)^r (1-t)^{n-r}, \tag{3}$$

where $f_r = f(t_r) = (f(-1),\,\ldots,\,f(1))$, $r = 0,\,\ldots,\,n$ — $n+1$-vector of samples of the function f at the points t_r.

Definition 1.2. *The coefficients of the powers of z in the polynomial $(1+z)^{n-r}(1-z)^r$ are obviously polynomials of r; they are called Krawtchouk polynomials of order n. In other words, the polynomial $(1+z)^{n-r}(1-z)^r$ is a generating function for Krawtchouk polynomials of order n* [3, ch. 5, §2]*:*

$$(1+z)^{n-r}(1-z)^r = \sum_{s=0}^{n} K_s^{(n)}(r)\, z^s. \tag{4}$$

Due to the trivial identity

$$(1+t)^r(1-t)^{n-r} = (1+t)^{n-(n-r)}(1-t)^{n-r},$$

the formula (3) can be rewritten to the form

$$B_n(f;\,t) = \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r} f_{n-r} \sum_{s=0}^{n} K_s^{(n)}(n-r) t^s, \tag{5}$$

where $f_r = f(t_r) = f(-1),\, f(1)$, $r = 0,\,\ldots,\,n$ and since $\binom{n}{r} = \binom{n}{n-r}$, we finally obtain

$$B_n(f;\,t) = \frac{1}{2^n}\sum_{s=0}^{n}\left(\sum_{r=0}^{n}\binom{n}{r}\, f_{n-r} K_s^{(n)}(r)\right)\cdot t^s, \tag{6}$$

which represents the expansion of the Bernstein polynomial in terms of the powers of the variable t. In the next section, we will provide a closed form for the coefficients of t^s in (6) using the definition of the MacWilliams transform, widely used in algebraic coding theory.

2. Bernstein polynomials and MacWilliams transform

Definition 2.1. *A square $(n+1)\times(n+1)$-matrix M_n, where*

$$(M_n)_{ij} = K_i^{(n)}(j), \quad 0 \le i, j < n \tag{7}$$

is called a MacWilliams matrix (see [4, p. 4, 18]*).*

For any column vector $u = (u_0, u_1, \ldots, u_n)$ of length $(n+1)$ its MacWilliams transform of order n is defined as the product

$$\mathcal{M}_n(u) = M_n\, u.$$

From the properties of Krawtchouk polynomials ([3], [4]), one can easily get the properties of MacWilliams matrices. Here are some of these properties:

Let $C = \operatorname{diag}\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right)$ and $\mathbf{I}$ is the identity matrix. Then the following relations hold:

1. Explicit formula: $K_s^{(n)}(r) = \sum_{l=0}^{s}(-1)^l\binom{n-r}{s-l}\binom{r}{l}$;
2. Free term of Krawtchouk polynomial: $K_r^{(n)}(0) = \binom{n}{r}$;
3. Orthogonality: $\sum_{i=0}^{n}\binom{n}{i}K_r^{(n)}(i)K_s^{(n)}(i) = 2^n\binom{n}{r}\delta_{r,\,s}$, i.e. $M_nCM_n^T = 2^nC$;
4. Involutiveness: $\sum_{i=0}^{n}K_r^{(n)}(i)K_i^{(n)}(s) = 2^n\delta_{r,\,s}$, i.e. $M_n^2 = 2^n\,\mathbf{I}$ and $M_n^{-1} = \frac{1}{2^n}M_n$
5. Reciprocity formula: $\binom{n}{r}K_s^{(n)}(r) = \binom{n}{s}K_r^{(n)}(s)$ i.e. $M_n^T = C^{-1}M_nC$

Some examples of MacWilliams matrices are shown below in 3.1.1 (see also [4])

Let ${}^{\beta}f = \left(\binom{n}{r}\cdot f_{n-r}\right)_{0\le r\le n}^{T}$ be the column vector of the samples of the function f with binomial weights, and let $\mathrm{T}_n(f)$ be the vector of coefficients of the Bernstein polynomial $B_n(f;\,t)$ in the basis t^s.

Then, it is easy to see that using the introduced notation, formula (6) can be written as

$$\mathrm{T}_n(f) = \frac{1}{2^n}\mathrm{M}_n\,{}^{\beta}f = \mathrm{M}_n^{-1}\,{}^{\beta}f, \tag{8}$$

which allows us to represent the Bernstein polynomial as

$$B_n(f;\,t) = \sum_{s=0}^{n}(\mathrm{T}_n(f))_s t^s. \tag{9}$$

Our previous considerations can be formulated as follows:

Proposition 2.1. *The $(n+1)$-dimensional vector of coefficients of the Bernstein polynomial $T_n(f)$ is a MacWilliams transform of the reverse vector of samples of the function f with the binomial weight, divided by 2^n.*

3. Pascal-MacWilliams pyramid

In this section, we will show that the set of MacWilliams matrices can be naturally represented as a three-dimensional pyramid, where the horizontal sections are the matrices M_n, and each such section is the algebraic sum of the shifts of the previous

section, similar to what happens for the rows of Pascal's triangle, which justifies the name of this pyramid [1].

For such representation, we will use the results of [4]. We introduce the following notation used in this work.

Definition 3.1. *For any matrix A, we define its zero-padded matrix $\overline{|A}$, where the horizontal (vertical) bar stands for $\mathbf{0}$ — a row (column) of zeros:*

$$\overline{|A} = \left[\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0} & A \end{array}\right].$$

The notation $\overline{A|}$, $\underline{|A}$, and $\underline{A|}$ have a similar meaning.

With previous notation, the construction of the Pascal-MacWilliams pyramid is given by the following theorem:

Theorem 3.1. *For the MacWilliams matrices M_n, the following recurrence relation holds (a detailed proof is given in [4, p. 7]):*

$$M_{n+1} = \left(\underline{M_n|} + \overline{M_n|} + \underline{|M_n} - \overline{|M_n}\right) \cdot \operatorname{diag}(1,\, 1/2,\, \ldots,\, 1/2,\, 1),\ n \geq 0,\ M_0 = [1]. \tag{10}$$

Example 3.1.1. *Let's list the MacWilliams matrices, computing them using formula (10):*

$$M_0 = [1];$$

$$M_1 = \left(\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\1&0\end{bmatrix} + \begin{bmatrix}0&1\\0&0\end{bmatrix} - \begin{bmatrix}0&0\\0&1\end{bmatrix}\right) \cdot \begin{bmatrix}1&0\\0&1\end{bmatrix} = \begin{bmatrix}1&1\\1&-1\end{bmatrix};$$

$$M_2 = \left(\begin{bmatrix}1&1&0\\1&-1&0\\0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0\\1&1&0\\1&-1&0\end{bmatrix} + \begin{bmatrix}0&1&1\\0&1&-1\\0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0\\0&1&1\\0&1&-1\end{bmatrix}\right) \cdot \begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1\\2&0&-2\\1&-1&1\end{bmatrix};$$

$$M_3 = \left(\begin{bmatrix}1&1&1&0\\2&0&-2&0\\1&-1&1&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\1&1&1&0\\2&0&-2&0\\1&-1&1&0\end{bmatrix} + \begin{bmatrix}0&1&1&1\\0&2&0&-2\\0&1&-1&1\\0&0&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0\\0&1&1&1\\0&2&0&-2\\0&1&-1&1\end{bmatrix}\right) \cdot \begin{bmatrix}1&0&0&0\\0&\frac{1}{2}&0&0\\0&0&\frac{1}{2}&0\\0&0&0&1\end{bmatrix} = \begin{bmatrix}1&1&1&1\\3&1&-1&3\\3&-1&-1&3\\1&-1&1&-1\end{bmatrix}.$$

4. Conclusion

1. Proposition 2.1 can be reformulated as follows: Let $V_n = \mathbb{Z}_2^n$ be the so-called dyadic group of dimension n, that is, an n-dimensional vector space over the field $\mathbb{Z}_2$. Let $f : V_n \to \mathbb{R}$ be a real-valued function on this space defined by the formula:
$$\forall v \in V_n \quad f(v) = f(|v|) = f_k, \tag{11}$$
where $k = |v|$ is the Hamming weight of the vector v, $0 \le k \le n$.

   Then Proposition 2.1 together with formula (11) show that problem of finding the coefficient-list $T_n(f)$ of the Bernstein polynomial is the problem of harmonic analysis on V_n, since the Krawtchouk polynomials $K_r^{(n)}$ are themselves the Fourier transforms (or, equivalently, the Hadamard transforms) on the group V_n of characteristic functions of Hamming spheres
$$S_r = \{v \in V_n \mid |v| = r\},\ 0 \le r \le n.$$
2. Similarly to how (as is well-known) Pascal's triangle can be considered as the result of successive states of a certain one-dimensional cellular automaton, the Pascal-MacWilliams pyramid can also be interpreted as the result of the operation of a similar but two-dimensional automaton, which was presented by one of the authors of this publication in the Wolfram Library Archive in 2004: https://library.wolfram.com/infocenter/MathSource/5223/

References

[1] Bondarenko Boris A. *Generalized Pascal triangles and pyramids.* The Fibonacci Association, 1993.

[2] Gerald E. Farin. *Curves and Surfaces for Computer-Aided Geometric Design. A Practical Guide.* Academic Press, 3rd edition, 2014.

[3] MacWilliams F.J. and Sloane N.J.A. *The theory of error correcting codes.* Northholland Publishing Company, 1977.

[4] Gogin Nikita and Hirvensalo Mika. *Recurrent construction of MacWilliams and Chebyshev matrices.* Fundam. Informaticae, 16(1–4):93–110, 2012.

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Construction and application of fully symmetric quadrature rules on the simplexes

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Abstract. A method for constructing fully symmetric quadrature rules of Gaussian type with positive weights, and with nodes lying inside the simplex and their applications are discussed.

Introduction

Substantial part of mathematical models in nuclear physics are formulated initially as the multidimensional elliptic boundary-value problems, for example, the consistent quadrupole-octupole vibration collective nuclear model [1]. To study such models a significant computer resource is needed because for its reduction, where the potential energy and components of the metric tensor are given by an order of 2×10^6 tabular values, to an algebraic problem the Monte-Carlo calculations of multidimensional integrals where conventionally applied. Some win can be achieved by application of the new economical computational schemes of the finite element method (FEM) [2].

The key problem in the implementation of the FEM schemes is the calculation of multidimensional integrals. It is well known [3] that as a result of applying the p-th order FEM to the solution of the discrete spectrum problem for the elliptic (Schrödinger) equation, the eigenfunction and the eigenvalue are determined with an accuracy of the order $p+1$ and $2p$, respectively, provided that all intermediate quantities are calculated with a sufficient accuracy. It follows that for the realization of the FEM of the order p, the corresponding integrals must be computed at least with an accuracy of the order $2p$. The most economical way of calculating of such integrals is the application of the quadratures of the Gaussian type.

In this talk, we restrict ourselves to constructing a system of nonlinear algebraic equations and numerical methods for solving it. The detailed description of construction of the fully symmetric quadrature rules with positive weights and with nodes lying in the simplex is given in [4].

1. Fully symmetric quadrature rules for the d-simplex

Let us construct the d-dimensional p-ordered quadrature rule

$$\int_{\Delta_d} V(x)dx=\frac{1}{d!}\sum_{j=1}^{N_{dp}} w_j V(x_{j1},\dots,x_{jd}), \quad x=(x_1,\dots,x_d), \quad dx=dx_1\cdots dx_d, \quad (1)$$

for integration over the d-dimensional standard unit simplex Δ_d with vertices $\hat{x}_j=(\hat{x}_{j1},\dots,\hat{x}_{jd})$, $\hat{x}_{jk}=\delta_{jk}$, $j=0,\dots,d$, $k=1,\dots,d$, which is exact for all polynomials of the variables $x_1,\dots,x_d$ of degree not exceeding p. In Eq. (1) N_{dp} is the number of nodes, w_j are the weights, and $(x_{j1},\dots,x_{jd})$ are the nodes.

We consider fully symmetric quadrature rules with positive weights and with nodes lying in the simplex (so-called PI-type) and for this will use the symmetric combinations of barycentric coordinates (BC) $(y_1,\dots,y_{d+1})$ that called orbits [4]. The orbit $S_{[i]}\equiv S_{m_1\dots m_{r_{di}}}$ contains the BC

$$(y_1,\dots,y_{d+1})=(\overbrace{\lambda_1,\dots,\lambda_1}^{m_1\text{ times}},\dots,\overbrace{\lambda_{m_{r_{di}}},\dots,\lambda_{m_{r_{di}}}}^{m_{r_{di}}\text{ times}}),$$
$$\sum_{j=1}^{r_{di}} m_j=d+1, \quad \sum_{j=1}^{r_{di}} m_j\lambda_j=1, \quad m_1\geq\cdots\geq m_{r_{di}}.$$

Substituting symmetric polynomials of degree p in (1) instead of $V(x)$, we obtain a system of nonlinear algebraic equations w.r.t unknowns $W_{i,j}$ and $\lambda_{i,jl}$:

$$\int_{\Delta_d} s_2^{l_2}s_3^{l_3}\times\cdots\times s_{d+1}^{l_{d+1}}dx=\frac{1}{d!}\sum_{i=0}^{M_d} P_{di}\sum_{j=1}^{K_{di}} W_{i,j}s_{i,j2}^{l_2}s_{i,j3}^{l_3}\times\cdots\times s_{i,jd+1}^{l_{d+1}}, \quad (2)$$
$$s_k=\sum_{l=1}^{d+1} x_l^k, \quad s_{i,jk}=\sum_{l=1}^{r_{di}} m_l\lambda_{i,jl}^k \quad 2l_2+3l_3+\cdots+(d+1)l_{d+1}\leq p,$$

where P_{di} is the number of different permutations of the BC corresponded to the orbit $S_{[i]}$. The number of independent equations for fully symmetric p-order quadrature rules is presented in Table 1.

$d\backslash p$	4	6	8	10	12	14	16	18	20
2	4	7	10	14	19	24	30	37	44
3	5	9	15	23	34	47	64	84	108
4	5	10	18	30	47	70	101	141	192
5	5	11	20	35	58	90	136	199	282
6	5	11	21	38	65	105	164	248	364

TABLE 1. The numbers E_{dp} of independent equations for fully symmetric p-order quadrature rules.

2. Numerical technique

We have chosen a modified Levenberg-Marquardt method [5, 6] to solve a system of nonlinear equations with convex constraints, that is more robust to the initial guess than Newton-type methods, and can be more stable than Newton-type method in the cases when the inverse problem becomes ill-posed.

Consider the problem of solving the constrained system of nonlinear equations

$$f_i(\mathbf{x}) = 0, \quad i = 1, \ldots, m, \quad \mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{X}, \tag{3}$$

and the corresponding minimization problem

$$\min_{\mathbf{x} \in \mathbf{X}} \|\mathbf{F}(\mathbf{x})\|^2, \quad \mathbf{F}(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T, \tag{4}$$

where $\mathbf{X} \subseteq R^n$ is a nonempty, closed and convex set. LM-type algorithm is an iterate method which, basically, solves at each iteration a linearization subproblem with the form

$$\min_{\mathbf{x}^k + \mathbf{h} \in \mathbf{X}} G_k(\mathbf{h}), \quad G_k(\mathbf{h}) = \frac{1}{2}\|\mathbf{F}(\mathbf{x}^k) + \mathbf{J}_k\mathbf{h}\|^2 + \frac{1}{2}\mu_k(\mathbf{h}, \mathbf{D}_k\mathbf{h}), \tag{5}$$

where $\mathbf{x}^k$ is the current iterate, $\mathbf{J}_k \in R^{m \times n}$ is a Jacobian of $\mathbf{F}(\mathbf{x})$ at $\mathbf{x} = \mathbf{x}^k$, $\mathbf{D}_k \in R^{n \times n}$ is a positive diagonal matrix and in most cases $\mathbf{D}_k = \mathrm{diag}(\mathbf{J}_k^T\mathbf{J}_k)$ or a unit matrix, and μ_k is a positive parameter. Note that $G_k(\mathbf{h})$ is a strictly convex quadratic function. Hence the solution $G_k(\mathbf{h})$ of subproblem (5) always exists uniquely, in particular for unconstrained case

$$\mathbf{h}^k = -(\mathbf{J}_k^T\mathbf{J}_k + \mu_k\mathbf{D}_k)^{-1}\mathbf{J}_k^T\mathbf{F}(\mathbf{x}^k). \tag{6}$$

Conclusion

Using the presented technique the quadrature rules up to 20-th order on the tetrahedron, 16-th order on 4-simplex, 10-th order on 5- and 6-simplexes are obtained [4]. For the convenience of their use, the INQSIM program for unpacking them in expanded form, and examples of their application are provided in JINRLIB Program Library [7]. The developed method is oriented on solving the six-dimensional elliptic boundary value problem by the finite element method for describing the discrete spectrum of the collective model of the atomic nuclei [1, 2].

References

[1] A. Dobrowolski, K. Mazurek, A. Góźdź, *Consistent quadrupole-octupole collective model*, Phys. Rev. C 94 (2016) 0543220-1-20.

[2] A.A. Gusev, S.I. Vinitsky, O. Chuluunbaatar, A. Góźdź, A. Dobrowolski, K. Mazurek, P.M. Krassovitskiy, *Finite element method for solving the collective nuclear model with tetrahedral symmetry*, Acta Physica Polonica B Proceedings Supplement 12 (2019) 589–594.

[3] G. Strang, G.J. Fix, *An analysis of the finite element method.* Prentice-Hall, Englewood Cliffs (1973).

[4] G. Chuluunbaatar, O. Chuluunbaatar, A.A. Gusev, S.I. Vinitsky, *PI-type fully symmetric quadrature rules on the 3-, ... , 6-simplexes*, Computers & Mathematics with Applications 124 (2022) 89–97.

[5] K. Levenberg, *A method for the solution of certain non-linear problems in least squares*, The Quarterly of Applied Mathematics 2 (1944) 164–168.

[6] D. Marquardt, *An algorithm for least squares estimation of parameters*, Journal of the Society for Industrial and Applied Mathematics 11 (1963) 431–441.

[7] G. Chuluunbaatar, O. Chuluunbaatar, A.A. Gusev, S.I. Vinitsky, *INQSIM - a program for converting PI-type fully symmetric quadrature rules on 2-,..., 6-simplexes from compact to expanded forms*, available at `http://wwwinfo.jinr.ru/programs/jinrlib/inqsim/indexe.html`.

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Investigation of the Stationary Motions of the System of Two Connected Bodies Moving along a Circular Orbit Using Polynomial Algebra Methods

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Abstract. Polynomial algebra methods are used to determine the equilibrium orientations of a system of two bodies connected by a spherical hinge that moves on a circular orbit. Primary attention is given to the study of equilibrium orientations of the two-body system in the plane perpendicular to the circular orbital plane. A method is proposed for transforming the system of trigonometric equations determining the equilibria into a system of polynomial equations, which in turn are reduced by calculating the resultant to a single algebraic equation of degree 12 in one unknown. By applying symbolic factorization, this algebraic equation is decomposed into three polynomial factors, each specifying a certain class of equilibrium configurations. The domains with an identical number of equilibrium positions are classified using algebraic methods for constructing a discriminant hypersurface. Using the proposed approach, it is shown that the system can have up to 48 equilibrium orientations in the plane perpendicular to the circular orbit.

Introduction

In our work, we apply polynomial algebra methods to investigate the equilibrium orientations of a system of two bodies (satellite and stabilizer) connected by a spherical hinge that moves in a central Newtonian force field along a circular orbit. Determining the equilibria for the system of bodies on a circular orbit is of practical interest for designing composite gravitational orientation systems of satellites that can stay on the orbit for a long time without energy consumption. The dynamics of various composite schemes for satellite–stabilizer gravitational orientation systems was discussed in detail in [1]. In [2], [3], [4] equilibrium orientations for the two-body system in the orbital plane were found in the case where the spherical hinge was positioned at the intersection of the principal central axes of inertia of the satellite and stabilizer, as well as in the case where the hinge was positioned on

the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer. In this work, we study the equilibrium orientations of the two-body system in the plane perpendicular to the circular orbital plane in the case when the hinge is positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer..

1. Investigation of Equilibrium Orientations

We consider a system of two bodies connected by a spherical hinge that moves along a circular orbit [4]. To write the corresponding equations of motion, we introduce the following right-handed rectangular coordinate systems. The orbital coordinate system is $OXYZ$. The $OZ-$ axis is directed along the radius vector that connects the Earth's center of mass with the center of mass of the two-body system O, the $OX-$ axis is directed along the linear velocity vector of the center of mass O, while the $OY-$ axis is directed along the normal to the orbital plane. The coordinate system of the ith body (i= 1, 2) is $O_i x_i y_i z_i$, where the axis of these coordinate systems are the principal central axes of inertia of the ith body. The orientation of coordinate system $O_i x_i y_i z_i$ with respect to the orbital coordinate system is determined using aircraft angles [1].

Suppose that (a_i, b_i, c_i) are the coordinates of spherical hinge in the coordinate system $O_i x_i y_i z_i$; A_i, B_i, C_i are the principal central moments of inertia of the each bodies; $M = M_1 M_2/(M_1 + M_2)$; M_i is the mass of the ith body.

Using the expressions of the kinetic energy of the two-body system and the force function that determines the action of the Earth's gravitational field on the two-body system in the case where $c_1 = c_2 = 0$ and its equilibrium orientations are in a plane perpendicular to the orbital plane (then, the coordinates of spherical hinge in the coordinate system of each body are given by $(a_i, b_i, 0)$) the equations of motion for this system we can written in the form of Lagrange equations of the second kind [1]. Then from Lagrange equations we can obtain the stationary trigonometric system which allows us to determine equilibrium orientations for the system of two bodies connected by the spherical hinge in the orbital coordinate system:

$$\begin{aligned}\big((B_1 - A_1)/M\big)\sin x_1 \cos x_1 + (a_1 \sin x_1 + b_1 \cos x_1)(a_1 \cos x_1 - b_1 \sin x_1) - \\ -(a_1 \cos x_1 - b_1 \sin x_1)(a_2 \sin x_2 + b_2 \cos x_2) = 0, \\ \big((B_2 - A_2)/M\big)\sin x_2 \cos x_2 + (a_2 \sin x_2 + b_2 \cos x_2)(a_2 \cos x_2 - b_2 \sin x_2) - \\ -(a_2 \cos x_2 - b_2 \sin x_2)(a_1 \sin x_1 + b_1 \cos x_1) = 0,\end{aligned} \quad (1)$$

where x_1 and x_2 are two of the aircraft angles.

The trigonometric system (1) cannot be solved analytically for two unknown aircraft angles. To solve system (1), we use the universal approach whereby the sines and cosines of angles x_i are replaced by their tangents $t_i = \tan(x_i)$.

As a result, we obtain from (1) the algebraic system of two equations in two unknowns t_1, t_2

$$\begin{aligned} \bar{a}_0 t_1^3 + \bar{a}_1 t_1^2 + \bar{a}_2 t_1 + \bar{a}_3 = 0, \\ \bar{b}_0 t_1^2 + \bar{b}_1 t_1 + \bar{b}_2 = 0, \end{aligned} \tag{2}$$

where $\bar{a}_i, \bar{b}_i$ are polynomials depending on six system parameters.

By using the resultant approach to eliminate t_1 from system (2) and symbolic computations in Wolfram Mathematica 12.1 to find the determinant of the resultant matrix, we obtain a twelfth-order algebraic equation in one unknown t_2, which upon factorization, turns into a product of three polynomials: $P(t_2) = P_1(t_2)P_2(t_2)P_3(t_2) = 0$. Here $P_1(t_2)$, $P_2(t_2)$ are second-order polynomials and $P_3(t_2)$ is an eighth-order polynomial, the coefficients of which are polynomials in six system parameters.

By the definition of the resultant, each root of equation $P(t_2) = 0$ corresponds to one common root of system (2) . The algebraic equation obtained has the even number of real roots, which does not exceed 12. By substituting real root of algebraic equation $P(t_2) = 0$ into the equations of system (2) , we find the common root of these equations. It can be shown that four equilibrium solutions of the original system correspond to each real root of equations (2).

Since the total number of real roots of $P(t_2) = 0$ does not exceed 12, the satellite–stabilizer system in the plane perpendicular to the orbital plane can have no more than 48 equilibrium orientations in the orbital coordinate system. Using obtained equations for each set of system parameters, we can determine all equilibrium orientations of the satellite–stabilizer system in the orbital coordinate system.

To investigate the number of equilibrium solutions for the satellite–stabilizer system, we define domains with equal numbers of real roots of $P_3(t_2) = 0$ in the space of the six parameters. For this purpose, we construct a discriminant hypersurface of this polynomial, which defines the boundary of the domains with equal numbers of real roots.

Conclusion

The use of polynomial computer algebra methods allowed us to solve the classical problem of space flight mechanics in a fairly simple form.

References

[1] V.A. Sarychev, *Problems of orientation of satellites*, Itogi Nauki i Tekhniki, Ser. "Space Research", Vol. 11. Moscow: VINITI, 1978.

[2] V.A. Sarychev, *Relative equilibrium orientations of two bodies connected by a spherical hinge on a circular orbit*, Cosmic Research. 1967. Vol. 5, No. 3. P. 360–364.

[3] S.A. Gutnik, V.A. Sarychev, *Symbolic investigation of the dynamics of a system of two connected bodies moving along a circular orbit*, In: England M., Koepf W., Sadykov T.M., Seiler W.M., Vorozhtsov E.V. (eds.) CASC 2019. LNCS, Vol. 11661, P. 164–178. Springer, Cham (2019)

[4] S.A. Gutnik, V.A. Sarychev, *Research into the Dynamics of a System of Two Connected Bodies Moving in the Plane of a Circular Orbit by Applying Computer Algebra Methods*, Computational Mathematics and Mathematical Physics. 2023. Vol. 63, No.1, P. 106–114.

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Intergration of new mathematical ideas into engineering curriculum: case of tropical mathematics

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Abstract. The case of introduction of Tropical Mathematics at the Faculty of Computer Science and Technology, St. Petersburg Electrotechnical University "LETI", is considered. Using it as an example, a tentative scheme of introduction of a new area of research to the existing education process is presented.

Introduction

Most results taught in basic mathematical courses at engineering higher education establishments (such as algebra or calculus) date back to XVIII – XIX centuries. In this aspect the discrete line of courses developed at LETI ("Discrete Mathematics and Computer Science", "Combinatorics and Graph Theory", "Mathematical Logic and Theory of Algorithms") are a refreshing exception, as many results discussed in these courses date to the second half of XX century or even to XXI century. But acquainting students with current state of mathematics and its applications which are most relevant to the students remains a burning question. In order to develop applied areas and introduce new ideas into engineering practice as soon as possible, it is important to introduce familiarisation with modern areas of research into the educational process. Such introduction is beset with a variety of obstacles, beginning with a rather limited available time and ending with a certain conservatism of the system of education. In 2019 the deptartment of Algorithmic Mathematics initiated the development of pegagogical technologies of introduction of new mathematical ideas into general education process. A new area of research must satisfy a set of criteria to be eligible for such an introduction:

1. Recently emerged areas have an advantage: they usually have a more moderate learning curve, and there probably are easy unsolved problems.
2. The area must be relevant for our students in their professional capacity, i.e. it must have enough applications, the relevance of which should be obvious

(and ideally should be heard ofeven by non-specialists) and easily confirmed by examples. This also alleviates the problem of motivating the students.

3. The area should be immersed in context the students are in, both educationally and professionally. This means that the area should be grounded in what our students learned before, it should tie in with what they are learning at the moment, and finally it should be applicable to their professional area of expertise. Ideally such area of research should provide a whole spectrum of opportunities, ranging from purely applied use in the industry for those who will use it only as one of possible tools of profession, to the opportunity to actually pursue research in it.
4. The area should have researchers willing to cooperate with the university. Introduction of a new area means it is necessary to grow and educate a cadre of teachers first. We propose that collaboration with leading researchers can be a successful center of crystallization of such a process. Of course, using leading researchers to teach fresh students is a waste of resources, but they can and should help with source material, read an introductory course for teachers, oversee seminars and act as scientific advisor for whose who want to pursue research, both students and teachers.

Tropical mathematics fit these criteria ideally. The area is young (Wikipedia dates its emergence to early 2000's), although many relevant ideas percolated long before it. The learning curve is not too steep, especially for students that paid attention at algebra and algebraic structures. Just saying "neural nets" is enough to establish its relevance. It also helps to bring the students into the context, as does the emergence of familiar terms, starting with the most basic, such as "semiring" or "piecewise-linear function". Students who have already covered graph theory can use the problem of finding the shortest paths in a graph as a relevant and manageable example, as it has a very compact and elegant interpretation in tropical terms. And last but not the least, D. Y. Grigoriev and N. N. Vasiliev have agreed to collaborate with us. Thus was made the decision that tropical mathematics was coming to LETI. By this moment the department also had members willing to pursue reseacrh in this area.

1. Stages of integration

The experimental integration of tropical mathematics into LETI's educational process was planned as a sequence of stages.

1.1. Stage 0

D. Y. Grigoriev read a small series of open lectures on tropical mathematics at LETI. The lectures proved to be of interest both to students and the faculty. Objectives at this point were:

- Familiarize with the term "tropical mathematics";
- Gauge the students´ response to the presented area;

- Notify the students that faculty is interested in such activities;
- Show the students — albeit rather superficially — what modern mathematical research looks like.

Results of stage 0 were:

- Recorded lectures of D. Y. Grigoriev, which were made available and which are, if views are any indicator, in demanhd;
- Quite substantial discussion during the lectures;
- Emergence of a group of interested students, which comprised the core of one branch of experiment during stage 1.

1.2. Stage 1

Stage 1 was comprised of three branches developing in parallel:

1. Student seminar on tropical mathematics and neural nets. This seminar alternated reports on tropical mathematics and neural nets. Articles on tropical mathematics were supplied by D. Y. Grigoriev. Faculty members also attended this seminar, acting as audience and consultants. Seminar persisted in this form for several years.
2. Seminar on algorithmic mathematics, supervised by N. N. Vasiliev, with mixed audience including students, graduate students and faculty. Some reports were on tropical mathematics. Seminar persists to this day.
3. D. Y. Grigorievś and N. N. Vasilievś collaboration with departmentś members, master students and graduate students. This collaboration is ongoing.

Results of stage 1 were:

- Familiarisation with the term “tropical mathematics”;
- Students learned enough to discuss tropics themselves, and acquired a habit of doing so. Simultaneously, an environment formed which encouraged such discussion, analysis and idea exchange;
- Emergence of a group of senior students who could supervise or consult junior students;
- Emergence of department members who could answer questions on the topic of tropical mathematics or act as an advisor for a work over specific problem, an alternative exam, or even a barchelorś or masterś thesis. It is important to stress that these department members were known as such to the students.

1.3. Stage 2

During the second stage (which is currently underway) a regular schedule of seminar work was established. Student seminars in small groups (up to 15 members) are regularly conducted, with authors as supervisors. For 4 years this seminar was conducted as an elective course for senior year barchelor students. This year the seminar also took place for second-year master students of our department (it should be noted that it was the first batch of master students our department had). Beside that, each year several freshmen take up tropical mathematics as a topic for alternative exam. Usually they are supervised by senior students, but one

time this group was too big, and authors had to step in as supervisors. Results of stage 2 are:

- Tropical mathematics has firmly secured a place in the curriculum, both formally and psychologically.
- Neural nets as the main application considered completely solve the problem of motivation and relevance.
- Number of students acquainted with tropical mathematics has grown considerably. The environment that was formed during the previous stage has also grown. For example, the students´ university "IT LETI" (a departments spin-off project) has a course on tropical mathematics.
- Methodics of teaching tropical mathematics to students of LETI are in the process of development.

Conclusion

All above is a work already done. Plans for the next stage include scaling the teaching of tropical mathematics both in the size of student group and the amount of alotted time. Tropical mathematics are slated to be offered as an elective for all senior-year barchelor students of FCST. "Basics of tropical mathematics" course is planned to be introduced for barchelors, while our department opens its own barchelor program. A number of difficulties is expected to arise as a consequence of such a scaling. Ways of mitigating them are being developed.

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How many roots of a random polynomial system on a compact Lie group are real?

B. Kazarnovskii

A finite linear combination of matrix elements of a finite dimensional real representation π of a Lie group K is said to be a real π-polynomial on K. If K is compact then any π-polynomial uniquely extends to a holomorphic function on the complexification $K_{\mathbb{C}}$ of K. For example, any trigonometric polynomial

$$f_m(\theta) = c + \sum_{1\le k\le m,\ \alpha_k,\beta_k\in\mathbb{R}} \alpha_k \cos(k\theta) + \beta_k \sin(k\theta)$$

on the 1-dimensional torus K (that is the unit circle $\{e^{i\theta}\colon \theta \in \mathbb{R}\} \subset \mathbb{C}$) extends uniquely to a Laurent polynomial

$$P_m(z) = c + \sum_{k\le m,\ a_k\in\mathbb{C}} a_k z^k + \bar{a}_k z^{-k}$$

on $\mathbb{C} \setminus 0$.

For a system of n π-polynomials, where $n = \dim(K)$, we consider the proportion of real roots, that is the ratio of the number of roots in K to the number of roots in $K_{\mathbb{C}}$. The source of these calculations is the following result by M. Kac [Ka]: *the expected proportion of real zeros of a random real polynomial of degree m asymptotically equals* $\frac{2}{\pi}\frac{\log m}{m}$.

Replacing ordinary polynomials with Laurent ones (see [K1]) and then with arbitrary π-polynomials on a compact Lie group leads to an unexpected result. It turns out that for growing representation π and random system of π-polynomials, the expected proportion of real roots converges not to 0, but to a nonzero constant. The limit is calculated in terms of the volumes of some compact convex sets that determine the growth of the representation π. For a 1-dimensional torus K the limit is $1/\sqrt{3}$.

References

[Ka] M. Kac. On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. 49 (1943), 314–320 (Correction: Bull. Amer. Math. Soc., Volume 49, Number 12 (1943), 938–938)

[K1] B. Ya. Kazarnovskii. How many roots of a system of random trigonometric polynomials are real? Sbornik:Mathematics (213:4), 2022, 27–37 (in Russian)

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Constructive Versions of Quantum Mechanics

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Abstract. The standard formulation of quantum mechanics is essentially non-constructive, since it is based on continuous unitary groups and number fields $\mathbb{R}$ and $\mathbb{C}$. This descriptive flaw does not allow one to study some fine details of the structure of quantum systems and sometimes leads to artifacts.

In [1–3], we considered a modification of quantum mechanics based on permutation representations of finite groups in Hilbert spaces over cyclotomic fields. This *permutation quantum mechanics* (PQM) "can accurately reproduce all of the results of conventional quantum mechanics" [4] in the permutation invariant *standard subspace* of the Hilbert space. Unitary evolution in PQM is generated by a permutation of *ontic* elements, which form a basis of the Hilbert space. By decomposing the permutation into a product of disjoint cycles, we can split the Hilbert space into a direct sum of subspaces, in each of which the evolution generated by a cyclic permutation occurs independently. Thus, in an N-dimensional Hilbert space, it suffices to consider the evolutions generated by cycles of length N. Such a cycle generates the group $\mathbb{Z}_N$. Since any projective representation of a cyclic group is trivial, to describe quantum mechanical phenomena it is necessary to consider the product $\mathbb{Z}_N \times \tilde{\mathbb{Z}}_N$, where $\tilde{\mathbb{Z}}_N\,(\simeq \mathbb{Z}_N)$ is the Pontryagin dual group to $\mathbb{Z}_N$. Note that we have only changed the description slightly, without introducing any additional external information: if X and Z are matrices representing generators of $\mathbb{Z}_N$ and $\tilde{\mathbb{Z}}_N$, respectively, then Z is simply the diagonal form of X, obtained by the Fourier transform.

In fact, we have come to the Weyl–Schwinger version of quantum mechanics, which is sometimes called *finite quantum mechanics* (FQM). FQM arose as a result of Weyl's correction of Heisenberg's canonical commutation relation, which cannot be realized in finite-dimensional Hilbert spaces. Weyl's canonical commutation relation has the form

$$XZ = \omega ZX,\ \omega = \mathrm{e}^{2\pi \mathbf{i}/N},$$

where X and Z are the matrices mentioned above. Weyl proved that the X and Z are generators of a projective representation of $\mathbb{Z}_N \times \mathbb{Z}_N$ in the N-dimensional Hilbert space. The orthonormal bases associated with the matrices X and Z are *mutually unbiased bases*, a concept introduced by Schwinger.

FQM, constructive by its nature, requires mathematical tools that differ significantly from those used in traditional continuous theory: number theory, Galois field theory, complex Hadamard matrices, finite geometries, etc.

At the same time, in FQM, it is possible to pose and solve problems that are important for fundamental quantum theory and quantum informatics, but which are difficult or even impossible to formulate within the framework of standard quantum mechanics. Let us give examples of problems in which the structure of the decomposition of the dimension of the Hilbert space into prime numbers is essential, which does not make sense in continuous quantum mechanics:

- decomposition of a quantum system into smaller subsystems;
- calculation of sets of mutually unbiased bases (sets of orthonormal bases in Hilbert space, measurements in which give maximum information about the quantum state);
- construction of symmetric information-complete positive operator-valued measures (SIC-POVM, a symmetric set of vectors in a Hilbert space, important for quantum measurement theory and related to Hilbert's 12th problem).

Modern problems of quantum physics and quantum informatics require a detailed analysis of the "fine structure" of quantum systems, which cannot be carried out using traditional approximate methods of quantum mechanics. However, exact methods are complex and often involve open (unsolved) mathematical problems. In these circumstances, a natural approach is to use computer calculations based on the methods of computer algebra and computational group theory.

References

[1] Kornyak V.V. *Quantum models based on finite groups.* IOP Conf. Series: Journal of Physics: Conf. Series **965**, 012023, 2018. arXiv:1803.00408 [physics.gen-ph]

[2] Kornyak V.V. *Modeling Quantum Behavior in the Framework of Permutation Groups.* EPJ Web of Conferences **173**, 01007, 2018. arXiv:1709.01831 [quant-ph]

[3] Kornyak V.V. *Mathematical Modeling of Finite Quantum Systems.* In: Adam G. *et al* (eds) MMCP 2011. LNCS, **7125**. Springer, 2012. arXiv:1107.5675 [quant-ph]

[4] *Banks T.* Finite Deformations of Quantum Mechanics. arXiv:2001.07662 [hep-th], 20 p., 2020.

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F-polynomials & Newton polytopes

Gleb Koshevoy and Denis Mironov

Abstract. We provide an effective algorithmic method for computation of Gross-Keel-Hacking-Kontsevich potential, F-polynomials and Bernstein-Kazhdan decoration function and its complexity bounds. For simply laced Lie algebras we make conjecture and provide experimental evidence for lattice points of Newton polytopes for Gross-Keel-Hacking-Kontsevich potential.

1. F-polynomials and Gross-Hacking-Keel-Kontsevich potentials

Let G be a group with the Lie algebra of simply-laced type , B_+ and B_- be its Borel subgroups, with the set of simple roots α_a, $a \in I$, W the Weyl group. The Gross-Hacking-Keel-Kontsevich potential (GHKK for short) W_{GHKK} is a function on the double Bruhat cell $G^{w_0,e} = B_- \cap B_+\overline{w_0}B_+$, defined using cluster algebra's tools [5]. Because of validity the Fock-Goncharov conjecture in such cases [4], we get the polyhedral parametrization of canonical bases of the ring of regular functions on G/B arising from the tropicalizations of the potential.

Specifically, the ring of regular functions on the double Bruhat cell is endowed with the cluster algebra structure. Namely, for a reduced decomposition $\mathbf{i}$ of the longest element $w_0 \in W$ with length N, let $\Sigma_{\mathbf{i}}$ be a corresponding X-cluster seed and $Q_{\mathbf{i}}$ be the corresponding quiver (due to [1]). Then W_{GHKK} is a polynomial in the cluster variables $\Sigma_{\mathbf{i}}$([9]).

The frozen variables of this cluster algebra (and corresponding vertices of quiver) are labeled by the set $-I \cup I$. A seed Σ with underlying quiver Q is *optimal for a frozen vertex* $a \in -I \cup I$ if after deleting arrows between frozen vertices and a, the vertex a becomes a source of the quiver Q.

For the optimal seed Σ, the ath part of the GHKK-potential is equal to the value of the corresponding frozen cluster variable,

$$W_a = Y_a. \tag{1}$$

For a frozen $a \in I$, there exists an appropriate reduced word $\mathbf{i}'$, such that seed $\Sigma_{\mathbf{i}'}$ is optimal for a.

Because of that, for a given reduced decomposition $\mathbf{i}$, one can compute the half

$$W'_{GHKK} = \sum_{a \in I} W_a$$

of the GHKK-potential

$$W_{GHKK} = \sum_{a \in -I \cup I} W_a$$

using cluster mutations corresponding to 3-braid moves between the reduced decompositions of w_0 (for l and k, such that $a_{lk} = -1$, $s_k s_l s_k = s_l s_k s_l$). Namely, for computing W_a, we apply a sequence of cluster mutations corresponding to 3-braid moves which transform $\Sigma_{\mathbf{i}}$ into an optimal seeds for a, then W_a is the X-cluster variable at the frozen vertex labeled by a in the optimal seed computed in the variables of the seed $\Sigma_{\mathbf{i}}$. In variables of the seed $\Sigma_{\mathbf{i}}$, such an X-cluster variable is equal to the specification of the F-polynomial (see [3, 8]) and takes the form

$$W_a = Y_1^{c_{1a}(t)} \cdots Y_N^{c_{Na}(t)} \prod_i F_i(t)(Y_1, \cdots, Y_N)^{b_{ia}(t)}. \tag{2}$$

In the above formula we take notations of [8], where t means the end vertex of the path in the mutation graph from the optimal seed for a to $\Sigma_{\mathbf{i}}$ and Y_j's are cluster variables of $\Sigma_{\mathbf{i}}$.

Precisely (see [9]) W_a is of the form product of the frozen $Y_a(t)$ and an F-polynomial.

From [4] we can compute full GHKK-potential:

$$W_{GHKK} = W'_{GHKK} + \sum_{i \in I} Y_{i_s}^{-1}(1 + Y_{i_{s-1}}^{-1}(1 + Y_{i_{s-2}}^{-1}(1 + Y_{i_{s-3}}^{-1}(\cdots))))), \tag{3}$$

where $i_1, i_2, \ldots i_s$ are indices of i in reduced decomposition $\mathbf{i}$.

2. Newton polytopes

We are interested of properties of the Newton polytopes of the individual terms W_a, $a \in I$, of the half-potential W_{GHHK}, as well as the Newton polytope of W_{GHHK}.

Fei in [2] conjectured that the Newton polytope of an F-polynomial has no interior integer points.

For minuscule weight $a \in I$, the validity of this conjecture for F-polynomials corresponding to terms W_a follows from Remark 5.17 [6]. Namely in such a case, the Newton polytope is a geometric realisation of a distributive lattice of the corresponding decorated graph DG_a. Since such a polytope is a convex hull of a subset of the vertices of a unit cube the claim follows.

We state the following

Conjecture 1. *For a simply-laced group G, and any reduced decomposition $\mathbf{i}$ of w_0, the Newton polytope of W'_{GHKK} is contains no interior integer lattice points.*

Conjecture 2. *For a simply-laced group G, and any reduced decomposition $\mathbf{i}$ of w_0, the Newton polytope of W_{GHKK} contains one interior integer lattice point.*

This conjectures are motivated by properties of affine Calabi-Yau manifolds and their mirror pairs should be affine too (thus rendering W'_{GHKK} void). Note that validity of this conjecture implies that the Newton polytopes of F-polynomials for frozens are void.

We made computer verification of Conjecture 1 for A_n, $n = 3, 4, 5, 6, 7$, D_n, $n = 4, 5, 6, 7$, E_6, E_7 and Conjecture 2 for A_n, $n = 3, 4, 5$, D_4 and some cases of D_5.

Note that the straightforward cluster computation of W_{GHKK} is time consuming, because the division of Laurent polynomials in many variables is slow.

We use another approach. Namely, because of Theorem 1 in [4], we compute W_{GHKK} by applying the algorithm of [7] described in [11] for computing the Berenstein-Kazhdan decoration function Φ_{BK}.

Theorem 1. *For simply-laced G, and a given reduced decomposition* $\mathbf{i}$*, the Newton polytopes Φ_{BK} and W_{GHKK} are isomorphic under a unimodular transformation.*

Corollary 1. *The Newton polytopes Φ_{BK} contain one or zero interior integer lattice points if and only if the Newton polytopes W'_{GHKK} is contain one or zero interior integer lattice points.*

Thus, for the numeric verification of Conjectures we compute the Newton polytope Φ_{BK} using the algorithm [7] and Polymake.

We establish a bound on the complexity of the algorithm computing $\Delta_{w_0\Lambda_i, s_i\Lambda_i}$ and W_j with respect to the number of monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i}$. Let K be the number of such monomials, and r be the rank of the Lie algebra.

Total complexity of generating the monomials of $\Delta_{w_0\Lambda_i, s_i\Lambda_i}$ is of complexity

$$O(r^4K) \sim O(r^2 * \textbf{length of string representation}).$$

W_j computation is bounded by multiplication complexity and number of edges in Gs: $O(r^2) * O(K * r^2) \sim O(r^4K)$. For a fixed r, this complexity is the lowest possible complexity being linear with respect to actual complexity to print out the answer.

Complexity of lattice point counting in polytopes has theoretical exponential upper bound with respect of number of inequalities defining polytope [10].

Actual computing speed is mostly determined by speed of Polymake operations. Average time of $\Delta_{w_0\Lambda_i, s_i\Lambda_i}$ and W_j computation for single simple root of D_6 with W'_{GHKK} polytope checking is around 2 seconds, for E_7 – 8 seconds. Checking Conjecture 2 for full W_{GHKK} polytope is much slower and can take from several hours to days even for D_5. For comparison, one computation for single simple root of D_6 without any Polymake operations takes around 70ms. For all computations we used a PC with dual 3.8 Ghz Intel® Xeon® Gold 5222 CPU running Ubuntu Linux.

References

[1] A.Berenstein, S.Fomin, and A.Zelevinsky, Cluster algebras.III. Upper bounds and double Bruhat cells. Duke Math. J. 126(1), 1–52 (2005)

[2] J. Fei. Combinatorics of F-polynomials Preprint. arXiv:1909.10151.

[3] S.Fomin and A. Zelevinsky, Cluster algebras.IV. Coefficients. Compos. Math.143,112–164(2007)

[4] V. Genz, G. Koshevoy, and B. Schumann, Polyhedral parametrizations of canonical bases & cluster duality, Advances in Mathematics 369 (2020), p. 107178

[5] M. Gross, P. Hacking, S.Keel, and M. Kontsevich, Canonical bases for cluster algebras. J. Am. Math. Soc. 31, 497–608 (2018)

[6] Y.Kanakubo, G.Koshevoy and T.Nakashima, An algorithm for Berenstein-Kazhdan decoration functions and trails for minuscule representations, J. of Algebra (2022)

[7] Y.Kanakubo, G.Koshevoy and T.Nakashima, An algorithm for Berenstein-Kazhdan decoration functions and trails for classical groups, arXiv:2207.08065

[8] Keller B., Cluster algebras and derived categories, arXiv:1202.4161

[9] G.Koshevoy and B.Schumann, Redundancy in string cone inequalities and multiplicities in potential functions on cluster varieties, J. of Algebraic Combinatorics (2022)

[10] J.Loerab, R.Hemmeckeb, J.Tauzera, R.Yoshida, Effective lattice point counting in rational convex polytopes, J. of Symbolic Computation, 38, 1273–1302 (2004)

[11] G. Koshevoy, D. Mironov, F-Polynomials and Newton Polytopes, SYNACS22, EasyChair Preprint no. 8975 (2022)

[12] https://github.com/mironovd/crystal

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An Algorithm for Solving Two-Sided Linear Vector Equations in Tropical Algebra

Nikolai Krivulin

Abstract. We consider a two-sided vector equation that is defined in terms of tropical algebra as $\boldsymbol{Ax} = \boldsymbol{By}$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are given matrices, $\boldsymbol{x}$ and $\boldsymbol{y}$ are unknown vectors. We propose a new procedure to solve this equation, which is based on the minimization of the distance between vectors of tropical vector spaces generated by the columns of the given matrices. The procedure produces a pair of vectors that provide the minimum distance between the spaces. If the two-sided equation has nontrivial solutions, the obtained vectors present a solution. Otherwise, these vectors compose a pseudo-solution that minimizes the deviation between both sides of the equation.

Introduction

We consider a vector equation that is defined in terms of tropical algebra as

$$\boldsymbol{Ax} = \boldsymbol{By},$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are given matrices, $\boldsymbol{x}$ and $\boldsymbol{y}$ are unknown vectors. This equation has unknowns on both sides and is usually referred to as the two-sided equation.

In tropical (idempotent) algebra, which deals with the theory and application of semirings and semifields with idempotent addition [1, 2, 3, 4, 5], solving the two-sided equation is a challenging problem from both analytical and the numerical perspectives. Since the first works of P. Butkovič [6, 7, 8] on the two-sided linear vector equation, the solution of this equation is still a topic of interest as can be seen in more recent papers [9, 10, 11, 12]. Existing approaches usually employ computational procedures based on iterative algorithms (see an overview of the state-of-art on the solution techniques given in Ch. 7 of [4]).

Available solution methods and techniques include combinatorial reduction algorithms [7], the elimination method [8], an algorithm of residuated functions for partially ordered sets [13], the alternating method [9], a combinatorial algorithm for the equation in the rational case [14], the method of bivariate equations and

inequalities [11], methods for the equation with square matrices of special type [12] and others. However, these approaches often turn out to be not efficient enough for practical problems because of high computational complexity, rather restrictive assumptions or for other reasons. Therefore, the development of new methods that are able to supplement and complement existing approaches to solving the two-sided equation under consideration seems to be a rather urgent work.

In this paper, we propose a new procedure to solve the two-sided equation by minimization of the distance between vectors of tropical vector spaces generated by the columns of the given matrices. The procedure produces a pair of vectors that minimize the distance between the spaces. If the equation has nontrivial solutions, the obtained vectors present a solution. Otherwise, these vectors compose a pseudo-solution that minimizes the deviation between both sides of the equation.

The execution of the procedure consists in constructing a sequence of vectors that are pseudo-solutions of the two-sided equation in which the left and right sides are alternately replaced by constant vectors. Unlike the alternating algorithm [9], in which the corresponding inequalities are solved one by one instead of equations, the proposed procedure uses a different argument, looks simpler, and allows one to establish natural criteria for completing calculations. If the equation has no solutions, the procedure also finds a pseudo-solution and determines the value of the error associated with it, which can be useful in solving approximation problems.

1. Preliminary Definitions and Notation

In this section, we present basic definitions and notation to provide a formal framework for the description and solution of the two-sided linear equation in the tropical algebra setting. Further details on the theory and applications of tropical algebra can be found in a range of works, including [1, 2, 3, 4, 5].

1.1. Idempotent Semifield

Consider a set $\mathbb{X}$ that is closed under addition $\oplus$ and multiplication $\otimes$, and it includes the zero $\mathbb{0}$ and the identity $\mathbb{1}$. Assume that $(\mathbb{X}, \oplus, \mathbb{0})$ is an idempotent commutative monoid, $(\mathbb{X} \setminus \{\mathbb{0}\}, \otimes, \mathbb{1})$ is an Abelian group, and multiplication $\otimes$ distributes over addition $\oplus$. The algebraic structure $(\mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is usually referred to as an idempotent semifield.

In the semifield, addition is idempotent and multiplication is invertible: for each $x \in \mathbb{X}$ the equality $x \oplus x = x$ holds, and if $x \neq \mathbb{0}$, there exists an inverse x^{-1} such that $xx^{-1} = \mathbb{1}$ (here and hereafter the multiplication sign $\otimes$ is suppressed).

The power notation with integer exponents is thought of in the sense of the multiplication $\otimes$. Additionally, it is assumed that equation $x^p = a$ has a unique solution x for any $a \in \mathbb{X}$ and integer $p > 0$ to allow for the powers with rational exponents, which makes the semifield algebraically closed.

Addition induces a partial order: $x \leq y$ if and only if $x \oplus y = y$. It is assumed that this order extends to a total order, which makes the semifield linearly ordered.

An example of linearly ordered algebraically closed idempotent semifields is the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, +)$, also known as $(\max, +)$-algebra. It has addition defined as max, multiplication as $+$, zero as $-\infty$ and identity as 0. The power x^y corresponds to the arithmetic product xy. The inverse x^{-1} of any $x \neq \mathbb{0}$ coincides with the opposite number $-x$. The order relation induced by addition agrees with the natural linear order on $\mathbb{R}$.

As another example, one can consider $\mathbb{R}_{\min} = (\mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times)$ (min-algebra), where $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$. It is equipped with the operations $\oplus = \min$ and $\otimes = \times$, which have the neutral elements $\mathbb{0} = +\infty$ and $\mathbb{1} = 1$. The notions of powers and inverses have the standard meaning. The partial order associated with addition is opposite to the natural linear order on $\mathbb{R}$.

1.2. Algebra of Matrices and Vectors

Let $\mathbb{X}^{m\times n}$ be the set of matrices over $\mathbb{X}$ with m rows and n columns. A matrix with all entries equal to $\mathbb{0}$ is the zero matrix. A matrix without zero rows and columns is called regular. A square matrix with the entries equal to $\mathbb{1}$ on the diagonal and to $\mathbb{0}$ elsewhere is the identity matrix.

Addition and multiplication of matrices and multiplication of matrices by scalars follow the standard entrywise rules with the arithmetic addition and multiplication replaced by $\oplus$ and $\otimes$.

A matrix that consists of one column (row) is a column (row) vector. All vectors are considered column vectors unless otherwise indicated. The set of column vectors with n elements is denoted by $\mathbb{X}^n$. A vector with all elements equal to $\mathbb{0}$ is the zero vector. A vector is called regular if it has no zero element.

For any nonzero column vector $\boldsymbol{x} = (x_i)$, a multiplicative conjugate row-vector $\boldsymbol{x}^- = (x_i^-)$ is defined, where $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$, and $x_i^- = \mathbb{0}$ otherwise.

1.3. Tropical Vector Space

Consider a system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{X}^m$. A vector $\boldsymbol{b} \in \mathbb{X}^m$ is a linear combination of these vectors if $\boldsymbol{b} = x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n$ for some $x_1, \ldots, x_n \in \mathbb{X}$. The set of linear combinations $\mathcal{A} = \{x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n |\ x_1, \ldots, x_n \in \mathbb{X}\}$ is closed under vector addition and scalar multiplication and referred to as the tropical vector space generated by the system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

For any vector $\boldsymbol{a} = (a_i)$ in a tropical space $\mathcal{A}$, consider its support given by $\operatorname{supp}(\boldsymbol{a}) = \{i |\ a_i \neq \mathbb{0},\ 1 \leq i \leq m\}$. For any nonzero vectors $\boldsymbol{a} = (a_i)$ and $\boldsymbol{b} = (b_i)$ such that $\operatorname{supp}(\boldsymbol{a}) = \operatorname{supp}(\boldsymbol{b})$, we define the distance function

$$d(\boldsymbol{a}, \boldsymbol{b}) = \bigoplus_{i \in \operatorname{supp}(\boldsymbol{a})} \left(b_i^{-1}a_i \oplus a_i^{-1}b_i\right) = \boldsymbol{b}^-\boldsymbol{a} \oplus \boldsymbol{a}^-\boldsymbol{b}.$$

If $\operatorname{supp}(\boldsymbol{a}) \neq \operatorname{supp}(\boldsymbol{b})$, we consider the function to take a value greater that any $x \in \mathbb{X}$ and write $d(\boldsymbol{a}, \boldsymbol{b}) = \infty$. If $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{0}$, then we set $d(\boldsymbol{a}, \boldsymbol{b}) = \mathbb{1}$.

In the context of the semifield $\mathbb{R}_{\max,+}$ where $\mathbb{1} = 0$, the function d coincides for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^m$ with the Chebyshev metric

$$d_\infty(\boldsymbol{a}, \boldsymbol{b}) = \max_{1\leq i\leq m} \max(a_i - b_i, b_i - a_i) = \max_{1\leq i\leq m} |b_i - a_i|.$$

For an arbitrary idempotent semifield $\mathbb{X}$, the function d can be considered as a generalized metric with values in the set $[\mathbb{1}, \infty)$.

2. Distance Between Vectors and Solution of Equations

Let $\mathcal{A}$ be a tropical vector space generated by nonzero vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{X}^m$. Any vector $\boldsymbol{a} \in \mathcal{A}$ can be represented as a linear combination $\boldsymbol{a} = x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n$ with coefficients $x_1, \ldots, x_n \in \mathbb{X}$ and hence as the matrix-vector product $\boldsymbol{a} = \boldsymbol{A}\boldsymbol{x}$ with the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and the vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$.

The distance from a vector $\boldsymbol{b}$ to the vector space $\mathcal{A}$ is given by

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{a} \in \mathcal{A}} d(\boldsymbol{a}, \boldsymbol{b}) = \min_{\boldsymbol{x} \in \mathbb{X}^n} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \min_{\boldsymbol{x} \in \mathbb{X}^n} (\boldsymbol{b}^- \boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^- \boldsymbol{b}).$$

If the vector $\boldsymbol{b}$ is regular, then the minimum over all $\boldsymbol{x} \in \mathbb{X}$ on the right-hand side can be replaced by the minimum over only regular $\boldsymbol{x}$ (see, e.g. [15]) to write

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{x} > \boldsymbol{0}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}).$$

As it is easy to see, the equality $d(\mathcal{A}, \boldsymbol{b}) = \mathbb{1}$ corresponds to the condition $\boldsymbol{b} \in \mathcal{A}$, while the inequality $d(\mathcal{A}, \boldsymbol{b}) \neq \mathbb{1}$ means that $\boldsymbol{b} \notin \mathcal{A}$.

Suppose $\boldsymbol{A} \in \mathbb{X}^{m \times n}$ is a regular matrix, and $\boldsymbol{b} \in \mathbb{X}^m$ is a regular vector. Define the function

$$\Delta_{\boldsymbol{A}}(\boldsymbol{b}) = (\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^-)^- \boldsymbol{b}.$$

The next result is obtained in [16] (see also [15]).

Lemma 1. *Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ a regular vector. Then*

$$\min_{\boldsymbol{x} > \boldsymbol{0}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})},$$

where the minimum is achieved at $\boldsymbol{x} = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}(\boldsymbol{b}^- \boldsymbol{A})^-$.

If $\mathcal{A}$ is a vector space generated by the columns of the matrix $\boldsymbol{A}$, then the distance from a vector $\boldsymbol{b}$ to $\mathcal{A}$ is calculated as $d(\mathcal{A}, \boldsymbol{b}) = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}$. The vector in $\mathcal{A}$, which is closest to $\boldsymbol{b}$, takes the form $\boldsymbol{y} = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^-$.

Note that the condition $\boldsymbol{b} \in \mathcal{A}$ leads to the equality $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) = \mathbb{1}$, while $\boldsymbol{b} \notin \mathcal{A}$ to the inequality $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) > \mathbb{1}$.

Consider the problem to find regular vectors $\boldsymbol{x}$ that solve the equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{1}$$

The equation has the unknown vector on one side and hence is called one-sided.

The solution of equation (1) can be described as follows [16, 15].

Theorem 2. *Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ a regular vector. Then:*

1. *If $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) = \mathbb{1}$, then equation (1) has regular solutions; the vector $\boldsymbol{x} = (\boldsymbol{b}^- \boldsymbol{A})^-$ is the maximal solution.*
2. *If $\Delta_{\boldsymbol{A}}(\boldsymbol{b}) > \mathbb{1}$, then no regular solution exists; the vector $\boldsymbol{x} = \sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}(\boldsymbol{b}^- \boldsymbol{A})^-$ is the best approximate solution in the sense of the distance function d.*

Note that $\sqrt{\Delta_{\boldsymbol{A}}(\boldsymbol{b})}$ has the meaning of the minimum achievable deviation between the left and right sides of (1), measured on the scale of the function d.

Suppose $\boldsymbol{A} \in \mathbb{X}^{m\times n}$ and $\boldsymbol{B} \in \mathbb{X}^{m\times k}$ are given regular matrices, and $\boldsymbol{x} \in \mathbb{X}^n$ and $\boldsymbol{y} \in \mathbb{X}^k$ are unknown regular vectors. Let us examine the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{y}. \tag{2}$$

Let $\mathcal{A}$ be the tropical vector space generated by the columns of $\boldsymbol{A}$, and $\mathcal{B}$ the space generated by the columns of $\boldsymbol{B}$. Define the distance between the spaces

$$d(\mathcal{A},\mathcal{B}) = \min_{\boldsymbol{a}\in\mathcal{A},\boldsymbol{b}\in\mathcal{B}} d(\boldsymbol{a},\boldsymbol{b}) = \min_{\boldsymbol{x}>\boldsymbol{0},\boldsymbol{y}>\boldsymbol{0}} d(\boldsymbol{A}\boldsymbol{x},\boldsymbol{B}\boldsymbol{y}).$$

The equality $d(\mathcal{A},\mathcal{B}) = \mathbb{1}$ means that the spaces $\mathcal{A}$ and $\mathcal{B}$ have nonempty intersection, and hence equation (2) has a solution $(\boldsymbol{x},\boldsymbol{y})$. If the distance satisfies the inequality $d(\mathcal{A},\mathcal{B}) > \mathbb{1}$, then the spaces have no common point (and thus the equation has no solution), while its value shows the minimum distance between vectors of the spaces (minimal deviation between both sides of the equation).

3. Solution Procedure for Two-Sided Equation

Consider a solution procedure that constructs a sequence of vectors from the spaces $\mathcal{A}$ and $\mathcal{B}$. The vectors are taken alternatively in both spaces so that after selecting a vector in one space, the next vector is found in the other space to minimize the distance between this space and the former vector. The vectors found in each space $\mathcal{A}$ and $\mathcal{B}$ are determined by coefficients in their decompositions as linear combinations of columns in the respective matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

Let $\boldsymbol{x}_0 \in \mathbb{X}^n$ be a regular vector and $\boldsymbol{a}_0 = \boldsymbol{A}\boldsymbol{x}_0 \in \mathcal{A}$. By applying Theorem 2, we find the minimum distance from the vector $\boldsymbol{a}_0$ to the vectors in $\mathcal{B}$ to be

$$d(\boldsymbol{a}_0,\mathcal{B}) = \sqrt{\Delta_0}, \qquad \Delta_0 = \Delta_{\boldsymbol{B}}(\boldsymbol{A}\boldsymbol{x}_0) = (\boldsymbol{B}((\boldsymbol{A}\boldsymbol{x}_0)^{-}\boldsymbol{B})^{-})^{-}\boldsymbol{A}\boldsymbol{x}_0.$$

This minimum distance is attained at a vector $\boldsymbol{b}_1 \in \mathcal{B}$ that is given by

$$\boldsymbol{b}_1 = \boldsymbol{B}\boldsymbol{y}_1, \qquad \boldsymbol{y}_1 = \sqrt{\Delta_0}((\boldsymbol{A}\boldsymbol{x}_0)^{-}\boldsymbol{B})^{-}.$$

The minimum distance from $\boldsymbol{b}_1$ to the vectors in $\mathcal{A}$ is equal to

$$d(\boldsymbol{b}_1,\mathcal{A}) = \sqrt{\Delta_1}, \qquad \Delta_1 = \Delta_{\boldsymbol{A}}(\boldsymbol{B}\boldsymbol{y}_1) = (\boldsymbol{A}((\boldsymbol{B}\boldsymbol{y}_1)^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{B}\boldsymbol{y}_1$$

and is achieved for a vector $\boldsymbol{a}_2 \in \mathcal{A}$ such that

$$\boldsymbol{a}_2 = \boldsymbol{A}\boldsymbol{x}_2, \qquad \boldsymbol{x}_2 = \sqrt{\Delta_1}((\boldsymbol{B}\boldsymbol{y}_1)^{-}\boldsymbol{A})^{-}.$$

In the same way, we obtain the distance $d(\boldsymbol{a}_2,\mathcal{B})$ by calculating Δ_2, which is then used for finding the vectors $\boldsymbol{y}_3$ and $\boldsymbol{b}_3$. Next, we calculate Δ_3 to evaluate the distance $d(\boldsymbol{b}_3,\mathcal{A})$, and find the vectors $\boldsymbol{x}_4$ and $\boldsymbol{a}_4$.

We continue the procedure to form a sequence of vectors $\boldsymbol{a}_0,\boldsymbol{b}_1,\boldsymbol{a}_2,\boldsymbol{b}_3,\boldsymbol{a}_4,\ldots$ taken alternatively from $\mathcal{A}$ and $\mathcal{B}$ to minimize the distance between successive vectors. At the same time, a sequence of pairs $(\boldsymbol{x}_0,\boldsymbol{y}_1),(\boldsymbol{x}_2,\boldsymbol{y}_3),\ldots$ is generated that provides successive approximations to the solution of equation (2).

Let us examine the sequence $\Delta_0, \Delta_1, \Delta_2, \ldots$ and first note that it is bounded from below since $\Delta_i \geq \mathbb{1}$ for all $i = 0, 1, 2 \ldots$ Furthermore, after some algebra, we can verify that $\Delta_{i+1} \leq \Delta_i$, which says that the sequence is nonincreasing. As a result, we conclude that this sequence converges to a limit $\Delta_* \geq \mathbb{1}$.

We observe that each element of the last sequence represents the squared distance between a vector of one of the spaces $\mathcal{A}$ and $\mathcal{B}$ and the nearest vector in the other space. Therefore, the equality $\Delta_i = \mathbb{1}$ for some i means that the spaces $\mathcal{A}$ and $\mathcal{B}$ have nonempty intersection, while equation (2) has regular solutions. Moreover, if i is even, then the intersection contains the vector $\boldsymbol{a}_i = \boldsymbol{A}\boldsymbol{x}_i$, and the pair of vectors $(\boldsymbol{x}_i, \boldsymbol{y}_{i+1})$ is a solution of the equation. In the case when i is odd, the intersection contains $\boldsymbol{b}_i = \boldsymbol{B}\boldsymbol{y}_i$, and the pair $(\boldsymbol{x}_{i+1}, \boldsymbol{y}_i)$ is a solution.

Reaching the equality $\Delta_i = \mathbb{1}$ indicates that sequence $\Delta_0, \Delta_1, \ldots$ converges to $\Delta_* = \Delta_i$, which can be used in numerical computations as a stop criterion for iterations. If $\mathcal{A}$ and $\mathcal{B}$ do not intersect, then the inequality $\Delta_* > \mathbb{1}$ holds, whereas equation (2) does not have regular solutions. In this case, the procedure stops as soon as there is a repeating element in any of the sequences $\boldsymbol{x}_0, \boldsymbol{x}_2, \ldots$ or $\boldsymbol{y}_1, \boldsymbol{y}_3, \ldots$

The above described solution procedure can be summarized as follows.

Algorithm 1. *Solution of the two-sided equation* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{x}$*:*

1. *Input regular matrices* $\boldsymbol{A}, \boldsymbol{B}$*; set* $i = 0$*; fix a regular vector* $\boldsymbol{x}_0$*.*
2. *Calculate*
$$\Delta_i = (\boldsymbol{B}((\boldsymbol{A}\boldsymbol{x}_i)^- \boldsymbol{B})^-)^- \boldsymbol{A}\boldsymbol{x}_i, \qquad \boldsymbol{y}_{i+1} = \sqrt{\Delta_i}((\boldsymbol{A}\boldsymbol{x}_i)^- \boldsymbol{B})^-.$$
3. *If* $\Delta_i = \mathbb{1}$ *or* $\boldsymbol{y}_{i+1} = \boldsymbol{y}_j$ *for some* $j < i$*, then set*
$$\Delta_* = \Delta_i, \qquad \boldsymbol{x}_* = \boldsymbol{x}_i, \qquad \boldsymbol{y}_* = \boldsymbol{y}_{i+1},$$
and stop; otherwise set $i = i + 1$*.*
4. *Calculate*
$$\Delta_i = (\boldsymbol{A}((\boldsymbol{B}\boldsymbol{y}_i)^- \boldsymbol{A})^-)^- \boldsymbol{B}\boldsymbol{y}_i, \qquad \boldsymbol{x}_{i+1} = \sqrt{\Delta_i}((\boldsymbol{B}\boldsymbol{y}_i)^- \boldsymbol{A})^-.$$
5. *If* $\Delta_i = \mathbb{1}$ *or* $\boldsymbol{x}_{i+1} = \boldsymbol{x}_j$ *for some* $j < i$*, then set*
$$\Delta_* = \Delta_i, \qquad \boldsymbol{x}_* = \boldsymbol{x}_{i+1}, \qquad \boldsymbol{y}_* = \boldsymbol{y}_i,$$
and stop; otherwise set $i = i + 1$*.*
6. *Go to step* 2*.*

If upon completion of the algorithm, we have $\Delta_* = \mathbb{1}$, then equation (2) has regular solutions including the obtained pair of vectors $(\boldsymbol{x}_*, \boldsymbol{y}_*)$. In the case when $\Delta_* > \mathbb{1}$, the equation has no regular solution, while Δ_* indicates the minimum deviation between both sides of the equation, which is attained at $(\boldsymbol{x}_*, \boldsymbol{y}_*)$.

References

[1] V. N. Kolokoltsov, V. P. Maslov. *Idempotent Analysis and Its Applications*, Mathematics and Its Applications, vol. 401. Dordrecht, Springer, 1997.

[2] J. S. Golan. *Semirings and Affine Equations Over Them*, Mathematics and Its Applications, vol. 556. Dordrecht, Springer, 2003.

[3] B. Heidergott, G. J. Olsder, J. van der Woude. *Max Plus at Work*, Princeton Series in Applied Mathematics. Princeton, NJ, Princeton Univ. Press, 2006.

[4] P. Butkovič. *Max-linear Systems*. Springer Monographs in Mathematics. London, Springer, 2010.

[5] D. Maclagan, B. Sturmfels. *Introduction to Tropical Geometry*. Graduate Studies in Mathematics, vol. 161. Providence, AMS, 2015.

[6] P. Butkovič. On certain properties of the systems of linear extremal equations // *Ekonom.-Mat. Obzor*, 1978. Vol. 14, N 1. P. 72–78.

[7] P. Butkovič. Solution of systems of linear extremal equations // *Ekonom.-Mat. Obzor*, 1981. Vol. 17, N 4. P. 402–416.

[8] P. Butkovič, G. Hegedüs. An elimination method for finding all solutions of the system of linear equations over an extremal algebra // *Ekonom.-Mat. Obzor*, 1984. Vol. 20, N 2. P. 203–215.

[9] R. A. Cuninghame-Green, P. Butkovič. The equation $A \otimes x = B \otimes y$ over (max,+) // *Theoret. Comput. Sci.*, 2003. Vol. 293, N 1. P. 3–12.

[10] S. Gaubert, R. D. Katz. The tropical analogue of polar cones // *Linear Algebra Appl.*, 2009. Vol. 431, N 5. P. 608–625.

[11] E. Lorenzo, M. J. de la Puente. An algorithm to describe the solution set of any tropical linear system $A \odot x = B \odot x$ // *Linear Algebra Appl.*, 2011. Vol. 435, N 4. P. 884–901.

[12] D. Jones. On two-sided max-linear equations // *Discrete Appl. Math.*, 2019. Vol. 254, N 3. P. 146–160.

[13] R. A. Cuninghame-Green, K. Zimmermann. Equation with residuated functions // *Comment. Math. Univ. Carolin.*, 2001. Vol. 42, N 4. P. 729–740.

[14] P. Butkovič, K. Zimmermann. A strongly polynomial algorithm for solving two-sided linear systems in max-algebra // *Discrete Appl. Math.*, 2006. Vol. 154, N 3. P. 437–446.

[15] N. K. Krivulin. On solution of a class of linear vector equations in idempotent algebra // *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2009. Vol. 5, N 3. P. 63–76. (in Russian)

[16] N. K. Krivulin. On solution of linear vector equations in idempotent algebra // *Mathematical Models. Theory and Applications. Issue 5*. St. Petersburg, VVM Publishing, 2004. P. 105–113. (in Russian)

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Reversible difference schemes for classical nonlinear oscillators

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Abstract. The report considers the properties of finite-difference schemes for classical oscillators that define a one-to-one correspondence between initial and final values (reversible difference schemes). The results of computer experiments with these schemes and their justification are given.

At PCA'14 [1], we started with a puzzle: why are finitely integrable dynamical systems integrable in elliptic functions? Strictly speaking, the answer to this question was not found in the continuous theory, but Painlevé made an important observation: any dynamical system that defines a birational correspondence between initial and final values on algebraic integral varieties is integrable in classical transcendental functions, usually in elliptic ones.

When we proceed to finite differences, the situation changes radically. In our talk at PCA'21 [2], it was shown that any dynamical system with a quadratic right-hand side admits a reversible difference scheme, i.e., a scheme that defines a birational correspondence between the initial and final positions of the system. In this case, there is no need to restrict the consideration to the integral manifold, since difference schemes define the Cremona transformations. Approximate solution of the Cauchy problem

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}), \quad \vec{x}(0) = \vec{x}_0 \tag{1}$$

is obtained by successively applying the Cremona transformation $\vec{R}$ to point $\vec{x}_0$:

$$\vec{x}_n = \vec{R}^n \vec{x}_0 \simeq \vec{x}(n\Delta t).$$

Reversible difference schemes can be constructed for a wide class of nonlinear dynamic with quadratic right-hand side, which includes both all classical nonlinear oscillators integrable in elliptic functions and systems that are not integrable in classical transcendental functions, e.g., asymmetric tops.

In the computer experiments presented in [2], we were surprised to see that the points of approximate solutions found by reversible schemes for classical oscillators line up into curves. This report will present the results of the study of these curves.

Elliptic oscillators correspond to the special case, when the points of not only exact but also approximate solutions lie on elliptic curves. We have written out the equations of these curves explicitly for the Jacobi oscillator in [3], having obtained a kind of difference analogue of the Lagutinski theory.

The curves themselves depend on the time sampling step Δt. The reduction of the Cremona transformation to integral curves (which are inevitably invariants of this transformation) defines a birational transformation on an elliptic curve. This transformation is always described using elliptic integrals of the first type, which gives us a description of the approximate solution in the form of a quadrature

$$\int_{\vec{x}_n}^{\vec{x}_{n+1}} v(\vec{x}, \Delta t) dx_1 = \Delta t,$$

where vdx_1 is an elliptic integral of the first kind on the appropriate elliptic curve. This description is quite analogous to that obtained in the continuous theory by separation of variables.

The appearance of quadrature leads to the periodicity of motion. For approximate solutions, the very concept of periodicity can be introduced in different ways. First of all, we have shown how to choose the step Δt so that the solution is a periodic sequence of points with a given period. We then showed that the approximate solution can be represented as a set of values of a meromorphic doubly periodic function:

$$\vec{x}_n = \wp(n\Delta t).$$

Thus, the appearance of integral curves, on which the points of approximate solutions lie, leads to the periodicity of the approximate solution.

The discrete and continuous theories of elliptic oscillators are described by the same formulas: the quadrature describes the transition from initial to final data, the motion is periodic, it is described by meromorphic functions, and so on. The whole difference lies in the fact that in the discrete theory the birational transformation describing the transition from the old position of the system to the new one is continued to the Cremona transformation. Hermite was absolutely right in assuming that Cremona's theory of transformations encompasses the theory of elliptic functions. This is a special case of the general theory of dynamical systems approximated by reversible difference schemes, i.e. Cremona transformations.

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References

[1] *Malykh M. D.* On transcendental functions arising from integrating differential equations in finite terms // Journal of mathematical sciences, 209, no. 6 (2015). DOI 10.1007/s10958-015-2539-6

[2] *Baddour A., Malykh M. & Sevastianov L.* On Periodic Approximate Solutions of Dynamical Systems with Quadratic Right-Hand Side. // J. Math. Sci. 261, 698–708 (2022). DOI 10.1007/s10958-022-05781-4

[3] *Ayryan E. A., Gambaryan M. M., Malykh M. D., & Sevastianov L. A.* On the trajectories of dynamical systems with a quadratic right-hand side calculated by reversible difference schemes // Representation theory, dynamical systems, combinatorial methods. XXXIV, Zapiski POMI, 517, POMI, St. Petersburg, 2022, 17–35 (in Russian)

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Implementation of A.N. Krylov series convergence acceleration in the CAS Sage

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Abstract. The problem of calculating the sum of the Fourier series in computer algebra systems is considered. The work presents some functions of the «Kryloff for Sage» software package. This package is designed to accelerate the convergence of Fourier series for problems of mathematical physics. In some cases, it is possible to obtain an expression in finite terms.

The solutions of the problems of mathematical physics are usually represented in the form of Fourier series. At the same time, many initial-boundary value problems for the wave equation can be solved in finite terms. An attempt to get a presentation in finite terms using standard series summation functions built into computer algebra systems leads to some difficulties, see [1]. The expressions for sums obtained in computer algebra systems contain special functions and numerous branches. These difficulties are related to the fact that the Fourier series for the wave equation are not the analytical functions of their arguments. Therefore they are not elementary functions in the sense of Liouville theory.

To solve this problem, it is proposed to consider not the task of summing up the series in finite terms form, but the task of accelerating its convergence. We will use the method of accelerating the convergence systematically described in A.N. Krylov works. As Krylov noted, this method «often leads to the representation of the sum of the proposed series inclosed form under the guise piecewice function» [2]. For example, this is true for several Green's functions of the wave equation on the segment [3]. The guise piecewice function will not be elementary in the sense of Liouville theory. However, it will be elementary function in the modern sense of this concept [4].

In this work the first functions of the software package «Kryloff for Sage» in CAS Sage [5] are presented. We consider a problem when the function represented by a given Fourier series is not known. We are implemented in Sage the variants of this method, which can lead to the definition of the desired function in finite terms [6, 7].

References

[1] M.D. Malykh, K.Yu. Malyshev, *Solving the hyperbolic equation in elementary functions*. Computer Algebra, Moscow, 2021.

[2] A.N. Krylov, *O nekotorykh differentsial'nykh uravneniyakh matematicheskoy fiziki* [On Some Differential Equations of Mathematical Physics]. Moscow, Leningrad, GITTL, 1950. (in Russian).

[3] K.Yu. Malyshev, *Representation of Green's functions of the wave equation on a segment in finite terms*, Izvestiya of Saratov University. Mathematics. Mechanics. Informatics, 2022, vol. 22, iss. 4, pp. 430–446. (in Russian).

[4] V. A. Il'in , È. G. Poznyak, *Osnovy matematicheskogo analiza, ch. 1* [Fundamentals of MathematicalAnalysis, Pt. 1]. Moscow, Fizmatlit, 2005. (in Russian).

[5] Symbolic and numeric computing environment Sage, 2021. `https://www.sagemath.org/`.

[6] A. B. Nersesyan, *Kvazipolinomy tipa Bernulli i uskorenie skhodimosti ryadov Fur'e kusochno-gladkih funkcij* [Quasipolinomas such as Bernoulli and Fourier series convergence acceleration of a piecewice-smooth functions], Dokl. Nats. Akad. Nauk Armen., 2004, vol. 104, pp. 273-279.

[7] L. V. Kantorovich, V. I. Krylov, *Priblizhonnye metody vysshego analiza* [Approximate methods of Higher Analysis]. Moscow, Leningrad, GITTL, 1950. (in Russian).

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Approximation of the zeros of the Riemann zeta function by rational functions

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Abstract. We define rational functions $R_N(a, d_0,\ d_1,\ \ldots,\ d_N)$ and demonstrate by numerical examples the following property. Let $d_0,\ d_1,\ \ldots,\ d_N$ be equal respectively to the value of the Riemann zeta function and the values of its first N derivatives calculated at a. If a is not too far from a zero ρ of the zeta function, then the value of $R_N(a, d_0,\ d_1,\ \ldots,\ d_N)$ is very close to ρ. For example, for $N = 10$ and $a = 0.6 + 14\mathrm{i}$ we have $|R_{10}(a, d_0,\ d_1,\ \ldots,\ d_{10}) - \rho_1| < 10^{-14}$ where $\rho_1 = 0.5 + 14.13...\mathrm{i}$ is the first non-trivial zeta zero.

Also we define rational functions $R_N(a, d_0,\ d_1,\ \ldots,\ d_N, n)$ which (under the same assumptions) have values which are very close to $n^{-\rho}$, that is, to the summands from the Dirichlet series for the zeta function calculated at its zero.

In the case when a is, say between two consecutive zeros, ρ_l and ρ_{l+1}, functions $R_N(a, d_0,\ d_1,\ \ldots,\ d_N, n)$ approximate neither $n^{-\rho_l}$ nor $n^{-\rho_{l+1}}$; nevertheless, they allow us to approximate the sum $n^{-\rho_l} + n^{-\rho_{l+1}}$ and the product $n^{-\rho_l}\, n^{-\rho_{l+1}}$ and hence to calculate both $n^{-\rho_l}$ and $n^{-\rho_{l+1}}$ by solving corresponding quadratic equation.

Introduction

The celebrated *Riemann zeta function* $\zeta(s)$ can be defined for a complex number s with real part greater than 1 via a *Dirichlet series*,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1}$$

and analytically extended to the whole complex plane. B. Riemann established a deep relationship between the zeros of this function and the prime numbers. Namely, he found an explicit expression for $\pi(x)$ – the number of primes below x – via these zeros.

The outstanding *Riemann Hypothesis* predicts that all the non-real zeros of the zeta function are lying on the *critical line* $\Re(s) = 1/2$. The hypothesis is equivalent to the assertion that

$$\pi(x) \quad = \quad \int_2^x \frac{\mathrm{d}y}{\ln(y)} + O(\ln(x)\sqrt{x}). \tag{2}$$

We address the following problem. Suppose that

$$d_k = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta(s)\bigg|_{s=a}, \quad k = 0, \ldots, N \tag{3}$$

where number a is not far from a zero ρ of the zeta function. *How numbers* $a, d_0, d_1, \ldots, d_N$ *could be used for calculating this* ρ *with high precision?*

1. Standard method

A straightforward way is to consider polynomial

$$T_N(s) = \sum_{k=0}^{N} \frac{d_k}{k!}(s-a)^k \tag{4}$$

(an initial fragment of Taylor series) and solve algebraic equation

$$T_N(s) = 0. \tag{5}$$

This approach has several drawbacks.

First, unless $N \le 4$, there is no "explicit" expression for the roots of this equation.

Second, we cannot analytically pinpoint which of the N roots of the equation is the desired approximation to ρ.

Third, ρ should be closer to a than the pole of the zeta function at $s = 1$.

2. Our method

We define

$$d_{m,k} = \frac{\mathrm{d}^k}{\mathrm{d}s^k}\zeta_m(s)\bigg|_{s=a} = \sum_{l=0}^{m} \binom{m}{l} \big(-\ln(n)\big)^{m-l} d_l, \tag{6}$$

$$T_{m,N}(s) = \sum_{k=0}^{N} \frac{d_{m,k}}{k!}(s-a)^k = b_{m,N,0} + \sum_{k=1}^{N} b_{m,N,k} s^k, \tag{7}$$

$$T_{m,N}(s_1, \ldots, s_N) = b_{m,N,0} + \sum_{k=1}^{N} b_{m,N,k} s_k, \tag{8}$$

and solve linear system

$$T_{m,N}(s_1, \ldots, s_N) = 0, \quad m = 1, \ldots, N. \tag{9}$$

Clearly, $s_1, \ldots, s_n$ are rational functions of $a, d_1, \ldots, d_N$.

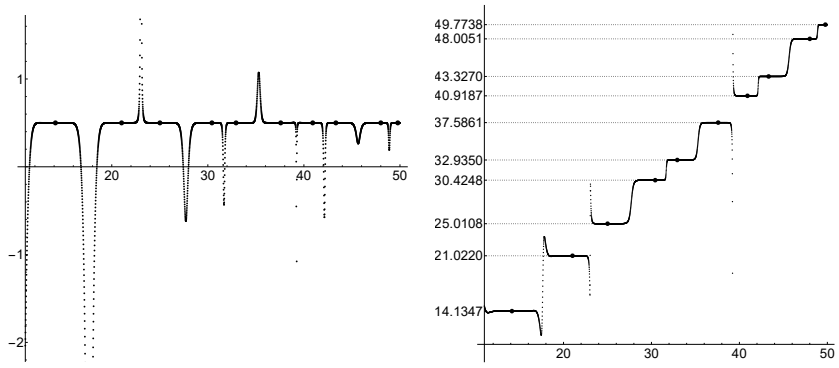

FIGURE 1. The small dots represent the real (on the left-hand plot) and the imaginary (on the right-hand plot) parts of s_1 from solutions of systems (9) for $N = 10$ and $a = 0.4 + \mathrm{i}\tau$, τ running from 10 to 50 with step 0.01. The ten larger dots on each plot represent the ten initial zeta zeros; the abscissas of the dots are equal to the imaginary parts of these zeros on both plots; the ordinates of the dots are equal respectively to the real and imaginary parts of the zeros.

Calculations demonstrate (see [1, 3]) that when a is not too far from a zero ρ of the zeta function, s_n is rather close to ρ_1. This is illustrated for $n = 1$ by Fig.1. We see that $\Re(s_1)$ is mainly equal to $1/2$. As for $\Im(s_1)$, it looks almost as a step function with levels equal to the imaginary parts of the zeros of the zeta function.

Calculations suggest the following guess.

Conjecture A. For all a except for a set of zero measure there is a zero ρ of the zeta function such that for all k the value of s_k from the solution of system (9) tends to ρ^k as $N \to \infty$.

In order to define functions $R_N(a, d_0,\ d_1,\ \ldots,\ d_N, n)$ we replace polynomials (4) by Dirichlet polynomials

$$D_{m,N}(s) \quad = \quad \sum_{n=1}^{N} c_{m,N,n} n^{-s} \tag{10}$$

having the same initial Taylor expansion as $\zeta_m(s)$. Polynomials (8) are replaced by linear polynomials

$$Q_{m,N}(q_2, \ldots, q_N) \quad = \quad c_{m,N,1} + \sum_{n=2}^{N} c_{m,N,n} q_n \tag{11}$$

and $R_N(a, d_0,\ d_1,\ \ldots,\ d_N, n)$ is defined as the value of q_n from the solution of system

$$Q_{m,N}(q_2, \ldots, q_N) = 0, \quad m = 1, \ldots, N-1. \tag{12}$$

So defined q_n approximate very well $n^{-\rho}$ when a is not too far from a zero ρ of the zeta function (see [2, 3]). This suggest the following guess.

Conjecture B. For all a except for a set of zero measure there is a zero ρ of the zeta function such that for all $n > 1$ the value of q_n from the solution of system (12) tends to ρ^n as $N \to \infty$.

In the case when a is, say between two consecutive zeros, ρ_l and ρ_{l+1}, the value of q_n is close neither to $n^{-\rho_l}$ nor to $n^{-\rho_{l+1}}$, but

$$\frac{q_n q_{n^2} - q_{n^3}}{q_n^2 - q_{n^2}} \approx n^{-\rho_l} + n^{-\rho_{l+1}}, \qquad \frac{q_{n^2}^2 - q_n q_{n^3}}{q_n^2 - q_{n^2}} \approx n^{-\rho_l} n^{-\rho_{l+1}}, \tag{13}$$

and we can calculate approximations to both $n^{-\rho_l}$ and to $n^{-\rho_{l+1}}$ by solving corresponding quadratic equation.

References

[1] Matiyasevich, Yu. V., Hunting zeros of Dirichlet series by linear algebra. II. POMI Preprints, 01, 18pp. 2022, DOI: 10.13140/RG.2.2.20434.22720.

[2] Matiyasevich, Yu. V., Hunting zeros of Dirichlet series by linear algebra. III (in Russian), POMI Preprents 03, 31pp. 2022, DOI: 10.13140/RG.2.2.28325.99044; extended English abstract http:logic.pdmi.ras.ru/~yumat/publications/papers/139_paper/eng_abstract_ext.pdf.

[3] Matiyasevich, Yu., On some algebraic ways to calculate zeros of the Riemann zeta function, in *Algebraic Informatics. CAI 2022*. Poulakis, D. and Rahonis, G., eds., Springer, *Lecture Notes in Computer Science*, 13706, 15–25, 2022, DOI: 10.1007/978-3-031-19685-0_2.

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Maxima in Teaching Basic Matrix Algebra

Tatiana Mylläri and Aleksandr Mylläri

Abstract. We discuss usage of computer algebra system Maxima in teaching basic matrix algebra in St. George's University.

Introduction

Modern technology changes the way to do mathematics and to teach mathematics [1, 2]. As we have reported earlier [3, 4], St. George's University School of Arts and Sciences comprise mainly of local and Caribbean students with poor background in mathematics. It is real challenge to teach College Mathematics, especially topics that are new to the students. Quite often students have problems dealing with the material that they have studied earlier and are supposed to know, but when the concepts are really new, students put a mental barrier and the process goes really hard.

One of the examples is matrices and systems of the linear equations. This looks very complicated to students especially because there is only a small number of classes to consider this material, and it is really new kind of problems for them. And that's the moment when Computer Algebra Systems (CAS) come to help. We have reported earlier that Maxima was chosen for using in teaching Math in SGU since it is powerful and free of charge. In the class we explain basic operations on matrices for small sizes. After students get the basic knowledge, Maxima is recommended to use to check the answers received manually or to work with more complicated problems. For example, students can find inverse matrix for matrices with the size 3x3 or higher, solve not only linear systems of the order 2 or 3, but also higher size systems.

After students start to use Maxima they realize that in fact it is not so complicated as they were thinking a priory and students feel more relaxed and confident when dealing with this kind of problems. Also, teacher has more time to consider additional examples because it is enough to consider only simple cases in the class.

Maxima for Windows has very convenient GUI wxMaxima, and it is very easy to declare matrix as it is presented on Figure 1. Matrix elements could be

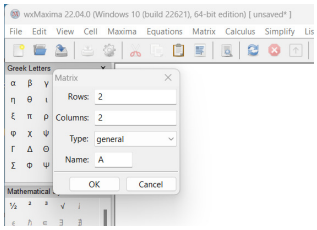

FIGURE 1. Declaration of matrix A

some parameters, not necessarily numbers (Figure 2). Using Maxima, it is easy to

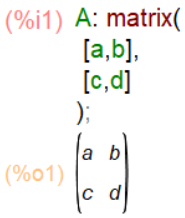

FIGURE 2. Initialization for the matrix

calculate the determinant of the matrix (Figures 3, 4). By default, the command for the determinant appears with the % (result of the last operation), but the % can be replaced by the name of the matrix (Fig. 4, right). We can use Maxima

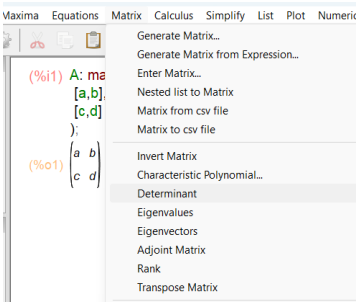

FIGURE 3. GUI for calculation of the determinant of the matrix

```
(%i2) determinant(%);
(%o2) a d−b c
```

```
(%i3) determinant(A);
(%o3) a d−b c
```

FIGURE 4. Calculation of the determinant of the matrix

```
(%i5) invert(A);
(%o5) matrix([ d/(a d−b c), −b/(a d−b c) ],
             [ −c/(a d−b c), a/(a d−b c) ])
```

FIGURE 5. Inverse of the matrix

to find the inverse of the matrix (Figure 5). It is also very easy to calculate some algebraic expressions for matrices (Figure 6).

```
(%i6) B: matrix(
        [1,2,3],
        [4,−3,−5]
      );
(%o6) matrix([1, 2, 3], [4, −3, −5])
```

```
(%i7) C: matrix(
        [−2,1,3],
        [2,−4,5]
      );
(%o7) matrix([−2, 1, 3], [2, −4, 5])
```

```
(%i10) 2·B+3·C;
(%o10) matrix([−4, 7, 15], [14, −18, 5])
```

FIGURE 6. Algebraic expression of matrices

Maxima can be used also to find the transpose matrix or to calculate the product of matrices (Figure 7).

```
(%i11) D:transpose(C);
(%o11) matrix([−2, 2], [1, −4], [3, 5])
```

```
(%i12) E: B.D;
(%o12) matrix([9, 9], [−26, −5])
```

FIGURE 7. Transpose matrix and product of matrices

We hope that with the use of technology students will be motivated to learn Math concepts in the learning environment.

References

[1] Vavilov N. A. *Computers as fresh mathematical reality. I. Personal account. Computer Tools in Education (2)*, pp. 5-26, 2020.

[2] Vavilov N. A., Halin V.G., Yurkov A. V. *Mathematica for non-mathematician:* University textbook, Moscow, 2021 (in Russian)

[3] Hypolite S., Mylläri A. *Use Of Computer Algebra Systems In Teaching Math At SGU.* Computer tools in education, pp. 96 - 100, No 3, 2021.

[4] Hypolite S., Mylläri T. *Teaching Math in SGU with Computer Algebra Systems.* In International Conference Polynomial Computer Algebra 2022, St. Petersburg department of Steklov Institute of Mathematics, RAS„ pp. 37 -39, 2022.

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Envy-free division in the presence of a dragon

Gaiane Panina

Following a novel approach, where the emphasis is on configuration spaces and equivariant topology, we prove several results addressing the envy-free division problem in the presence of an unpredictable (secretive, non-cooperative) player, called *the dragon*. There are two basic scenarios.

1. There are $r-1$ players and a dragon. Once the "cake" is divided into r parts, the dragon makes his choice and grabs one of the pieces. After that the players should be able to share the remaining pieces in an envy-free fashion.

2. There are $r+1$ players who divide the cake into r pieces. A ferocious dragon comes and swallows one of the players. The players need to cut the cake in advance in such a way that no matter who is the unlucky player swallowed by the dragon, the remaining players can share the tiles in an envy-free manner.

The talk is based on the joint work with Rade Živaljević.

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Real stability of spanning tree enumerator

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Abstract. Kruskal's theorem states that a sum of product tensors constitutes a unique tensor rank decomposition if the so-called k-ranks of the product tensors are large. We prove a more general result in which the k-rank condition of Kruskal's theorem is weakened to the standard notion of rank, and the conclusion is relaxed to a statement on the linear dependence of the product tensors. As a corollary, we prove that if n product tensors form a circuit, then they have rank greater than one in at most $n-2$ subsystems. This generalizes several recent results in this direction, and is sharp. The proof of the main result is based on the matroid ear decomposition technique.

References

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Some observations on degree 3 and 4 exponential sums over finite fields

N. V. Proskurin

Abstract. By numerical experiments, it is discovered some strictures in distribution of cubic and quartic exponential sums of additive type in finite fields. Concerning the cubic sums, we give a theoretical explanation for that. For the quartic sums, we observe numerically that Euler's deltoid play role in their distribution.

Introduction

Consider the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of prime order p, its additive character

$$x \mapsto e_p(x) = \exp(2\pi i x/p), \quad x \in \mathbb{F}_p,$$

a one-variable polynomial f over $\mathbb{F}_p$ and an additive type exponential sum

$$S_p(f) = \sum_{x \in \mathbb{F}_p} e_p\big(f(x)\big).$$

The Weil inequality $|\,S_p(f)\,| \le (\deg f - 1)\,\sqrt{p}$ is valid for all the sums whenever $p \nmid \deg f$. That means, the points

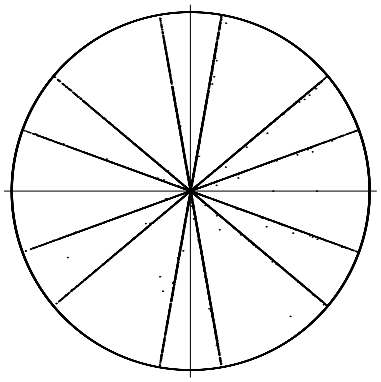
$$E_p(f) = \frac{1}{(\deg f - 1)\,\sqrt{p}}\, S_p(f)$$

are located in the unit disk $D = \big\{ z \in \mathbb{C} \;\big|\; |z| \le 1 \big\}$. See [1], [2].

Given a one-variable polynomial f over $\mathbb{Z}$, consider f as a polynomial over each of $\mathbb{F}_p$ just by reduction its coefficients $\bmod p$. Then one may look on distribution of the points $E_p(f)$ (with prime $p = 2, 3, 5, 7 \dots$) in the disk D. We have used computer algebra systems PARI and MAPLE to study numerically the sums $S_p(f)$ for lot of polynomials f of degree 3 and 4.

Cubic sums

Consider one instructive sample in [3]. On the picture below we have plotted the real coordinate axis, the imaginary coordinate axis, the unit disk $D \subset \mathbb{C}$, and the points $E_p(f) \in D$ for the polynomial $f(x) = 6x^3 + 3x^2 + 4x$ and for all prime $p \leq 100000$.



The points $E_p(f)$ are concentrated mainly along few lines passing through the point 0. One has a similar picture for other polynomials as well. The number of lines depends on f.

To state our results, let us agree to write $\{t\}$ for the fractional part of $t \in \mathbb{R}$. We have proved [4] the following two propositions.

Consider a cubic polynomial $f(x) = ax^3 + bx^2 + cx$ over $\mathbb{Z}$. Let l be an integer, $\gcd(l, 3a) = 1$, *and let p be any prime under the conditions* $lp + 1 \equiv 0 \bmod 27a^3$ *and $p \nmid 6a$. If $S_p(f) \neq 0$, then the real axis forms the angle*

$$\theta_p = 2\pi \left\{ \frac{b\,(2b^2 - 9ac)}{27a^2} \Big(l + \frac{1}{p} \Big) \right\}$$

with the line passing through the points 0 *and $S_p(f)$.*

This proposition implies easily the second one.

Consider a cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ over $\mathbb{Z}$. The points $E_p(f)$ are concentrated along the lines that pass through the point 0 *and intersect the real axis under the angles*

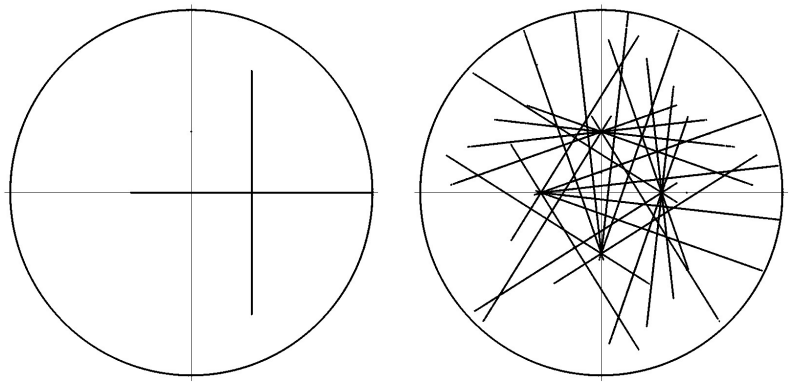
$$\theta = 2\pi \left\{ \frac{b\,(2b^2 - 9ac)}{27a^2}\, l \right\}$$

with $l \in \mathbb{Z}$ under the condition $\gcd(l, 3a) = 1$.

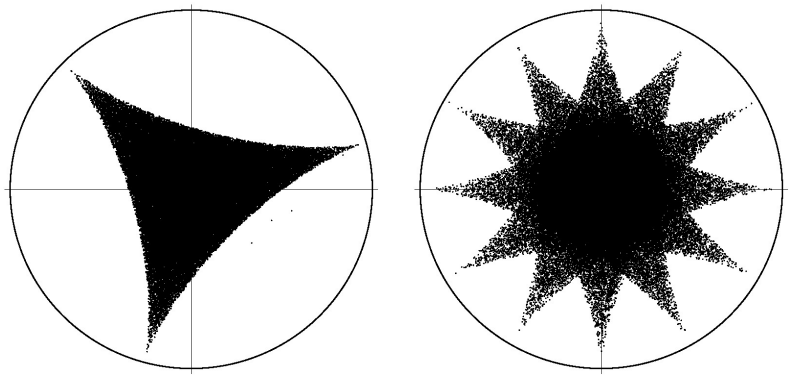
This result gives us full description of the asters attached to cubic polynomials in [3]. Also, it shows that there are no at all the clusters considered in [3].

Quartic sums

For some quartic polynomials f, we have find empirically that almost all of the points $E_p(f)$ are located on few intervals in D. Let us look on two samples. On the pictures below we have plotted the real and imaginary coordinate axes, the disk $D \subset \mathbb{C}$, and the points $E_p(f) \in D$ for chosen polynomials f and for all prime $p \leq 480000$. The sums $S_p(f)$ with $f(x) = x^4$ are nothing but the biquadratic Gauss sums. By known explicit formulas, one has either $E_p(f) = i/3$ or $E_p(f) \in [-1/3, 1]$ or $E_p(f) \in [1/3 - 2i/3, 1/3 + 2i/3]$ according to $p \equiv 3 \pmod 4$ or $p \equiv 1 \pmod 8$ or $p \equiv 5 \pmod 8$. This case is represented by the left-hand side picture below.



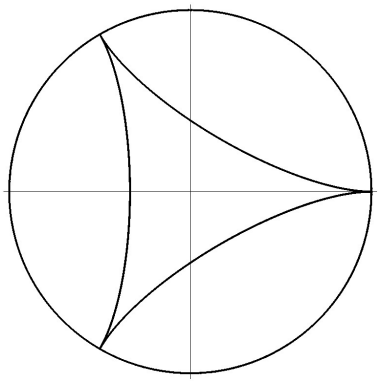
The right-hand side picture represents similarly the case $f(x) = 7x^4 + x^2$. Assume $r \in \mathbb{Z}$ and $\gcd(r, 56) = 1$. It seems reasonable to expect that the points $E_p(f)$ with $p \equiv r \pmod{56}$ form one of 24 intervals shown on the picture.
There are other polynomials f with entirely different distribution of the points $E_p(f)$. Two typical samples are given on the pictures below.



For the polynomial $f(x) = 4x^4 + 8x^3 + 3x^2 + 6$, the points $E_p(f)$ with $p \le 1000000$ forms the left-hand side picture. We see that the points $E_p(f)$ are located within some three-cusped curve. The right-hand side picture is formed similarly for $f(x) = 7x^4 + 1$. For a lot of polynomials f, we have similar pictures "formed by 1, 2, 4, 8 triangles bounded by the same three-cusped curve".

We conjecture that the three-cusped curve discussed is the Euler deltoid considered in 1745 *in connection with an optical problem.*

The deltoid can be defined as the curve consisting of the points $z = x + iy \in \mathbb{C}$ satisfying $3(x^2 + y^2)(x^2 + y^2 + 2) = 8x^3 - 24xy^2 + 1$ with $x, y \in \mathbb{R}$.



Also, the deltoid can be created by a point on the circumference of a circle of radius 1/3 as it rolls without slipping along the inside of a circle of radius 1.

References

[1] J.-P. Serre, *Majorations de sommes exponentielles*, Société Mathématique de France, Asterisque 41–42, p. 111–126, 1977.

[2] S. A. Stepanov, *Arithmetic of algebraic curves*, Moscow, 1991 (in Russian). English translation: Springer–Verlag, 1995.

[3] N. V. Proskurin, *On some cubic exponential sums*, Zap. Nauchn. semin. POMI, vol. 502, 122–132, 2021 (in Russian).

[4] N. V. Proskurin, *Distribution of cubic exponential sums*, Zap. Nauchn. semin. POMI, vol. 511, 161–170, 2022 (in Russian).

[5] N. V. Proskurin, *On quartic exponential sums*, Zap. Nauchn. semin. POMI, vol. 517, 162–175, 2022 (in Russian).

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The phase portrait of the pendulum model of resonance in (a, λ) coordinates

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Abstract. The model of pendulum is widely used to study resonance in celestial mechanics. Usually it is considered in $(\lambda, \dot{\lambda})$ coordinates where it is symmetric. However, when we rebuild it in (λ, a) coordinates, where λ is resonance argument, a is semimaajor axis the symmetry is loses. The migration rate (and rate of the resonance approach) is expressed in terms of semimajor axis. For example, Yarkovsky effect generates a linear drift in semimajor axis. Moreover, if we accept adiabatic changes of orbit when approach to resonance we can locally assume linear dependence semimajor axis on time in a very common case. Therefore this representation has some advantages.

Introduction

In detail, the pendulum model of resonance is described in [1]. The Hamiltonian of the three body problem in case of resonance can be reduced to the form:

$$H = 1/2\alpha\Lambda^2 + \mu A\cos\lambda \tag{1}$$

where:

$$\beta = \frac{3}{2k^2}\left[\frac{(j-k)^2}{ma^2} + \frac{j^2}{m_p a_p^2}\right] \tag{2}$$

$$\varepsilon(n^2)^{1-k/4} f_d \frac{a^{3-k}}{a_p}\frac{m_p}{m_c} m^{1-k/2} \tag{3}$$

$$m = \frac{m_p m_c}{m_p + m_c} \tag{4}$$

Here f_d - the according resonance term of the perturbation function expansion, m_c - central mass, m, a, n – mass, semimajor axis and mean motion of the asteroid,m_p, a_p, n_p – mass, semimajor axis and mean motion of the perturbing planet, j, k – integer, $\dot{\varpi}$ - secular perihelion precession rate of the asteroid.

The value of α is the measure of the neighborhood to the exact resonance.

1. The case of constant velocity

Usually, the phase portrait of the pendulum is drawn in $(\lambda, \dot{\lambda})$ coordinates, where it is obviously symmetric. However in the study of dynamic in the central gravity field it is natural to use other coordinates: radius vector r (or semimajor axis a) and longitude. In the model of resonance it is possible to identify λ with longitude. Therefore it is naturally to redraw the phase portrait in a, λ coordinates where it becomes asymmetric, Figure 1.

The exact solution of the pendulum equation is not suitable, because it contained the elliptic integral. By this reason, we consider asymptotic solution[2]. After the substitution $\dot{\lambda} = n - n_{res} = \delta_n$ for the circulation solution we have:

$$\dot{\lambda} = \Omega + \frac{\Omega}{\Omega^2}\cos\Omega t = \delta_n \pm \frac{1}{\delta_n}\cos\delta_n t \tag{5}$$

Denote $a - a_{res} = \delta_a$ and using the expansion:

$$\delta_n = \pm 3/2\delta_a\frac{n_{res}}{a_{res}} + \frac{15}{8}\delta_a^2\frac{n_{res}}{a_{res}^2} + ... \tag{6}$$

Finally we obtain:

$$\dot{\lambda} = a_{res}^{-3/2} \pm 3/2\delta_a\frac{n_{res}}{a_{res}} + \frac{15}{8}\delta_a^2\frac{n_{res}}{a_{res}^2} + ... \tag{7}$$

This result allows us to rebuild the phase portrait in $(a, \lambda\ )$ coordinates (fig.1). Evidently, the curves with $\dot{\lambda}$ which are symmetric in $(\dot{\lambda}, \lambda)$ coordinates are not symmetric in (a, λ) coordinates.

This representation has some advantages. The migration rate (and rate of the resonance approach) is expressed in terms of semimajor axis. For example, Yarkovsky effect generates a linear drift in semimajor axis.

As it is followed from expression above, λ_{res} is approximately proportional time. Therefore we can replace the horizontal x-axis (λ axis) in the phase portrait by the time axis. As consequence, the linear drift in the semimajor axis can be expressed as a tangent (slope) of the trajectory. On our opinion, such expression can be useful in the process of the resonance capture modelling.

Moreover, if we accept adiabatic changes of orbit when approach to resonance we can locally assume linear dependence semimajor axis on time in a very common case.

Conclusion

The pendulum model of the mean motion resonance is considered in (λ, a) coordinates where it is not symmetric. This representation has some advantages. The migration rate (and rate of the resonance approach) is expressed in terms of semimajor axis. For example, Yarkovsky effect generates a linear drift in semimajor axis. Our method allow to simple numerical modeling of the process of approach of the small body to resonance.

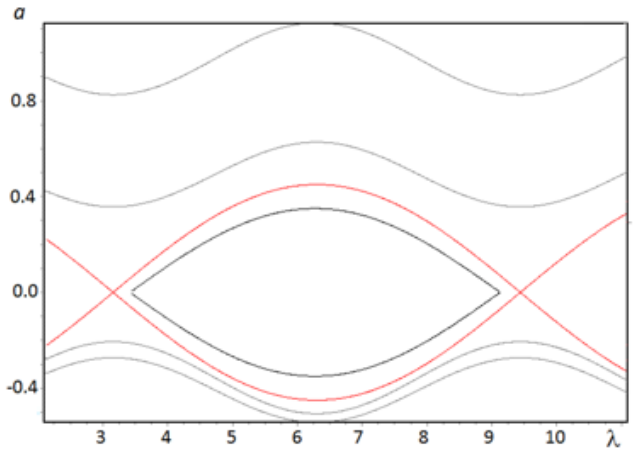

FIGURE 1. The phase portrait of asymmetric pendulum in a, λ coordinates

References

[1] Murray, C. D., Dermott, S. F., *Solar System Dynamics* Cambridge Univ. Press, Cambridge, 606 p. 1999.

[2] Moiseev N. N., Asymptotic Methods in Nonlinear Mechanics [in Russian], Moscow, Nauka 1981.

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Computing the Complement to the Amoeba of a Multivariate Polynomial

Timur Sadykov

We propose a method for computing and visualizing the amoeba of a Laurent polynomial in several complex variables, which is applicable in arbitrary dimension. The algorithms developed based on this method are implemented as a free web service (http://amoebas.ru) which enables interactive computation of amoebas of bivariate polynomials and provides a set of precomputed amoebas and their cross-sections in higher dimensions. The correctness and running time of the proposed algorithms are tested against a set of optimal polynomials in two, three, and four variables, which are generated by means of Mathematica computer algebra system. The developed program code makes it possible, in particular, to generate optimal hypergeometric polynomials in an arbitrary number of variables supported in an arbitrary zonotope given by a set of generating vectors. The talk is based on a joint work with T.Zhukov.

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A Normal Form of Derivations for Quantifier-Free Sequent Calculi With Nonlogical Axioms

Alexander Sakharov

1. Introduction

The core of AI systems is domain knowledge that is commonly expressed as logical programs or knowledge base rules. A great deal of research has been devoted to logical characterizations of these systems. These characterizations give formal descriptions of otherwise obscure systems and make their results explainable. AI systems are regularly categorized by various non-standard calculi. Models may not be available for these calculi. In this situation, the task of developing an inference method for a particular non-standard logic is often a complicated research project.

Arguably, sequents are the most common notation in the specification of proof theories. Sequent calculi have been used in formalizations of a variety of logics. Sequent calculi are suitable for establishing derivation properties. Nonetheless, they do not facilitate inference methods because some of their axioms and inference rules constitute infinite branching points in derivation search and because of the variety of rule choices at any derivation step.

We suggest sequent calculi with inference rules of certain forms as a framework for representing standard and non-standard logics having AI utility. Logical rules in these quantifier-free calculi are introduction rules for logical connectives. Domain knowledge is expressed by nonlogical axioms in the form of sequents in these calculi. Inference in these calculi can be restricted to a normal form for which a variant of the subformula property holds.

2. Sequent Calculi

We consider quantifier-free languages because typical AI languages such as logic programming and knowledge base languages exclude quantifiers [6]. Skolem functions serve as an alternative to quantifiers. Quantifiers are problematic for some non-standard logics. As usual, formulas are built recursively from atoms and logical connectives, atom arguments are terms built recursively from object

variables, constants, and functions. We limit connectives to unary and binary. The languages of particular calculi could be more restricted. A number of calculi related to AI are propositional - they do not include variables. Datalog [6] does not include functions.

Sequent calculi have axioms and inference rules [4]. Inference rules have one or more premises and one conclusion, each of them is a sequent. We limit the number of premises to two. Axioms are basically inference rules with zero premises. Sequent calculi include logical axiom $A \vdash A$ or a similar one. Upper-case Latin letters denote formulas in inference rules. Upper-case Greek letters are metavariables denoting formula multisets.

It is known that sequent calculi do not necessarily have an adequate expressiveness for some intricate logics. Sequent calculus extensions such as hypersequents [10] have been developed to address these unusual cases. These different extensions are not covered in this paper. It is fair to say that it is not realistic to have a universal language whose expressive power is sufficient for a variety of sequent calculus extensions. Allowing additional logical axioms makes it possible to express complicated logics as ordinary sequent calculi, but these axioms may compromise important properties of sequent calculi.

A substitution is a finite mapping of object variables to terms. The result of applying a substitution θ to a formula A is the expression $A\theta$ obtained from A by simultaneously replacing every occurrence of every variable from θ by the term with which the variable is associated. $A\theta$ is called an instance of A. The notions of substitution and instance can be extended onto sequents.

Nonlogical axioms are sequents containing formulas, no multiset or formula metavariables occur in them. Any nonlogical axiom with variables represents infinitely many sequents. Each of these sequents is an instance of the axiom. Nonlogical axioms represent knowledge base rules and facts (or logical programs). They express properties of concrete predicates and functions.

Usually, the outcome of inference is sequents $\vdash G$ where formula G is called a goal. Unlike goals for theorem provers, goals for AI systems as well as formulas in nonlogical axioms are shallow formulas. An axiom is called reducible if it has an instance with two or more identical formulas. A calculus is called consistent if sequent $\vdash$ is not derivable. Inconsistent calculi without nonlogical axioms are not worth investigating but it is acceptable for nonlogical axioms to be the source of inconsistency. Argumentation deals with inconsistent sets of nonlogical axioms [1].

Inference rules in sequent calculi are split into structural and logical. The structural rules are essentially universal for all of the calculi whereas logical rules vary. Given the multiset view of antecedents and succedents, the structural rules are weakening, contraction, and cut [4]. Some of these structural rules may be missing in some calculi. Some calculi restrict the number of formulas in succedents to one. We do not consider calculi with other constraints.

Every formula from the conclusion of a logical inference rule that is not identical to a formula from a premise is called principal. Every formula from premises that is not identical to a formula from the conclusion is called active.

All other formulas are called contexts. Let $\diamond$ denote a unary connective, $\circ$ denote a binary connective. Let $\diamond\Pi$ denote $\{\diamond A | A \in \Pi\}$.

Definition 1. *A logical inference rule is called an introduction rule if it has one of the following forms and does not have any applicability provisos.*

$$\frac{A, \Gamma \vdash \Pi}{\diamond A, \Gamma \vdash \Pi}\ L1 \qquad \frac{A \vdash \diamond\Pi}{\diamond A \vdash \diamond\Pi}\ LP \qquad \frac{\Gamma \vdash A, \Pi}{\Gamma \vdash \diamond A, \Pi}\ R1 \qquad \frac{\diamond\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\ RP$$

$$\frac{A, \Gamma \vdash \Pi}{\Gamma \vdash \diamond A, \Pi}\ F1 \qquad \frac{\Gamma \vdash A, \Pi}{\diamond A, \Gamma \vdash \Pi}\ B1$$

$$\frac{\Gamma \vdash}{\diamond\Gamma \vdash}\ LO \qquad \frac{\Gamma \vdash A}{\diamond\Gamma \vdash \diamond A}\ RL \qquad \frac{\vdash \Pi}{\vdash \diamond\Pi}\ RO \qquad \frac{A \vdash \Pi}{\diamond A \vdash \diamond\Pi}\ LR$$

$$\frac{A, B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ L2 \qquad \frac{\Gamma \vdash A, B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ R2 \qquad \frac{A, \Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ F2 \qquad \frac{A, \Gamma \vdash B, \Pi}{A \circ B, \Gamma \vdash \Pi}\ B2$$

$$\frac{A, \Gamma \vdash \Pi \quad B, \Gamma \vdash \Pi}{A \circ B, \Gamma \vdash \Pi}\ LA \qquad \frac{B, \Gamma \vdash \Pi \quad B, \Delta \vdash \Sigma}{A \circ B, \Gamma, \Delta \vdash \Pi, \Sigma}\ LM$$

$$\frac{\Gamma \vdash A, \Pi \quad \Gamma \vdash B, \Pi}{\Gamma \vdash A \circ B, \Pi}\ RA \qquad \frac{\Gamma \vdash A, \Pi \quad \Delta \vdash B, \Sigma}{\Gamma, \Delta \vdash A \circ B, \Pi, \Sigma}\ RM$$

$$\frac{B, \Gamma \vdash \Pi \quad \Delta \vdash A, \Sigma}{\Gamma, \Delta \vdash A \circ B, \Pi, \Sigma}\ FM \qquad \frac{B, \Gamma \vdash \Pi \quad \Delta \vdash A, \Sigma}{A \circ B, \Gamma, \Delta \vdash \Pi, \Sigma}\ BM$$

Definition 2. *A sequent calculus is called a L_A calculus if it has one logical axiom $A \vdash A$ and possibly nonlogical axioms, the cut rule, possibly the two weakening rules, possibly the two contraction rules, some introduction logical rules, and*

- *for every unary connective $\diamond$, the rules with this connective are limited to one R1 rule and possibly one L1 or LP rule, one RP rule and possibly one L1 rule, one F1 rule and possibly one B1 rule, one RL rule and one of LO/L1 rules, or one LR rule and one of RO/R1 rules,*
- *for every binary connective $\circ$, the rules with this connective are limited to one R2 rule and possibly one LA rule, one R2 rule and possibly one LM rule, one RA rule and possibly one L2 rule, one RM rule and possibly one L2 rule, one F2 rule and possibly one BM rule, or one FM rule and possibly one B2 rule.*

The idea of introduction rules is that every formula from a premise is a subformula of some formula from the conclusion. There are some non-standard logics that cannot be specified by calculi with introduction rules. One example of that is temporal logics. Their sequent calculi include logical rules in which some formulas in the conclusion are proper subformulas of ones in the premise [5]. The choice of the introduction rule forms is dictated by the desideratum of the subformula property. For that reason, rules with syntactic constraints on both antecedent and succedent contexts such as $S5$ rules are excluded [10]. No surprise that the introduction rules correspond to the calculi that enjoy cut admissibility in the absence of nonlogical axioms.

Clearly, the quantifier-free fragments of classical and intuitionistic first-order logics are L_A calculi. Other examples of L_A calculi include multiplicative linear

logic [2], the LK_{-c} calculi of evaluable non-Horn knowledge bases [7], modal logic $S4$ [10], standard deontic logic [1].

3. Normal Form

The object of this investigation is sets (families) of L_A calculi in which structural and logical inference rules are fixed and every calculus in the set has its own set of nonlogical axioms. Any calculus in a set corresponds to an AI system, predicates and functions are constants from a finite set determined by the domain of the system. Basically, such set of calculi corresponds to a logic for a variety of domains. We do not consider calculi without the cut rule. This rule plays the role of Modus Pones. Without Modus Pones, nonlogical axioms are useless.

Let $[\Gamma]$ denote the result of applying zero or more possible contractions to multiset Γ. If a calculus set does not include contraction, then $[\Gamma] = \Gamma$. If a calculus includes both weakening and contraction, then the $[\,]$ operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic. Let us modify the conclusion of the cut rule and all logical rules by applying $[\,]$ to both the antecedent and the succedent. For instance, cut and BM become

$$\frac{\Gamma \vdash A, \Delta \quad A, \Pi \vdash \Sigma}{[\Gamma, \Pi] \vdash [\Delta, \Sigma]}\ cut \qquad \frac{A, \Gamma \vdash \Pi \quad \Delta \vdash B, \Sigma}{[A \circ B, \Gamma, \Delta] \vdash [\Pi, \Sigma]}\ BM$$

Definition 3. *The calculi obtained from L_A by applying $[\,]$ to both antecedent and succedent in the conclusion of cut and logical inference rules are called L'_A.*

Proposition 1. *For any L_A calculus and its L'_A counterpart, any L_A derivation can be transformed into a L'_A derivation with the same endsequent and vice versa.*

Proposition 2. *The contraction rules are admissible in L'_A derivations for calculi with non-reducible nonlogical axioms.*

Theorem 1. *(normal form) For a consistent L'_A calculus with non-reducible nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into such derivation with the same endsequent and without contractions that the following holds:*

1) (weak subformula property) Every formula in the derivation is G, its subformula, or an instance of a formula from a nonlogical axiom or its subformula.

2) Every cut formula is an instance of a formula from a nonlogical axiom.

3) If one premise of cut is the conclusion of a logical rule, then the cut formula is principal in the logical rule and the other premise is a nonlogical axiom or the conclusion of another cut.

4) The conclusion of every weakening is the premise of $L2, R2, F2, B2, LA, RA$, or another weakening.

5) Every weakening formula is active in the first descendant $L2, R2, F2, B2$ rule or adds a formula to the context of a premise of the first descendant LA, RA rule from the context of the other premise of the latter rule.

Theorem 2. *For a L'_A calculus without LP, RP rules and a simplification order* [3], *every derivation of $\vdash G$ can be transformed into such normal-form derivation that every cut formula is maximal with respect to such formulas from both the succedent of the first premise and the antecedent of the second premise that are not G, its subformulas, or instances of proper subformulas of nonlogical-axiom formulas.*

Consider the following rules:

$$\frac{A,\Gamma\vdash\Pi}{A\circ B,\Gamma\vdash\Pi}\ L2^+ \qquad \frac{B,\Gamma\vdash\Pi}{A\circ B,\Gamma\vdash\Pi}\ L2^* \qquad \frac{\Gamma\vdash A,\Pi}{\Gamma\vdash A\circ B,\Pi}\ R2^+ \qquad \frac{\Gamma\vdash B,\Pi}{\Gamma\vdash A\circ B,\Pi}\ R2^*$$

$$\frac{A,\Gamma\vdash\Pi}{\Gamma\vdash A\circ B,\Pi}\ F2^+ \qquad \frac{\Gamma\vdash B,\Pi}{\Gamma\vdash A\circ B,\Pi}\ F2^* \qquad \frac{A,\Gamma\vdash\Pi}{A\circ B,\Gamma\vdash\Pi}\ B2^+ \qquad \frac{\Gamma\vdash B,\Pi}{A\circ B,\Gamma\vdash\Pi}\ B2^*$$

$$\frac{A,\Gamma\vdash\Pi \quad B,\Delta\vdash\Sigma}{A\circ B,\Gamma\cup\Delta\vdash\Pi\cup\Sigma}\ LA^* \qquad \frac{\Gamma\vdash A,\Pi \quad \Delta\vdash B,\Sigma}{\Gamma\cup\Delta\vdash A\circ B,\Pi\cup\Sigma}\ RA^*$$

Definition 4. *The calculi obtained from L'_A calculi with weakening by adding the $L2^+, R2^+, L2^*, R2^*, F2^+, B2^+, F2^*, B2^*$ rules and replacing the LA, RA rules with the LA^*, RA^* rules, respectively, are called L''_A. The L'_A calculi without weakening have identical L''_A counterparts.*

Proposition 3. *For any L'_A calculus and its L''_A counterpart, any L'_A derivation can be transformed into a L''_A derivation with the same endsequent and vice versa.*

Proposition 4. *For a consistent L'_A calculus with non-reducible nonlogical axioms, every derivation with endsequent $\vdash G$ can be transformed into a normal-form L''_A derivation with the same endsequent and without the weakening rules.*

4. Conclusion

The subformula property is a desirable property for any calculus. This property is a corollary and a primary reason for cut elimination. In general, cut elimination is not possible for sequent calculi with nonlogical axioms. The normal-form theorem shows that derivations can be limited to those satisfying the weak subformula property for a wide class of calculi with nonlogical axioms even though cut is not admissible in them. This theorem gives other constraints for inference rules. Weakening can be embedded into logical rules. Theorem 2 adapts ordered resolution [3] to sequent derivations. It states an additional constraint for the cut rule.

Due to the weak subformula property, the choices for A in the logical axiom $A\vdash A$, the choices for the weakening formulas, and the choices for the principal formulas of logical rules can be limited to the goal, its subformulas, and instances of formulas from nonlogical axioms and their subformulas. Given that the majority of formulas in nonlogical axioms are expected to be shallow, the sets of their subformulas are rather small.

The instantiation of nonlogical axioms, formulas in the logical axiom, and the weakening formulas is a potential source of infinite branching in inference

procedures. Fortunately, this problem is solved by using formulas from nonlogical axioms and their subformulas 'as is' after renaming object variables and by embedding unification in inference rules [9]. There exist efficient unification algorithms [8]. They are applicable to quantifier-free first-order formulas because these formulas can be treated as terms whose signature is extended with predicates and logical connectives.

References

[1] Ofer Arieli and Christian Straßer. Sequent-based logical argumentation. *Argument & Computation*, 6(1):73–99, 2015.

[2] Emmanuel Beffara. *Introduction to linear logic*. HAL CCSD, 2013.

[3] Laura Kovács and Andrei Voronkov. First-order theorem proving and vampire. In *International Conference on Computer Aided Verification*, pages 1–35, 2013.

[4] Sara Negri and Jan Von Plato. *Structural proof theory*. Cambridge University Press, 2001.

[5] Hirokazu Nishimura. A study of some tense logics by gentzen's sequential method. *Publications of the Research Institute for Mathematical Sciences*, 16(2):343–353, 1980.

[6] Stuart Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach*. Prentice Hall Press, 3rd edition, 2009.

[7] Alexander Sakharov. A logical characterization of evaluable knowledge bases. In *14th International Conference on Agents and Artificial Intelligence*, pages 681–688, 2022.

[8] Jörg H Siekmann. Unification theory. *Journal of Symbolic computation*, 7(3-4):207–274, 1989.

[9] Tanel Tammet. Resolution, inverse method and the sequent calculus. In *Kurt Gödel Colloquium*, volume 1289, pages 65–83, 1997.

[10] Heinrich Wansing. *Sequent systems for modal logics*, pages 61–145. Springer, 2002.

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Notes on Obstacles to Dimensionality Reduction

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Abstract. We consider the arrangement of vertices of the unit multidimensional cube and affine subspace as well as their orthographic projections onto coordinate hyperplanes. Upper and lower bounds on the subspace dimension are given under which some orthographic projection always preserves the incidence relation between the subspace and cube vertices. The proved upper bound is equal to the integer part of half the dimension of the ambient space.

Introduction

Let us consider the recognition problem whether there is a $(0,1)$-solution to a system of linear equations over a field K, where $\mathrm{char}(K) \neq 2$. From a geometric point of view, we consider the recognition problem whether a given affine subspace passes through a vertex of the multidimensional unit cube. Over the ring of integers, the problem is known as the multiple subset sum problem. It is NP-complete. Nevertheless, there are known many heuristic methods [1, 2, 3, 4].

Let us consider the n-dimensional affine space with a fixed system of Cartesian coordinates. The vertices of the unit n-dimensional cube are points with coordinates equal to either zero or one. These vertices are called $(0,1)$-points for short. As usual, a cube in the plane is called a square.

Let us formulate the second problem. Given an affine subspace L that is not incident to any $(0,1)$-point. Does there exist a projection onto a low-dimensional coordinate subspace that forgets some coordinates so that the image of the subspace L is also not incident to any $(0,1)$-point? It is important that the image of the cube is again a low-dimensional cube. The problem is closely related to pseudo-Boolean programming and various generalizations of the knapsack problem. Of course, such dimensionality reduction reduces the computational complexity. The imposed conditions ensure the correctness of reducing the original problem to a problem with a smaller number of variables. But in the worst case, any hyperplane cannot serve as an image of a subspace L under the considered restrictions.

In Euclidean space, for dimensionality reduction by means of some projection, one can use probabilistic algorithms based on the Johnson–Lindenstrauss lemma [5, 6]. However, this approach is not applicable over an arbitrary field.

For $n > m$, a projection $K^n \to K^m$ is so-called *orthographic* when the projection forgets some coordinates. The term was historically used to denote orthogonal projections from three-dimensional space onto a plane over reals. For an affine space over an arbitrary field K, the notion of orthogonality has no meaning. Nevertheless, using a fixed coordinate system, it is possible to define a special class of projections onto coordinate subspaces. We hope this term will not lead to misunderstanding.

Results

Theorem 1. *Given a positive integer s. There is an s-dimensional affine subspace $L \subset K^{2s+1}$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto a coordinate hyperplane, the image of the plane L passes through some $(0,1)$-point.*

Proof. Let us denote by A the point with coordinates $(1/2, 0, \dots, 0)$, where all but one of the coordinates are zeros. For $1 \le k \le s$, the point $A^{(2k)}$ has coordinates $(0, \dots, -1, 1, \dots)$, where $A^{(2k)}_{2k} = -1$, $A^{(2k)}_{2k+1} = 1$, and other coordinates are zeros. The point $A^{(2k+1)}$ has coordinates $(1, \dots, 1, -1, \dots)$, where $A^{(2k+1)}_1 = 1$, $A^{(2k+1)}_{2k} = 1$, $A^{(2k+1)}_{2k+1} = -1$, and other coordinates are zeros. All points A, $A^{(2k)}$, and $A^{(2k+1)}$ belong to an affine subspace L, which is defined by a system of linear equations: $1 - 2x_1 + x_2 + \dots + x_{2k} + \dots + x_{2s} = 0$ and other s equations $x_{2k} + x_{2k+1} = 0$, where $1 \le k \le s$.

The inequality $\dim L \le s$ holds because these equations are linearly independent. On the other hand, for all $1 \le k \le s$, three points A, $A^{(2k)}$, and $A^{(2k+1)}$ belongs to a straight line. All these lines intersect each other at the point A. The inequality $\dim L \ge s$ holds because the affine hull of these s straight lines is s-dimensional. Thus, $\dim L = s$.

Under the orthographic projection onto a coordinate hyperplane, the image of L passes through a $(0,1)$-point. This point is an image of some point from the set A, $A^{(2)}$, $\dots$, $A^{(2s+1)}$. Let us check that no $(0,1)$-point belongs to L. If all even coordinates vanish $x_{2k} = 0$, then $x_1 = 1/2$ in accordance with the first equation. Otherwise, for some k, both equalities $x_{2k} = 1$ and $x_{2k+1} = -1$ hold. □

Example 1. Let us consider three points in a three-dimensional affine space with coordinates $(0, 1, 1/2)$, $(1, 2, 0)$, and $(-1, 0, 1)$, respectively. These points belong to the same straight line L, which can be given by a system of two equations $x_2 = x_1 + 1$ and $x_3 = (-x_1 + 1)/2$. But under the orthographic projection onto any coordinate plane, the image of this set of three points contains some $(0,1)$-point.

Remark 1. The characteristic of the field K does not equal two because we use division by two.

Theorem 2. *Given a positive integer s. Over any infinite field K, there is an s-dimensional affine subspace $L \subset K^{2s}$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto a coordinate hyperplane, the image of the plane L passes through some $(0,1)$-point.*

Remark 2. In Theorem 2, the field K is infinite because the proof uses the Schwartz–Zippel lemma. In fact, the same theorem holds over the field having exactly three elements. It is unknown whether it holds over larger finite fields.

Example 2. Let us consider the plane in the four-dimensional affine space that is defined by the system of two equations $x_3 = x_1+x_2+1$ and $x_4 = (-x_1+x_2+1)/2$. A straightforward check shows that this plane does not pass through any $(0,1)$-point. However, this plane passes through the points $(-1,0,0,1)$, $(0,-1,0,0)$, $(0,1,2,1)$, $(0,0,1,1/2)$, each of which has exactly one coordinate different from both zero and one. Therefore, under the orthographic projection onto any coordinate hyperplane, the image of this plane is incident to some $(0,1)$-point.

Theorem 3. *For all straight lines $L \subset K^n$, where $n \geq 4$, if L is not incident to any $(0,1)$-point, then there is an orthographic projection onto some coordinate hyperplane such that the image of the line L is also not incident to any $(0,1)$-point.*

Theorem 4. *For all planes $L \subset K^n$, where $n \geq 6$, if L is not incident to any $(0,1)$-point, then there is an orthographic projection onto some coordinate hyperplane such that the image of the plane L is also not incident to any $(0,1)$-point.*

Remark 3. Over a finite field K, one can obtain a lower bound for the dimension of an affine subspace $L \subset K^n$ such that L does not pass through any $(0,1)$-point, but under the orthographic projection onto any coordinate hyperplane, the image of L is incident to some $(0,1)$-point. If the field K consists of q elements, then s-dimensional subspace has q^s elements. So, the bound is $s \geq \log_q n$.

Conclusion

Our results illustrate the high computational complexity of pseudo-Boolean programming because the reduction in the dimension of the ambient space by means of projection meets an obstacle in the worst case. Moreover, in small dimensions, we know the exact bound for the dimension of the subspace for which the discussed obstacle to dimensionality reduction exists. However, such an obstacle arises only for special arrangements of the affine subspace. In the general case and over an infinite field, there is an orthographic projection so that the image of the subspace is a hyperplane in a space of lower dimension and the image is not incident to any $(0,1)$-point. Of course, there may be many such projections. Moreover, for different projections, the computational complexity of checking whether no $(0,1)$-points is incident to the resulting subspace may be greater or less.

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References

[1] Kuzyurin N.N. An integer linear programming algorithm polynomial in the average case. In: Korshunov A.D. (eds.) Discrete Analysis and Operations Research. Mathematics and Its Applications, vol. 355, pp. 143–152. Springer, Dordrecht, 1996. https://doi.org/10.1007/978-94-009-1606-7_11

[2] Pan Y., Zhang F. Solving low-density multiple subset sum problems with SVP oracle. J. Syst. Sci. Complex. 2016. Vol. 29. P. 228–242. https://doi.org/10.1007/s11424-015-3324-9

[3] Seliverstov A.V. Binary solutions to large systems of linear equations (in Russian). Prikladnaya Diskretnaya Matematika. 2021. No. 52. P. 5–15. https://doi.org/10.17223/20710410/52/1

[4] Cacchiani V., Iori M., Locatelli A., Martello S. Knapsack problems – An overview of recent advances. Part II: Multiple, multidimensional, and quadratic knapsack problems. Comput. Oper. Res. 2022. Vol. 143. No. 105693. P. 1–14. https://doi.org/10.1016/j.cor.2021.105693

[5] D'Ambrosio C., Liberti L., Poirion P.-L., Vu K. Random projections for quadratic programs. Math. Program. 2020. Vol. 183. P. 619–647. https://doi.org/10.1007/s10107-020-01517-x

[6] Poirion P.-L., Lourenço B.F., Takeda A.: Random projections of linear and semidefinite problems with linear inequalities. Linear Algebra Appl. 2023. Vol. 664. P. 24–60. https://doi.org/10.1016/j.laa.2023.01.013

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TouIST, a Pedagogical Tool for Logic, Algebra, and Discrete Mathematics

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Abstract. TouIST is an automatic translator (developed at IRIT, Toulouse) that provides a simple language to generate logical formulas from a problem description. Coupled with SAT, QBF or SMT solvers, it allows us to model many static or dynamic combinatorial problems. This can be very helpful as a teaching support for logics, algebra and discrete mathematics. A series of examples (based on my personal experience of work with TouIST) will be presented.

References

[1] O. Gasquet, D. Longuin, E. Lorini, F. Maris, P. Régnier, S. Soloviev. TouIST, a Teacher- and Student-Friendly Language for Propositional Logic and Discrete Mathematics. Computer tools in education, no. 2, pp. 13-25, **2021** (Engl.); doi:10.32603/2071- 2340-2021-2-13-25
[2] O. Gasquet, D. Longin, F. Mari, P. Régnier, and M. Valais, "Compact Tree Encodings for Planning as QBF," *Inteligencia Artificial (Ibero-American Journal of Artificial Intelligence)*, vol. 21(62), pp. 103–114, 2018; doi: 10.4114/intartif.vol21iss62pp103-113

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Machine learning and moduli spaces of curves

Elira Curri and Tony Shaska

Abstract. We propose new methods to apply machine learning to various databases which have emerged in the study of the moduli spaces of algebraic curves. We find that with such methods one can learn many significant quantities to astounding accuracy in a matter of minutes and can also predict unknown results making this approach a valuable tool in pure mathematics.

Introduction

Artificial Intelligence and Machine Learning are some of the most active and exciting branches of science of the last few decades. These new technologies have made their way into economy, including engineering, medical science, finance, cybersecurity, etc. Can they be used for mathematical research?

The question is not new. After all science is all about collecting data and deducing conclusions. Machine Learning is about gathering data, training the data, and drawing conclusions. Depending on the kind of data we use different methods for machine learning: supervised learning, unsupervised learning, or a combination of the two.

So the first step is to gather the data. There have always been databases in mathematics, but the most famous databases of the XX-century were the Atlas of Finite Simple Groups, Cremona tables of elliptic curves, database of elliptic curves compiled by Birch and Swinnerton-Dyer which led to the famous Birch and Swinnerton-Dyer conjecture; see [2]. With the development of computer algebra toward the last quarter of the XX-century we saw different databases which had a huge impact on mathematics, for example the Small Library of Groups in Gap, the list of Calabi-Yau hypersurfaces, etc.

The goal of this work is to use new tools of machine learning to study the moduli space $\mathcal{M}_g$ of genus $g \geq 2$ curves defined over a field k. The moduli space of algebraic curves has been the focal point of algebraic geometry for the last few decades. With the development of the new computational tools it became necessary in the last few decades to reconsider the theory of invariants, in its classical form

or in the framework of the theory developed by Mumford with the intention of studying the arithmetic of moduli spaces. Naturally some of the first attempts focused on $\mathcal{M}_2$; see for example [6] and attempts to generalize to $g > 2$ [5]. From these attempts the concept of weighted Weil height was born; see [7].

The moduli space $\mathcal{M}_2$ as a case study.

The moduli space $\mathcal{M}_2$ of genus 2 curves is the most understood moduli space among all moduli spaces. This is mostly due to two main facts; first all genus two curves are hyperelliptic and therefore studying them it is easier than general curves, secondly even among hyperelliptic curves the curves of genus two have a special place since they correspond to binary sextics which, from the computational point of view, are relatively well understood compared to higher degree binary forms.

One of the main questions related to $\mathcal{M}_2$ has been to recover a nice equation for any point $\mathfrak{p} \in \mathcal{M}_2$. Since $\mathcal{M}_2$ is a coarse moduli space, such equation is not always defined over the field of moduli of $\mathfrak{p}$. Can we find a universal equation for genus two curves over their minimal field of definition? Can such equation provide a minimal model for the curve? Does the height of this minimal model has any relation to the projective height of the corresponding moduli point $\mathfrak{p} \in \mathcal{M}_2$? What is the distribution in $\mathcal{M}_2$ of points $\mathfrak{p}$ for which the field of moduli is not a field of definition? The answers to these questions are still unknown.

In [1] we provide a database of genus 2 curves which contains all curves with height $\mathrm{h} \leq 5$, curves with moduli height $\mathfrak{h} \leq 20$, and curves with automorphism and height ≤ 101. They are organized in three Python directories $\mathcal{L}_i$, $i = 1, 2, 3$. The database is build with the idea of better understanding $\mathcal{M}_2$, the distribution of points in $\mathcal{M}_2$ based on the moduli height, the distribution of points for which the field of moduli is not a field of definition. Even in genus $g = 2$ there are many technical issues that need to be addressed.

Let $\mathcal{X}$ be a genus two curve defined over $\mathbb{Q}$. The moduli point in $\mathcal{M}_2$ corresponding to $\mathcal{X}$ is given by $\mathfrak{p} = (i_1, i_2, i_3)$, where i_1, i_2, i_3 are absolute invariants as in [1]. Since i_1, i_2, i_3 are rational functions in terms of the coefficients of $\mathcal{X}$, then $i_1, i_2, i_3 \in \mathbb{Q}$. The converse isn't necessarily true. Let $\mathfrak{p} = (i_1, i_2, i_3) \in \mathcal{M}_2(\mathbb{Q})$. The universal equation of a genus 2 curve corresponding to $\mathfrak{p}$ is determined in [6], which is defined over a quadratic number field K. The main questions we want to consider is what percentage of the rational moduli points are defined over $\mathbb{Q}$? How can we determine a minimal equation for such curves?

For every point $\mathfrak{p} \in \mathcal{M}_2$ such that $\mathfrak{p} \in \mathcal{M}_2(k)$, for some number field K, there is a pair of genus-two curves $\mathcal{C}^{\pm}$ given by $\mathcal{C}^{\pm} : \ y^2 = \sum_{i=0}^{6} a_{6-i}^{\pm} x^i$, corresponding to $\mathfrak{p}$, such that $a_i^{\pm} \in K(d)$, $i = 0, \ldots, 6$; see [6].

In [1] were created three Python dictionaries: $\mathcal{L}_1$: curves with height ≤ 10, $\mathcal{L}_2$: curves with extra involutions, $\mathcal{L}_3$: curves with small moduli height. There are 20 697 curves in $\mathcal{L}_2$, such that for each h we have roughly 4 h curves. So it is expected that the number of curves of height $\leq h$, defined over $\mathbb{Q}$ is $\leq 4^{\frac{\mathrm{h}(\mathrm{h}+1)}{2}}$. Let $\mathfrak{p} \in \mathcal{M}_2(\mathbb{Q})$ be such that $\mathrm{Aut}(\mathfrak{p}) \cong V_4$. There is a genus 2 curve $\mathcal{X}$ corresponding to $\mathfrak{p}$ with equation $y^2 z^4 = f(x^2, z^2)$. We pick $f \in \mathbb{Z}[x, z]$, such that $f(x, z)$ is a

reduced binary form. From 20 292 such curves we found only 57 which do not have minimal absolute height. $\mathcal{L}_3$ is a list of all moduli points $[x_0 : x_1 : x_2 : x_3]$ of projective height $\leq \mathfrak{h}$ in $\mathbb{P}^3(\mathbb{Q})$, for some integer $\mathfrak{h} \geq 1$. Each such point correspond to the point $[J_2^5 : J_4 J_2^3 : J_6 J_2^2 : J_{10}]$.

What percentage of rational points $\mathfrak{p} \in \mathcal{M}_2(\mathbb{Q})$ with a fixed moduli height $\mathfrak{h}$ have $\mathbb{Q}$ as a field of definition, when $\mathfrak{h}$ becomes arbitrarily large? We confirm, as expected, that for large moduli height $\mathfrak{h} \in \mathcal{M}_2(\mathbb{Q})$, the majority of genus 2 curves not defined over $\mathbb{Q}$ and they don't have extra automorphisms.

Higher moduli:

Can we generalize the approach above to $\mathcal{M}_g$ for $g > 2$? Moreover, can we train a machine learning model to obtain reliable results for $g \geq 2$?

The moduli space $\mathcal{M}_2$ is a very good model for the hyperelliptic moduli $\mathcal{H}_g$. Many of the results of $g = 2$ have been realized to higher genus hyperelliptic curves already and we now know many general theorems for $\mathcal{H}_g$; see [3], [5], etc.

Moreover, generalizing from hyperelliptic curves to superelliptic curves gives a very important tool in understanding $\mathcal{M}_g$; see [5] for details. Using results from [4] and previous work of these authors we can determine fully the list of automorphisms groups and inclusions among the loci for any genus, hence obtaining a full stratification of the moduli space $\mathcal{M}_g$. About 75-80% of all cases come from superelliptic curves, for which we know a great deal.

A very important development in understanding $\mathcal{M}_g$ is the discovery of the weighted height on the weighted projective spaces. Hence, the most efficient way to create a database of points in $\mathcal{M}_g$ is to consider the corresponding weighted moduli space $\mathcal{W}_g$ and sort the points in this space via their weighted heights.

A great learning example is the case $g = 3$ for many reasons. It is the first case that we have non-hyperelliptic curves, so it is more general than $g = 2$, but also it is still a case that we fully understand. For example, we explicitly know invariants of binary octavics, which classify hyperelliptic genus 3 curves, and invariants of ternary quartics which classify non-hyperelliptic genus 2 curves. We have a full understanding of the list of groups of automorphisms and in each case we can write an explicit parametric equation for the corresponding family. There has been work in the last decade by several authors on the field of moduli of genus 3 curves and we can recover the equation of the curve over a minimal field of definition.

It needs to be pointed out that in this general approach the biggest difficulty comes from arithmetic invariant theory in the sense that we don't know an explicit way of describing a moduli point $\mathfrak{p} \in \mathcal{M}_g$. While GIT provides an elegant theoretical framework, explicit results are missing even for genus g as small as 4 or 5.

Conclusion

Our general philosophy is to build the skeleton of $\mathcal{M}_g$ using the superelliptic curves. After all, the majority of points in $\mathcal{M}_g$ with nontrivial $\mathrm{Aut}(\mathfrak{p})$ are superelliptic points. We can say a lot on these superelliptic points on the problem of field of moduli versus field of definition, determine if they have complex multiplication, and write down explicit equations for them.

In this talk we will describe what can be achieved and what are the challenges for fully understanding the arithmetic of the moduli space. Our goal is to bring this topic to the attention to mathematicians specialized on machine learning and artificial intelligence techniques and hopefully involve more people in this ambitious but exciting project.

References

1. L. Beshaj, R. Hidalgo, S. Kruk, A. Malmendier, S. Quispe, and T. Shaska, *Rational points in the moduli space of genus two*, Higher genus curves in mathematical physics and arithmetic geometry, Contemp. Math., vol. 703, Amer. Math. Soc., [Providence], RI, 2018, pp. 83–115. MR 3782461
2. B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. Reine Angew. Math. **218** (1965), 79–108. MR 179168
3. J. Gutierrez and T. Shaska, *Hyperelliptic curves with extra involutions*, LMS J. Comput. Math. **8** (2005), 102–115. MR **2135032 (2006b:**14049)
4. K. Magaard, T. Shaska, S. Shpectorov, and H. Völklein, *The locus of curves with prescribed automorphism group*, Sūrikaisekikenkyūsho Kōkyūroku (2002), no. 1267, 112–141, Communications in arithmetic fundamental groups (Kyoto, 1999/2001). MR 1954371
5. A. Malmendier and T. Shaska, *From hyperelliptic to superelliptic curves*, Albanian J. Math. **13** (2019), no. 1, 107–200. MR 3978315
6. Andreas Malmendier and Tony Shaska, *A universal genus-two curve from siegel modular forms*, SIGMA Symmetry Integrability Geom. Methods Appl. **13** (2017), Paper No. 089, 17. MR 3731039
7. Sajad Salami and Tony Shaska, *Local and global heights on weighted projective varieties and Vojta's conjecture*, arXiv preprint arXiv:2204.01624 (2022).

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Fractional order differentiation of Meijer G-functions and their cases

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Abstract. We describe the Riemann-Liouville-Hadamard integro-differentiation of an arbitrary function to arbitrary symbolic order α which is realised in the Wolfram Language.

Introduction

The fractional derivative is a generalization of the mathematical concept of a derivative [1]. There are several different ways to generalize this concept, but the most of them coincide in corresponding classes of functions. When not only fractional, but also negative orders of the derivative are considered, the term differ-integral can be used.

We will use notation $\mathcal{D}_z^\alpha[f(z)]$ for Riemann-Liouville-Hadamard differ-integral for all $\alpha \in \mathbb{C}$. By definition of $\mathcal{D}_z^\alpha[f(z)]$ we put

$$\mathcal{D}_z^\alpha[f(z)] = \begin{cases} f(z), & \alpha = 0; \\ f^{(\alpha)}(z), & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \underbrace{\int\limits_0^z dt ... \int\limits_0^t dt \int\limits_0^t}_{-\alpha \text{ times}} f(t)dt, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha < 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor \alpha \rfloor + 1 \quad \text{and} \quad \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0 \quad \text{and} \quad \alpha \notin \mathbb{Z}; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha) \neq 0, \end{cases} \tag{1}$$

where in the cases of divergence of integrals we use Hadamard finite part approach. Such construction is called *Riemann-Liouville-Hadamard fractional order derivative*. For "good enough" functions $f(z)$, provided convergence of above integrals at basic point $z=0$ it coincides with classical Riemann-Liouville definition, but for analytical functions can be extended to handle functions like $1/z$ or z^a or $(z^a)^b$ or e^{-z}/z or $\sqrt{z^2}/z$ or $\log(z^2)$ or $log(z)$ or $z^a\log^n(z)$, which are basic for building Taylor and Fourier series representations of more complicated functions like hypergeometric, Meijer G-function and Fox H-function.

For example, using (1), we obtain

$$\mathcal{D}_z^\alpha\left[\frac{1}{z}\right]=\begin{cases}(-1)^\alpha(1)_\alpha z^{-\alpha-1}, & \alpha\in\mathbb{Z},-1<\alpha;\\ \frac{z^{-\alpha-1}(-\psi(-\alpha)+\log(z)-\gamma)}{\Gamma(-\alpha)}, & \text{in other cases,}\end{cases}$$

$$\mathcal{D}_z^\alpha\left[z^\lambda\right]=\begin{cases}(-1)^\alpha(-\lambda)_\alpha z^{\lambda-\alpha}, & \alpha\in\mathbb{Z},\lambda\in\mathbb{Z},\lambda<0,\lambda<\alpha;\\ \frac{(-1)^{\lambda-1}z^{\lambda-\alpha}(\psi(-\lambda)-\psi(\lambda-\alpha+1)+\log(z))}{(-\lambda-1)!\Gamma(\lambda-\alpha+1)}, & \lambda\in\mathbb{Z},\lambda<0;\\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}z^{\lambda-\alpha}, & \text{in other cases,}\end{cases} \tag{2}$$

$$\mathcal{D}_z^\alpha\left[\frac{\sqrt{z^2}}{z}\right]=\frac{z^{-\alpha-1}\sqrt{z^2}}{\Gamma(1-\alpha)},$$

$$\mathcal{D}_z^\alpha\left[\log(z)\right]=\begin{cases}(-1)^{\alpha-1}(\alpha-1)!z^{-\alpha}, & \alpha\in\mathbb{Z},\alpha>0;\\ \frac{z^{-\alpha}(-\psi(1-\alpha)+\log(z)-\gamma)}{\Gamma(1-\alpha)}, & \text{in other cases,}\end{cases}$$

$$\mathcal{D}_z^\alpha\left[\frac{e^{-z}}{z}\right]=-\frac{z^{-\alpha}}{\Gamma(1-\alpha)}\,{}_2F_2(1,1;2,1-\alpha;-z)+$$

$$+z^{-\alpha-1}\begin{cases}(-1)^{-\alpha}(1)_\alpha, & \alpha\in\mathbb{Z}\wedge-1<\alpha;\\ \frac{\log(z)-\psi(-\alpha)-\gamma}{\Gamma(-\alpha)}, & \text{in other cases,}\end{cases}.$$

$$\mathcal{D}_z^\alpha\left[\log^2(z)\right]=\begin{cases}2(-1)^{\alpha-1}(\alpha-1)!z^{-\alpha}(-\psi(\alpha)+\log(z)-\gamma), & \alpha\in\mathbb{Z},\alpha>0;\\ \frac{z^{-\alpha}\left((\log(z)-\gamma)^2-\psi(1-\alpha)+\frac{\pi^2}{6}\right)-\psi'(1-\alpha)}{\Gamma(1-\alpha)}, & \text{in other cases.}\end{cases}$$

Here ψ is the digamma function, given by $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$, ψ' gives the derivative of the digamma function.

Let us note that in (2) in the case $\lambda\in\mathbb{Z}$, $\lambda<0$ and $\lambda\geq\alpha$ the integration of z^λ can produce $\log(z)$. In this case we use Hadamard finite part. For example, when $\lambda=-1$ and $\alpha=-1$

$$\int_\varepsilon^z\frac{dt}{t}=\log(z)-\log(\varepsilon)\Rightarrow\mathcal{D}_z^{-1}\left[z^{-1}\right]=f.p.\int_0^z\frac{dt}{t}=\log(z).$$

We consider two approaches to calculating $\mathcal{D}_z^\alpha[f(z)]$. The first approach is to use Loran series expansions near zero. The second approach is to present function f through Meijer G-function and then find $\mathcal{D}_z^\alpha[f(z)]$ as a fractional derivative of this Meijer G-function.

1. Calculation of fractional derivatives and integrals by series expansion

Let consider the first approach to calculating $\mathcal{D}_z^\alpha[f(z)]$. Series expansion allows us to find $\mathcal{D}_z^\alpha[f(z)]$ because differ-integral applied to each term of Taylor series expansions of all functions near zero. So if

$$f(z) = z^b \sum_{n=0}^{\infty} c_n z^n \quad \Rightarrow \quad \mathcal{D}_z^\alpha[f(z)] = \sum_{n=0}^{\infty} c_n \mathcal{D}_z^\alpha[z^{b+n}]. \tag{3}$$

Sum representations by formula (3) we meet for functions like

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

but sometimes series expansions include $\log(z)$ function as in the logarithmic case of $K_0(z)$:

$$K_0(z) = -\left(\log\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} \left(\frac{z}{2}\right)^{2n},$$

with n-th harmonic number $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ and γ is Euler–Mascheroni constant. It means, that we should consider more general series

$$f_L(z) = z^b \log^k(z) \sum_{n=0}^{\infty} c_n z^n,$$

and evaluate for arbitrary b, α and integer $k = 0, 1, 2, ...$ the following values

$$\mathcal{D}_z^\alpha[f_L(z)] = \sum_{n=0}^{\infty} c_n \mathcal{D}_z^\alpha \left[z^{b+n} \log^k(z)\right]. \tag{4}$$

So, in order to calculate (4) we should find $\mathcal{D}_z^\alpha \left[z^\lambda \log^k(z)\right]$ by the formula (1)

$$\mathcal{D}_z^\alpha \left[z^\lambda \log^k(z)\right] = \begin{cases} z^\lambda \log^k(z), & \alpha = 0; \\ (z^\lambda \log^k(z))^{(\alpha)}, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor \alpha \rfloor + 1 \quad \text{and} \quad \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha) \neq 0. \end{cases} \tag{5}$$

Here for $\alpha \in \mathbb{Z}$ and $\alpha > 0$

$$(z^\lambda \log^k(z))^{(\alpha)} = \sum_{j=0}^{\alpha} \binom{\alpha}{j} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-j+1)} z^{\lambda-j} \frac{d^{\alpha-j} \log^k(z)}{dz^{\alpha-j}}.$$

If some of integrals $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}}$, $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha-n+1}}$, $\int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha+1}}$, $\int\limits_0^z \frac{t^\lambda \log^k(t) dt}{(z-t)^{\alpha-n+1}}$ in (5) diverges we take Hadamard finite part of this integral.

2. The Meijer G-function and fractional calculus

It is known [2] (also see `ResourceFunction["MeijerGForm"]` in Wolfram Mathematica) that wide class of functions (hypergeometric type functions) can be defined as the functions, which generically can be represented through linear combinations of generalized Meijer G–function which is a very general special function of the form

$$G_{p,q}^{m,n}\left(z,r\,\middle|\,\begin{matrix}a_1,...,a_n,a_{n+1},...,a_p\\ b_1,...,b_m,b_{m+1},...,b_q\end{matrix}\right) =$$

$$= \frac{r}{2\pi i}\int\limits_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m}\Gamma(b_k+s)\prod\limits_{k=1}^{n}\Gamma(1-a_k-s)}{\prod\limits_{k=m+1}^{q}\Gamma(1-b_k-s)\prod\limits_{k=n+1}^{p}\Gamma(a_k+s)} z^{-\frac{s}{r}}\,ds, \tag{6}$$

where $r\in\mathbb{R}$, $r\neq 0$, $m\in\mathbb{Z}$, $m\geq 0$, $n\in\mathbb{Z}$, $n\geq 0$, $p\in\mathbb{Z}$, $p\geq 0$, $q\in\mathbb{Z}$, $q\geq 0$, $m\leq q$, $n\leq p$ (details about contour $\mathcal{L}$ separating "left" poles from "right" one see at `https://functions.wolfram.com/HypergeometricFunctions/MeijerG1/02/`).

Fractional order integral of this function with argument a z^r and parameter v can be described by the formula

$$\frac{1}{\Gamma(\alpha)}\int\limits_0^z (z-\tau)^{\alpha-1}\tau^{u-1}G_{p,q}^{m,n}\left(a\tau^r,\nu\,\middle|\,\begin{matrix}a_1,...,a_n,a_{n+1},...,a_p\\ b_1,...,b_m,b_{m+1},...,b_q\end{matrix}\right)d\tau = z^{\alpha+u-1}\times$$

$$\times H_{p+1,q+1}^{m,n+1}\left(a^{1/\nu}z^{r/\nu}\,\middle|\,\begin{matrix}\left(1-u,\frac{r}{\nu}\right),(a_1,1),...,(a_n,1),(a_{n+1},1),...,(a_p,1)\\ (b_1,1),...,(b_m,1),(b_{m+1},1),...,(b_q,1),\left(1-\alpha-u,\frac{r}{\nu}\right)\end{matrix}\right) \tag{7}$$

which is valid under corresponding conditions, providing convergence of above integral. Here $H_{p,q}^{m,n}$ is the Fox H-function defined by a Mellin–Barnes integral

$$H_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_1,A_1) & (a_2,A_2) & \dots & (a_p,A_p)\\ (b_1,B_1) & (b_2,B_2) & \dots & (b_q,B_q)\end{matrix}\right] =$$

$$= \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{j=n+1}^{p}\Gamma(a_j+A_js)} z^{-s}\,ds,$$

where $\mathcal{L}$ is a certain contour separating the poles of the two groups of factors in the numerator. If the function $f(z)$ can be written as a finite sum of generalized Meijer G-function applying the formula (7) we can find fractional integral or derivative of $f(z)$ in the form of the Meijer G-function or Fox H-functions. Then

we can write the Fox H-function as a simpler function if possible. Numerous examples of evaluation of fractional order integro derivatives users can find using `https://resources.wolframcloud.com/FunctionRepository/resources/FractionalOrderD` for example,

$$\mathcal{D}_z^{\alpha}[K_0(z)] = \frac{1}{2} G_{2,4}^{2,2}\left(\frac{z}{2}, \frac{1}{2}\middle| \begin{matrix} \frac{1-\alpha}{2}, -\frac{\alpha}{2} \\ -\frac{\alpha}{2}, -\frac{\alpha}{2}, 0, \frac{1}{2} \end{matrix}\right).$$

Conclusion

Despite the fact that there are a large number of different approaches to fractional integro-derivation, for example, Riemann-Liouville, Caputo, Grünwald–Letnikov and others [1], these approaches are not so different. Indeed, calculating various fractional derivatives of a power function x^p, we almost always get the same result. In this paper we considered two approaches to calculating an arbitrary power of a differential operator $\frac{d}{dx}$ which are suitable for a wide class of functions.

References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional integrals and derivatives*, Amsterdam: Gordon and Breach Science Publishers, 1993.

[2] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental functions (theory and algorithmic tables)*, Ellis Horwood Ltd, 1983.

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On a property of Young diagrams of maximum dimensions

Egor Smirnov-Maltsev

Abstract. The work is devoted to finding the Young diagrams of large dimensions, i.e. those which have large number of Young tableaux. The algorithm which modifies a diagram A_n of size n into another diagram A_{1n} of size n is proposed. It is proved that the dimension of A_{1n} is greater than or equal to the dimension of A_n. A criterion for a Young diagram of maximum dimension is formulated.

Introduction

This work is devoted to investigation of an open combinatorial problem [1] of finding a Young diagram of size n with the maximum dimension. In other words, the goal is to find a Young diagram which has the largest number of Young tableaux among all the diagrams of size n. The dimension is a rational function of diagram shape. So we can reformulate the problem as search for the maximum of this rational function. This talk is dedicated to an important property of a diagram of maximum dimension.

Let us define a basic subdiagram of a diagram A as the maximum symmetric subdiagram of A. So each diagram A consists of its basic subdiagram A_{sym}, boxes A_d located below the line $y = x$ and not included in the basic subdiagram, as well as boxes A_u located above $y = x$ and not included in the basic subdiagram. The study of Young diagrams of large dimension [2] has shown that a diagram A with the largest dimension has either no A_u or no A_d boxes. The idea of the algorithm proposed in this work has the similar nature with the previous algorithms [3]. It is assumed that the algorithm modifies a diagram A_n of size n into another diagram A_{1n} of size n so the dimension of A_{1n} is greater than or equal to the dimension of A_n.

The last statement can be proved using the hook length formula [4]:

$$\dim(A_n) = \frac{n!}{\prod\limits_{(i,j)\in A_n} h(i,j)}, \tag{1}$$

where A_n is a diagram of size n, $h(i,j)$ is the hook length of box (i,j) in diagram A_n. Our goal is to prove that $\dim(A_n) \leq \dim(A_{1n})$. Since we consider diagrams of the same size, it is enough to prove that

$$\prod_{(i,j)\in A_n} h(i,j) \geq \prod_{(i,j)\in A_{1n}} h_1(i,j), \tag{2}$$

where $h(i,j)$ is the hook length of a box (i,j) in diagram A_n, and $h_1(i,j)$ is the hook length of a box (i,j) in diagram A_{1n}.

1. Algorithm for modifying Young diagrams

In the first step of the algorithm, we transform a diagram A into the diagram A_1 which has no boxes located below the line $y = x$ and not included in the basic subdiagram A_1. In other words, A_1 consists only of its basic subdiagram A_{1sym} and boxes $A_{1u} \notin A_{sym}$ located above the $y = x$. Let us consider each row t containing boxes from A_d. We move half of such boxes in a row t to a column t. If the row t has $2l + 1$ boxes from A_d, we move $l + 1$ boxes. An example of this procedure is shown in Figure 1. The white boxes form the basic subdiagram of the diagram A, the moved boxes are highlighted in gray, and the remaining boxes are highlighted in black.

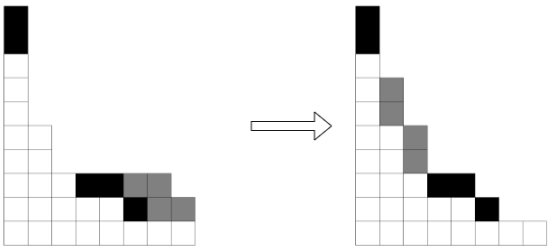

FIGURE 1. The first step of the algorithm

On the second step, we transform the diagram A_1 into a diagram A_2 that consists of its base subdiagram with single boxes added in some rows. All the added single boxes are located below the line $y = x$. An example of this step is illustrated in Figure 2. The meaning of the colors is the same as in the previous example.

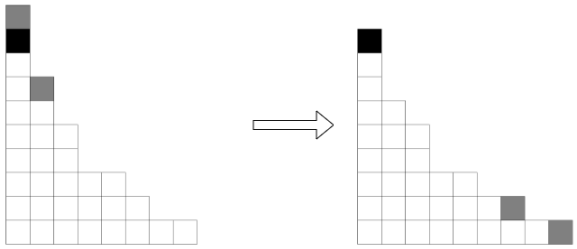

FIGURE 2. The second step of the algorithm

2. Dimensions of the original and modified diagrams

Here we prove that the dimension of a diagram does not decrease during each of the above transformations. Firstly, let us consider the second transformation. By formula 2, we need to prove that

$$\prod_{(i,j)\in A_1} h_1(i,j) \geq \prod_{(i,j)\in A_2} h_2(i,j). \tag{3}$$

Let t_i be all the columns that contain boxes from A_{1u}. Consider a column t_i for some i. t'_i is the row that is symmetric to the column t_i with respect to $y = x$. Then we can directly prove that the product of hook lengths for the boxes in t_i column and t'_i row in a diagram A_1 is not less than the product of hook lengths for the boxes in t_i column and t'_i row in a diagram A_2. Hook lengths for some boxes are counted several times but it does not affect the proof because the hook lengths of these boxes in the diagram A_1 are not greater than the hook lengths of these boxes in the diagram A_2. It is proved similarly that the product of hook lengths in the diagram A_1 for remaining boxes is greater than the product of hook lengths for these boxes in the diagram A_2.

The proof that the dimension of the diagram does not decrease during the first transformation comes from the previous statements. Particularly, it can be claimed that $\dim(A \setminus A_u) \leq \dim(A_1 \setminus A_u)$. Then, we add boxes one by one from A_u to both diagrams. The hook lengths product of boxes of A_1 grows faster than the hook lengths product of boxes of A_2 each time a box is added.

References

[1] R. M. Baer, P. Brock, *Natural Sorting over Permutation Spaces.* Math. Comp., 22:385–410, 1968.

[2] V. S. Duzhin, A.A. Chudnovskaya, *Searching for Young diagrams with large dimensions.* Computer tools in education.

[3] N. N. Vasilyev, V. S. Duzhin, *Building Irreducible Representations of a Symmetric Group S(n) with Large and Maximum Dimensions*. Information and Control Systems, 2015, №3, pp. 17-22.

[4] J. S. Frame, G. de B. Robinson, R. M. Thrall, *The hook graphs of the symmetric group*. Canad J. Math., 6 (1954), 316-325.

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Doubly-periodic string comparison

Nikita Gaevoy and Alexander Tiskin

Abstract. The longest common subsequence (LCS) problem is a textbook problem in string algorithms and bioinformatics. Given a pair of strings, the problem asks for the length of the longest string that is a subsequence in both input strings. In previous works, the second author developed a novel powerful approach to the LCS and related problems. This approach is based on the algebraic framework of the Hecke monoid, which can be visualised by manipulating braid-like objects that we call sticky braids. Among the many algorithmic problems that can be solved efficiently by the Hecke monoid approach, there is the natural problem of obtaining the LCS for a pair of strings, one or both of which has periodic structure. The case of one periodic string has been considered before; in this work, we extend the solution to the case where both input strings are periodic. The resulting algorithm for doubly-periodic LCS has been engineered by the first author while developing the content for an ICPC training camp. In numerical experiments, the code shows performance that allows one to process doubly-periodic inputs of sizes far beyond the reach of ordinary and singly-periodic LCS algorithms.

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Levi-Civita reduction in three-body problem

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Abstract. The constructed areas of possible motion of a three-body planar problem in the case of a spatial problem require tools to reduce the problem. To study a spatial problem, the space of forms is reduced to the space of forms of a plane problem. Since the three bodies are always in the same plane, the shape of the triangle is described using a tool already used in the plane problem. Levi-Civita reduction separates the variables responsible for the configuration of the three bodies from the variables describing the motion of the plane of the three bodies.

Introduction

In the planar three-body problem, a zero-velocity surface can be built. By fixing the energy constant ($h < 0$), we can construct such a surface in the form space, that is, in the factor space of the configuration space by transfers and rotation. Depending on the value of the angular momentum J, we get five topologically different types of surface, that is, five topologically different types of the area of possible motion [2].

1. Reduction

One can write the kinetic energy T in a planar problem reduced to the form space ξ_1, ξ_2, ξ_3 as

$$T = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} \tag{1}$$

and potential function U depends on $U(\xi_1, \xi_2, \xi_3)$ and can be written in the form

$$U = \frac{1}{\rho} D(\theta, \phi), \tag{2}$$

where ρ, θ, ϕ are spherical coordinates in the $\xi_1\xi_2\xi_3$ space.

We want to reduce the problem to the form space.

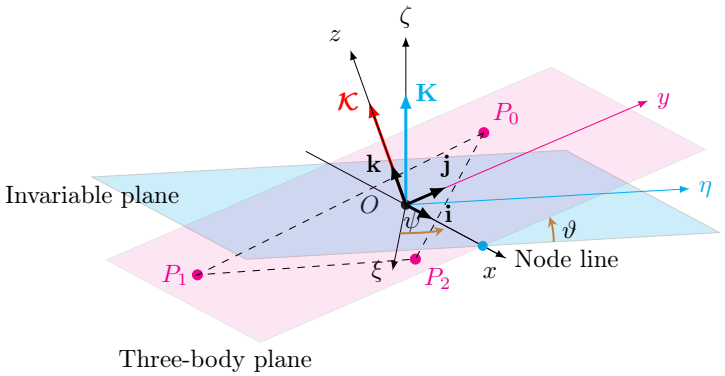

FIGURE 1. Coordinates ψ, ϑ of the plane of three bodies. $\mathbf{K}$ – vector of total angular momentum

In a planar three-body problem, the kinetic energy is

$$\begin{aligned} T &= \frac{4J^2+\dot\xi_1^2+\dot\xi_2^2+\dot\xi_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}, \\ J &= \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\,\frac{d\lambda}{dt} + \frac{\xi_2\frac{d\xi_3}{dt}-\xi_3\frac{d\xi_2}{dt}}{2(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)} \end{aligned} \tag{3}$$

To extend a planar problem to a spatial one, we will use the Levi-Civita reduction [1]. In Fig. 1, the angles ψ and ϑ determine the position of the plane of the three bodies relative to the Laplace invariable plane. Then the positions of the points are determined by these angles and the coordinates of the points in this plane.

2. Equations of motion

Let us express kinetic energy in the variables ψ, ϑ and x_i, y_i, z_i. For the coordinates x_i, y_i, z_i we have

$$\begin{aligned} x_i &= X_i\cos\psi + Y_i\sin\psi, \\ y_i &= (-X_i\sin\psi + Y_i\cos\psi)\cos\vartheta + Z_i\sin\vartheta, \\ z_i &= (X_i\sin\psi - Y_i\cos\psi)\sin\vartheta + Z_i\cos\vartheta, \end{aligned} \tag{4}$$

and for kinetic energy T

$$\begin{aligned} T = \mathcal{T}+\mathfrak{T}: \quad \mathcal{T} &= \tfrac12\sum_{i=1}^{3} m_i\left\{(\dot x_i - y_i\dot\psi\cos\vartheta)^2 + (\dot y_i + x_i\dot\psi\cos\vartheta)^2\right\}, \\ \mathfrak{T} &= \tfrac12\sum_{i=1}^{3} m_i\left\{(y_i\dot\vartheta - x_i\dot\psi\sin\vartheta)^2\right\}. \end{aligned} \tag{5}$$

Note that in conjugate momenta $P_i = \frac{\partial T}{\partial \dot{x}_i}$, $Q_i = \frac{\partial T}{\partial \dot{y}_i}$ the expression for $\mathcal{T}$ is equal to the following

$$\mathcal{T} = \frac{1}{2}\sum_{i=0}^{2}(P_i^2 + Q_i^2)/m_i. \tag{6}$$

Conclusion

The coordinates x_i, y_i, z_i define the position in the plane of three bodies, and therefore we can construct a form space similar to the planar three-body problem [2].

References

[1] Tulio Levi-Civita, *Sulla riduzione del problema dei tre corpi.* Atti Ist. Veneto di Sc., lett. ed arti, t. LXXIV, 1915.

[2] V. B. Titov, *Zero-Velocity Surface in the General Three-Body-Problem.* Vestnik St. Petersburg University, Mathematics, 2023, Vol. 56, No. 1.

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Describing classicality of states of a finite-dimensional quantum system via Wigner function positivity

Arsen Khvedelidze and Astghik Torosyan

Abstract. In the present report, within the phase-space formulation of quantum theory of $N-$level quantum system, three measures of classicality constructed out of the quasiprobability distributions will be discussed. All considered measures are based on the existence of the "classical states" defined as those whose Wigner function is positive semi-definite over the whole phase space. The variety of classicality measures originates from different ways of quantifying deviations of states from the subset of classical states. Algebraic and geometric descriptions of the set of classical states will be given in terms of the corresponding convex bodies located inside the simplex of density matrices eigenvalues. A few computational aspects of classicality measures will be discussed and exemplified for qubits, qutrits and quatrits.

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Bounded elementary generation of Chevalley groups and Steinberg groups

Boris Kunyavskiĭ, Andrei Lavrenov, Eugene Plotkin and Nikolai Vavilov

Abstract. This is a sequel of our talk at the PCA-2022, see [17] Here we state a definitive result which almost completely closes the problem of bounded elementary generation for Chevalley groups over arbitrary Dedekind rings of arithmetic type with uniform bounds. Namely, for every reduced irreducible root system Φ of rank ≥ 2 there exists a universal bound $L = L(\Phi)$ such that the simply connected Chevalley groups $G(\Phi, R)$ have elementary width $\leq L$ for all Dedekind rings of arithmetic type R. We also state two results concerning bounded elementary generation of the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$.

Introduction

In the present talk, we consider Chevalley groups $G = G(\Phi, R)$, their elementary subgroups $E(\Phi, R)$, and the corresponding Steinberg groups $\mathrm{St}(\Phi, R)$. over various classes of rings, mostly over Dedekind rings of arithmetic type (we refer to [40] for notation and further references pertaining to Chevalley groups, and to [2] for the number theory background).

Primarily, we are interested in the classical problem of estimating the width of $E(\Phi, R)$ with respect to the elementary generators $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$. We say that a group G is **boundedly elementarily generated** if $E(\Phi, R)$ has finite width $w_E(G)$ with respect to elementary generators.

This problem has attracted considerable attention over the last 40 years or so. Below, we reproduce the survey page from [17].

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• Bounded elementary generation always holds with obvious *small* bounds for 0-dimensional rings. This follows from the existence of such short factorisations as Bruhat decomposition, Gauß decomposition, unitriangular factorisation of length 4, and the like. On the other hand, bounded generation usually fails for rings of dimension ≥ 2. But for 1-dimensional rings it is problematic.

• Existence of arbitrary long division chains in Euclidean algorithm implies that $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{SL}(2,\mathbb{F}_q[t])$ are not boundedly elementary generated [7]. But this could be attributed to the exceptional behaviours of rank 1 groups.

• What came as a shock, was when Wilberd van der Kallen [15] established that bounded elementary generation — and thus also finite commutator width — fail even for $\mathrm{SL}(3,\mathbb{C}[x])$, a group of Lie rank 2 over a Euclidean ring! Compare also [9], for a slightly simplified proof.

An emblematic example of 1-dimensional rings are Dedekind rings of arithmetic type $R = \mathcal{O}_S$, for which bounded elementary generation of $G(\Phi, R)$ is intrinsically related to the positive solution of the congruence subgroup problem in that group.

For the **number case** the situation is well understood, even for rank 1 groups. Without attempting to give a detailed survey, let us mention some high points of this development. Apart from the rings $R = \mathcal{O}_S$, $|S| = 1$, with finite multiplicative group, such finiteness results are even available for $\mathrm{SL}(2, R)$.

• For all Chevalley groups of rank ≥ 2, after the initial breakthrough by Douglas Carter and Gordon Keller, [4, 5], later explained and expanded by Oleg Tavgen [37], and many others, we now know bounded elementary generation with excellent bounds depending on the type of Φ and the class number of R alone.

This leaves us with the analysis of the group $\mathrm{SL}(2, R)$, for a Dedekind ring $R = \mathcal{O}_S$, with infinite multiplicative group.

• At about the same time, jointly with Paige, Carter and Keller gave a model theoretic proof [unpublished], [6], somewhat refashioned by Dave Morris [27]. But as all model theoretic proofs, this proof gives no bounds whatsoever.

• On the other hand, another important advance was made by Cooke and Weinberger [8], who got excellent bounds, modulo the Generalised Riemann Hypothesis. The explicit unconditional bounds obtained thereafter seemed to be grossly exaggerated [23].

• Some 10 years ago Maxim Vsemirnov and Sury [43] considered the key example of $\mathrm{SL}\Big(2, \mathbb{Z}\Big[\frac{1}{p}\Big]\Big)$, obtaining the bound $w_E(\mathrm{SL}(2, R)) = 5$ *unconditionally*.

• This was a key inroad to the first complete unconditional solution of the general case with a good bound, in the work of Alexander Morgan, Andrei Rapinchuk and Sury [25]. The bound they gave is ≤ 9, but for the case when S contains at least one real or non-Archimedean valuation was almost immediately improved [with the same ideas] to ≤ 8 by Jordan and Zaytman [13].

However, the **function case** turned out to be much more recalcitrant, and was not fully solved until March 2023, apart from some important but isolated results.

• Until very recently the only published result was that by Clifford Queen [30]. Queen's main result implies that when R^* is infinite + *some additional assumptions* on R hold, the elementary width of the group $\mathrm{SL}(2, R)$ is 5. As shown in [16] this implies, in particular, bounded elementary generation of all Chevalley groups $G(\Phi, R)$ under the same assumptions on R, with plausible bounds.

• The case of the groups over the usual polynomial ring $\mathbb{F}_q[t]$ long remained open. Only in 2018 has Bogdan Nica [28] established bounded elementary generation of $\mathrm{SL}(n, \mathbb{F}_q[t])$, $n \geq 3$. Next, in [16] we established bounded elementary generation of $\mathrm{Sp}(l, \mathbb{F}_q[t])$, $l \geq 2$, and

• The next breakthrough came in the preprints of Alexander Trost [38, 39] where he established bounded elementary generation of $\mathrm{SL}(n, R)$, for the ring of integers R of an arbitrary global function field K. First with a bound of the form $L(d, q) \cdot |\Phi|$, where the factor L depends on q and of the degree d of K, and then with the uniform bound. His method in [39] is similar to Morris' approach in [27].

1. Bounded generation of $G(\Phi, R)$

Combining the methods of [16] and [39], we are now able to come up with a complete solution in the general case. An important — and unexpected! — aspect of this work is the existence of *uniform* bounds. In the symplectic case this result is new even for the number case. All details are to be found in our forthcoming paper [18].

Theorem A. *Let Φ be a reduced irreducible root system of rank $l \geq 2$. Then there exists a constant $L = L(\Phi)$, depending on Φ alone, such that for any Dedekind ring of arithmetic type R, any element in $\mathrm{G_{sc}}(\Phi, R)$ is a product of at most L elementary root unipotents.*

Roughly, the ingredients of the proof are as follows.

• For the **number case**, when R^* is infinite there is a definitive result by Morgan, Rapinchuk and Sury [25], with a small uniform bound $L \leq 9$, which can be improved in some cases.

Some uniform bound can be now easily derived by a version of the usual Tavgen's trick [37], Theorem 1, as described and generalised in [41, 33] and [16, 17].

• The uniform bound for $\mathrm{SL}(3, R)$ over imaginary quadratic rings was obtained by [27], see also [39]. Using the rank reduction methods based on Tavgen's lemma *and* stability, as in [16], we can reduce the analysis of $G(\Phi, R)$ for all *non-symplectic* root systems to $\mathrm{SL}(3, R)$.

• This leaves us with the analysis of $\mathrm{Sp}(2l, R)$, $l \geq 2$. What we haven't moticed when writing [16] is that bounded generation of $\mathrm{Sp}(2l, R)$, $l \geq 3$, also reduces to $\mathrm{SL}(3, R)$, with the help of the symplectic lemmas on switching long and

short roots [16]. Thus, only $\mathrm{Sp}(4, R)$ requires separate analysis, since the bound given by Tavgen [37] is not uniform, it depends on the degree and discriminant of the number field K. However, in this case using our Sp_4-lemmas from [16] we are able to give a new proof in the style of [27].

• For the **function case**, $\mathrm{SL}(2, R)$ is not completely solved, so we have to rely on the reduction to rank two systems instead. Luckily, for $\mathrm{SL}(3, R)$ the uniform bound is given by Trost [39], which again (with the help of reduction lemmas from [16]) provides uniform bounds for all other groups of rank ≥ 2, with the sole exception of $\mathrm{Sp}(4, R)$. The key ingredient for this, bounded extraction of square roots from Mennicke symbols, is also contained in [39]. For this last case we succeed in imitating the proof from [27, 39] with our Sp_4-lemmas from [16].

2. Bounded generation of $\mathrm{St}(\Phi, R)$

Also, we obtained partial results towards bounded generation for the corresponding Steinberg groups. Again, we are interested in the bounded generation in terms of the set

$$X = \{x_\alpha(r) \mid r \in R,\ \alpha \in \Phi\}$$

o elementary generators.

However, this case turned out to be much more demanding. Apart from the bounded generation of the Chevalley groups themselves, it depends on the deep results on the finiteness of the (linear) K_2-functor, and on bunch of other difficult results of K-theory, such as stability theorem for K_2, centrality of K_2, etc.

Here is our second main result. So far we have been able to establish it only for the simply-laced systems.

Theorem B. *Let Φ be a reduced irreducible simply laced root system of rank ≥ 2, and let R be a Dedekind ring of arithmetic type. If $\Phi = \mathsf{A}_2$ assume additionally that R^* is infinite. Then $\mathrm{St}(\Phi, R)$ is boundedly elementary generated.*

The idea is to derive this result from Theorem A. It suffices to establish that the kernel $K_2(\Phi, R)$ of the projection $\mathrm{St}(\Phi, R) \longrightarrow G(\Phi, R)$ is finite and thus bounded elementary generation of $G(\Phi, R)$ implies that of $\mathrm{St}(\Phi, R)$. Here are the main sources on which we rely in this proof.

• The *stable* linear $K_2(R)$ is finite, for the function case this is proven by Hyman Bass and John Tate [3] and for the number case by Howard Garland [10]. (These finiteness results were generalised to higher K-theory by Daniel Quillen and Günter Harder, see the survey by Chuck Weibel [44]).

• However, we need similar results for the unstable K_2-functors $K_2(\Phi, R)$. For the *linear* case $\mathrm{SL}(n, R)$ there is a definitive stability theorem by Andrei Suslin and Marat Tulenbaev [36]. However, injective stability for Dedekind rings only starts with $n \geq 4$, so that for $\mathrm{SL}(3, R)$ one has to refer to van der Kallen [14] instead, which accounts for the extra-condition in this case.

• However, for other embeddings there are no stability theorems in the form we need them and starting where we want them to start. For instance, in the even orthogonal case the theorem of Ivan Panin [29] starts with $\mathrm{Spin}(10, R)$, whereas we would like to cover also $\mathrm{Spin}(8, R)$. In any case, there are no similar results for the exceptional embeddings.

Thus, we have to prove to prove a comparison theorem relating $K_2(\Phi, R)$ to $K_2(\mathrm{A}_3, R)$. This is accomplished by a combination of two techniques. On the one hand there are partial stability results for Dedekind rings developed by Hideya Matsumoto [24] and *surjective* stability of K_2 for some embeddings, established by Michael Stein [34] and one of us Eugene Plotkin. On the other hand, there are powerful recent calculations used to prove the centrality of K_2 for all Chevalley groups, by Andrei Lavrenov, Sergei Sinchuk, and Egor Voronetsky [19, 32, 20, 21, 42, 22].

• An essential obstacle in the symplectic case is that $K_2(\mathrm{C}_l, R)$ is the Milnor—Witt K_2^{MW}, rather than the usual Milnor K_2^M, as for all other cases (compare [35] for an explicit connection between $\mathrm{K}_2\mathrm{Sp}(R)$ and $\mathrm{K}_(R)$). As is well known, it may fail to be finite, which means that our approach does not work at all in this case. It does not mean that the result itself fails, but the proof would require an entirely different idea.

But even for non-symplectic multiply laced systems, where our approach could theoretically work, we were unable to overcome occurring technical difficulties related to the K_2-stability and comparison theorems. At least, as yet.

However, using specific calculations of $K_2(\Phi, \mathbb{F}_q[t])$ and $K_2(\Phi, \mathbb{F}_q[t, t^{-1}])$ by Eiichi Abe, Jun Morita, Jürgen Hurrelbrink and Ulf Rehmann [1, 11, 26, 31] we were able to establish similar results over $\mathbb{F}_q[t]$ and $\mathbb{F}_q[t,\, t^{-1}]$ also for the multiply laced systems, even the symplectic ones.

Theorem C. *Let Φ be a reduced irreducible root system, and $R = \mathbb{F}_q[t,\, t^{-1}]$ or $R = \mathbb{F}_q[t]$. In the latter case assume additionally that $\Phi \neq \mathsf{A}_1$. Then $\mathrm{St}(\Phi,\, R)$ is boundedly elementary generated.*

In the present talk we do not touch further closely related problems, such as commutator width or verbal width, or even relative versions of our results. Some indications and references can be found in [16, 17], more are coming in [18].

Acknowledgements. In the preliminary version of the present work Theorem A was stated in a weaker form, with some exceptions in the symplectic case. We are grateful to Sergei Gorchinsky, Denis Osipov, and Dmitry Timashev, or the invitations to give talks at the seminars in Moscow, and the subsequent discussions that framed our mind to lift all remaining conditions.

References

[1] E. Abe, J. Morita, *Some Tits Systems with Affine Weyl Groups in Chevalley Groups over Dedekind Domains*, J. Algebra **115** (1988), 450–465.

[2] H. Bass, J. Milnor, J-P. Serre, *Solution of the congruence subgroup problem for* $\mathrm{SL}_n (n \geq 3)$ *and* $\mathrm{Sp}_{2n} (n \geq 2)$. Inst. Hautes Études Sci. Publ. Math. **33** (1967), 59–137.

[3] H. Bass, J. Tate, *Milnor ring of a global field*, in: H. Bass (ed), 'Classical' Algebraic K-Theory, and Connections with Arithmetic, Lecture Notes Math. **342**, Springer Berlin, Heidelberg (1973).

[4] D. Carter, G. E. Keller, *Bounded elementary generation of* $\mathrm{SL}_n(\mathcal{O})$. Amer. J. Math. **105** (1983), 673–687.

[5] D. Carter, G. E. Keller, *Elementary expressions for unimodular matrices.* — Commun. Algebra **12** (1984), 379–389.

[6] D. Carter, G. E. Keller, E. Paige, *Bounded expressions in* $\mathrm{SL}(2, \mathcal{O})$, preprint Univ. Virginia, 1985, 1–21.

[7] P. M. Cohn, *On the structure of the* GL_2 *of a ring.* Inst. Hautes Études Sci. Publ. Math. **30** (1966), 5–53.

[8] G. Cooke, P. J. Weinberger, *On the construction of division chains in algebraic number rings, with applications to* SL_2. Commun. Algebra **3** (1975), 481–524.

[9] I. V. Erovenko, $\mathrm{SL}_n(F[x])$ *is not boundedly generated by elementary matrices: explicit proof.* Electronic J. Linear Algebra **11** (2004), 162–167.

[10] H. Garland *A Finiteness Theorem for* K_2 *of a Number Field*, Ann. Math. **94** (1971), no. 3, 534–548.

[11] J. Hurrelbrink, *Endlich präsentierte arithmetische Gruppen und* K_2 *über Laurent-Polynomringen*, Math. Ann. **225** (1977), no. 2, 123–129.

[12] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang, *Commutator width in Chevalley groups.* Note Mat. **33** (2013), 139–170.

[13] B. W. Jordan, Y. Zaytman, *On the bounded generation of arithmetic* SL_2. Proc. Nat. Acad. Sci. USA **116** (2019), 18880–18882.

[14] W. van der Kallen, *Stability for* K_2 *of Dedekind rings of arithmetic type*, in: E. Friedlander, M. Stein (eds), Algebraic K-Theory Evanston 1980, Lecture Notes Math. **854**, Springer Berlin, Heidelberg (1981).

[15] W. van der Kallen, $\mathrm{SL}_3(\mathbb{C}[X])$ *does not have bounded word length.* In 'Algebraic K-theory, Part I (Oberwolfach 1980)', 357–361, Lecture Notes in Math. **966**, Springer, 1982.

[16] B. Kunyavskiĭ, E. Plotkin, N. Vavilov *Bounded generation and commutator width of Chevalley groups: function case*, `arXiv:2204.10951v1 [math.GR], 22 Apr 2022`, 54p.

[17] B. Kunyavskiĭ, E. Plotkin, N. Vavilov *Bounded elementary generation of Chevalley groups*, PCA-2022, 02–07 May 2022, St Petersburg, EIMI, 109–120.

[18] B. Kunyavskiĭ, A. Lavrenov, E. Plotkin, N. Vavilov *Bounded elementary generation of Chevalley groups and Steinberg groups*, 2023.

[19] A. Lavrenov, *Another presentation for symplectic Steinberg groups*, J. Pure Appl. Alg. **219** (2015), no 9, 3755–3780.

[20] A. Lavrenov, S. Sinchuk, *On centrality of even orthogonal* K_2, J. Pure Appl. Alg. **221** (2017), no 5, 1134–1145.

[21] A. Lavrenov, S. Sinchuk, *A Horrocks-Type Theorem for Even Orthogonal* K_2, Doc. Math. **25** (2020) 767–809.

[22] A. Lavrenov, S. Sinchuk, E. Voronetsky, *Centrality of* K_2 *for Chevalley groups: a pro-group approach*, *Israel Math. J.* (2023), *in press.*

[23] B. Liehl, *Beschränkte Wortlänge in* SL_2. Math. Z. **186** (1984), 509–524.

[24] H. Matsumoto, *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. Éc. Norm. Sup. **2** (1969) 1–62.

[25] A. V. Morgan, A. S. Rapinchuk, B. Sury, *Bounded generation of* SL_2 *over rings of* S*-integers with infinitely many units.* Algebra Number Theory **12** (2018), 1949–1974.

[26] J. Morita, U. Rehmann, *Symplectic* K_2 *of Laurent polynomials, associated Kac–Moody groups and Witt rings*, Math. Z. **206** (1991), no. 1, 57–66.

[27] D. Morris, *Bounded generation of* $\mathrm{SL}(n, A)$ (*after D. Carter, G. Keller, and E. Paige*). New York J. Math. **13** (2007), 383–421.

[28] B. Nica, *On bounded elementary generation for* SL_n *over polynomial rings.* Israel J. Math. **225** (2018), 403–410.

[29] I. Panin, *Stabilization for orthogonal and symplectic algebraic* K*-theories*, Leningrad Math. J. **1** (1990), no. 1, 741–764.

[30] C. Queen, *Some arithmetic properties of subrings of function fields over finite fields.* Arch. Math. **36** (1975), 51–56.

[31] U. Rehmann, *Präsentationen von Chevalleygruppen über* $k[t]$, preprint (1975), `https://www.math.uni-bielefeld.de/~rehmann/Publ/praesentation.pdf`.

[32] S. Sinchuk, *On centrality of* K_2 *for Chevalley groups of type* E_ℓ, J. Pure Appl. Alg. **220**:2 (2016), no. 2, 857–875.

[33] A. Smolensky, B. Sury, N. Vavilov, *Gauss decomposition for Chevalley groups, revisited.* Intern. J. Group Theory **1** 2011, 3–16.

[34] M. R. Stein, *Stability theorems for* K_1, K_2 *and related functors modeled on Chevalley groups.* Japan J. Math. (N.S.) **4** (1978), 77–108.

[35] A. Suslin, *Torsion in* K_2 *of fields*, K-Theory **1** (1987) 5–29.

[36] A. Suslin, M. Tulenbaev, *Stabilization theorem for the Milnor* K_2*-functor*, Journal of Soviet Mathematics **17**:2 (1981), no. 2, 1804–1819.

[37] O. Tavgen, *Bounded generation of Chevalley groups over rings of algebraic* S*-integers.* Izv. Akad. Nauk SSSR Ser. Mat. **54** (1990), no. 1, 97-–122; English transl. in Math. USSR Izv. **36** (1991), 101–128.

[38] A. A. Trost, *Bounded generation by root elements for Chevalley groups defined over rings of integers of function fields with an application in strong boundedness.* `arXiv:2108.12254v1 [math.GR], 27 Aug 2021`, 27p.

[39] A. Trost, *Elementary bounded generation for* SL_n *for global function fields and* $n \geq 3$, *preprint* (2022), `arXiv:2206.13958`.

[40] N. Vavilov, E. Plotkin, *Chevalley groups over commutative rings.* I: *Elementary calculations.* Acta Appl. Math. **45** (1996), 73–113.

[41] N. A. Vavilov, A. V. Smolensky, B. Sury, *Unitriangular factorizations of Chevalley groups.* Zap. Nauchn. Semin. POMI **388** (2011), 17–47; English transl. in J. Math. Sci., New York **183** (2012), 584–599.

[42] E. Voronetsky, *Centrality of odd unitary* K_2*-functor* (2020), *preprint.*

[43] M. Vsemirnov, *Short unitriangular factorizations of* $\mathrm{SL}_2(\mathbb{Z}[1/p])$. Q. J. Math. **65** (2014), 279–290.

[44] Ch. Weibel, Algebraic K-theory of rings of integers in local and global fields. Friedlander, Eric M. (ed.) et al., Handbook of K-theory. Vol. 1 and 2. Berlin, (2005), 139–190 .

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Scaled entropy and metric dynamics; recent achievements

Anatoly Vershik

Scaled entropy generalizes the known entropy Shannon-Kolmogorov was proposed by the speaker in the early 2000s, but, as it turned out recently, it was actually guessed by Shannon in the appendices to his classic work, which was not noticed by absolutely all followers. The development of this concept in recent years has been contained in the works of P.Zaritsky, G.Veprev and the author. A detailed review of these authors will be published in the journal "Uspekhi Matematicheskikh Nauk" In V78:3 (2023) under the title "Dynamics of metrics and scaled entropy". In the report an overview of this article and related issues will be presented. This work is carried out as part of a project supported by an RNF grant 21-11-00152

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Summing-up Involutive Bases Computations Experience

Denis A. Yanovich

Abstract. In this talk, I want to briefly summarize 25 years of experience in the involutive basis computations. I will shortly consider the theory behind and will talk about various data structures, computational approaches, and technologies utilized to go from monomial ideals calculations through polynomial equations system solving to the solution of system of difference equations and from a single-threaded program to distributed computations on supercomputers. This talk will be presented in memory of teacher and scientific advisor V. P. Gerdt.

References

[1] Gerdt V.P., Blinkov Yu.A. *Involutive Bases of Polynomial Ideals*, Math. Comp. Sim. **45**, 1998, pp.519-542

[2] Gerdt V.P., Blinkov Yu.A. and Yanovich D.A. *Construction of Janet Bases. I. Monomial Bases* In: Computer Algebra in Scientific Computing / CASC'01, V.G.Ganzha, E.W.Mayr and E.V.Vorozhtsov (eds.), Springer, Berlin, 2001, pp.249–263.

[3] Yanovich D.A. *Parallelization of an Algorithm for Computation of Involutive Janet Bases*, Programming and Computer Software, 2002, **28**, no. 2, pp. 66-69

[4] Gerdt V.P., Yanovich D.A. *Parallel Computation of Janet and Gröbner Bases over Rational Numbers*, Programming and Computer Software, 2005, **31**, no. 2, pp. 73-80

[5] Gerdt V.P. *Gröbner Bases Applied to Systems of Linear Difference Equations* Physics of Particles and Nuclei Letters, 2008, **5**, no. 3, pp. 248-254

[6] Yanovich D.A. *Evaluation of Parallel Computation of Gröbner and Janet bases using MPI*, "Polynomial Computer Algebra 2011", pp.124–127.

[7] Yanovich D.A. *Evaluation of parallel computations of Grobner and involutive bases on the massive SMP computer*, Mathematical Modeling and Computational Physics (MMCP'2013): Book of Abstracts of the International Conference (Dubna, July 8-12, 2013)

[8] Yanovich D.A. *Computing Gröbner and Involutive Bases for Linear Systems of Difference Equations*, 2018, EPJ Web of Conferences 173, 05018

[9] Yanovich D.A. *Computation of Involutive and Gröbner Bases Using the Tableau Representation of Polynomials*, Programming and Computing Software, 2020, **46**, pp. 162–166

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Mathematics for non-mathematicians: memories of the future

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Abstract. The report discusses the problematic issues of modernization of mathematical education in higher education institutions on the basis of the experience of delivering training courses that use the capabilities of computer mathematics at the Department of Information Systems in Economics of St. Petersburg State University. The main result of the work performed is the development and justification of the author's approach to teaching mathematics to students, which makes it possible to combine key mathematical knowledge with calculations based on modern systems of symbolic computing and computer algebra. The implementation of the approach is described in the publicly available "Mathematica for a non-mathematician" textbook published under the auspices of the Moscow Center for Continuous Mathematical Education. The ideas of the approach have given the authors the victory in the competition of innovative educational projects of the Government of St. Petersburg. They formed the basis for a new project aimed at developing a domestic system of computer mathematics for science and education.

Background

2023 marks 20 years since the Department of Information Systems in Economics of St. Petersburg State University was established. For a number of years, it has been an administering department in the 'Applied Informatics in Economics' specialty. Starting from 2011, the department has been providing training of bachelors and masters in 'Business Informatics'. Graduating students of the department obtain the 'computational economist' qualification and have fundamental skills both in the field of computer science and mathematics and in the field of economics.

The mathematical education of business informaticians is an important component of the education plan. For a number of years of existence of the specialist program at the department, the 'Mathematics and computer' training course was taught to the students, where the key mathematical ideas were discussed by using

the modern systems of symbolic computing and computer algebra. For those years, a number of textbooks have been published [3-11] which have accumulated the approaches developed by the authors and numerous problems the solving of which shows the breakthrough opportunities of computer mathematics both in computation and in visualization of the obtained results, compared with the traditional methods of teaching mathematical science.

As an example, Figure 1 represents building of the plot of function $\cos(x^2-y^2)$ in the system Mathematica.

```
Plot3D[Cos[x^2 - y^2], {x, -Pi, Pi}, {y, -Pi, Pi},
 MeshStyle -> AbsoluteThickness[1],
 BoxStyle -> AbsoluteThickness[0.8],
 PlotPoints -> 60]
```

FIGURE 1. Building a graph by commands of Mathematica

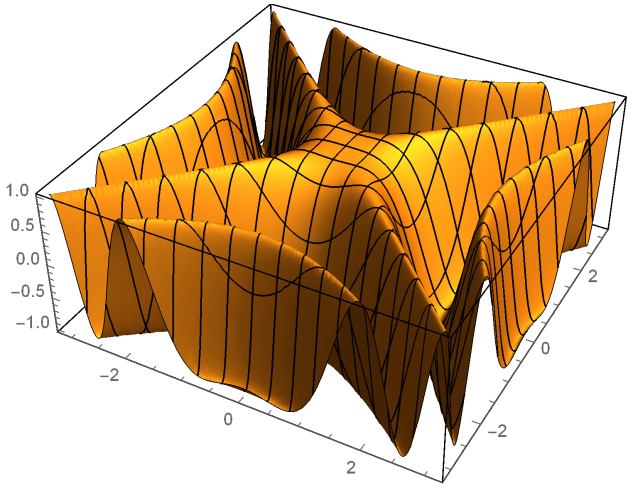

FIGURE 2. Command execution result

Regretfully, upon transition to the two-level system of higher education, there was no room for the "Mathematics and computer" course in the bachelor's degree program in 'Business Informatics'. However, the work performed for the specialist program in 2005-2011 has not fallen into oblivion[1] , and in 2018 the authors'

[1]Complete list of our books of that time is available at `https://web.archive.org/web/20160917081245/http://www.spbu-bi.ru/ru/science/publications.html`

application for the textbook 'Mathematics for non-mathematicians' was awarded a grant by the Vladimir Potanin Charitable Foundation which implements large-scale projects in the sphere of education and culture. Thanks to such support, the textbook [1] was created which manifests the opinion shared by the co-authors that mathematics may not be taught by drilling in routine operations which will never be applied by many students in their future life. The modern mathematical packages in skilled hands will solve equations and perform computations in a better way. The textbook has recalled to life the work of the early 2000s when the authors delivered lectures in 'Mathematics and computers' at the Department of Economics of St. Petersburg State University. The application prepared on the basis of the textbook received the Reward of St. Petersburg Government for winning the competition among innovative projects in the sphere of science and education (2021)[2].

The authors are sure that the developed textbook makes it possible for students of non-mathematical disciplines to get an insight into the opportunities provided by professional mathematical investigation tools which are a real alternative to the wide-spread office software. If published, the textbook will supplement the teaching-learning base of courses of quantitative methods for processing of economic information.

1. Key ideas of the textbook

- By using computer algebra systems, even today it is possible to conduct all computations which are standard for mathematics and its applications. All implications of this fact not only have not been recognized but even have not started to be considered seriously.
- The main general-purpose computer algebra systems are first of all programming languages of a very high level, near-living languages in their expressive power, and they should be learned exactly as languages but not as standard computer applications.
- Mathematicians are prone to underestimate the extent of dependence of the development of mathematics on the environment, firstly on computing facilities available. Even today the development of computer algebra has a dramatic impact on investigations in a lot of spheres of pure mathematics such as theory of groups, combinatorics, theory of numbers, commutative algebra, algebraic geometry, etc. In the near future, this influence will cover mathematics in its entirety and result in fundamental revision of the main areas of research, reappraisal of values, and a complete change in the mathematicians' work style.
- The furious antagonism caused among methodologists and many teachers of mathematics by the development of computer algebra is due to the fact that even in the nearest 10-15 years further development of such systems

[2] https://math-cs.spbu.ru/news/news-12-10-2021/

will result in complete devaluation of all traditional computing skills and the necessity of complete revision of teaching mathematics at the secondary and higher school levels.

- The furious antagonism caused among many representatives of Computer Science by the development of computer algebra is due to the fact that those systems also completely devaluate substantially all traditional programming skills. By using such systems, any informed amateur can write a program in a few minutes, while a similar program written in the algorithmic language Fortran or the currently popular dialects C and Python would require serious efforts of a professional programmer.

2. Prospects

The prospects for the development of mathematical education on the basis of use of systems of symbolic computing and computer algebra are described in the report 'The skies are falling [EVER MORE RAPIDLY]: mathematics for non-mathematicians' made at the joint seminar with Moscow State University 'Mathematics and informatics at secondary and higher school' held on March 9, 2023 (under the leadership of Academician A.L. Semenov and others). Presentation of the report and the video record of the workshop are available through the link. The recent events, including the unjustified sanctions of western vendors against the Russian Federation in relation to information and software products for science and education, have put a challenging problem of strategic security of domestic scientific research. The authors think that the creation of a competitive system of symbolic computing and computer algebra is real. We announced that by sending the project 'Computer mathematics: concepts of architectural, language and algorithmic support of computer algebra and quantum informatics systems' to the RSF competition of fundamental studies. The purpose of the project is to develop and implement new concepts of Computer Mathematics, namely: the basics of architectural, language and algorithmic support of computer algebra and quantum informatics systems for interface matching with languages, research style, and presentation of results, peculiar for modern mathematics. To achieve the purpose, it is planned to conduct large-scale comprehensive research in a number of inter-related areas, in particular:

- in the field of methodology: develop modern algorithms of symbolic computing and solve the critical difficulties of mathematical interpretation of their results, develop and practically promote the original methodology of using specialized mathematical packages for all levels of mathematical education in Russia, above all in training of researchers, mathematicians, and non-mathematicians;
- in the field of software: develop an image of a competitive mathematical package which would have the merits of modern foreign systems of computer

mathematics and implement the authors' experience of using such software tools;

- in the field of educational content: review, develop and design an educational content aimed at teaching substantiated and efficient application of modern computer facilities to various user categories;
- in the field of qualimetry and metrology: develop mathematical methods of valuation and forecasting of the statuses of Russian higher education institutions in focused ratings in mathematics and computer sciences in order to prepare and substantiate managerial decisions on improvement of their global competitiveness.

3. Scientific challenge to be solved by the project

Approximately 30 years ago, Doron Zeilberger stated that computers were becoming a thing of the same value for mathematics as telescopes and microscopes were for astronomy and biology in the XVII-th century. At the same time, speaking of the role of computers in mathematics, a lot of people confine themselves to the role of numerical calculations in applications on the one hand and to the formal derivation systems (automatic verification of theorems, verification of proofs, etc.) on the other. In those areas, especially in the first one, Russia has a fully developed school and major achievements. Meanwhile, computer mathematics is hardly limited to the above. In our opinion, in particular, systems of symbolic computing, especially computer algebra systems, will become far more significant both for mathematics itself and for its applications in the near future. In particular, for the recent years it has become clear that for a lot of real industrial projects not the applied mathematics and numerical methods are strongly sought for, but different branches of fundamental mathematics and advanced computer technology. Strong research groups are working in those areas, especially in Dubna, Moscow, and St. Petersburg; they have a vast experience of creating specialized packages focused on the performance of special types of computations for specific applications, usually in mathematics, physics, and astronomy, and partly for engineering applications. However, the functions implemented in such packages do not cover any wide branches of mathematics, and the packages themselves cannot be used directly in mathematical education. On top of that, quite often Russian mathematicians do not trust in the capabilities of symbolic computing systems, it is customary to point to "errors of systems of computer algebra" which, in our opinion, are absolutely imaginary and which result, on the one hand, from the failure to understand the basic principles of computer calculations and on the other hand, from objective difficulties of interpreting their results in traditional mathematical terms. It should be honestly acknowledged that the Russian mathematics in this respect is perceptibly in arrears of the world level. As concerns not specialized packages but full-fledged general-purpose computer algebra systems (general-purpose CAS), there are just a few of them in the world. Of course, there

is a great number of extremely flexible and powerful specialized systems such as GAP, Magma, CoCoA, Singular, Pari, Lie and others, specially created for computations in specific fields such as theory of numbers, theory of groups, theory of representations, commutative algebra, algebraic geometry, etc. On the other hand, there is a great number of elementary systems, including very interesting ones, which are used at elementary steps of teaching mathematics, at the level of junior and secondary school. What practically does not exist is the intermediary step - systems of computer algebra which would cover a wide range of different branches of mathematics on an average-high level. If we set aside experimental, rudimentary and knowingly obsolete CAS, currently there are quite a few such modern systems, actually four: Maple, Mathematica, Axiom, and SageMath. This being said, Axiom, after the death of its author Richard Janks, has not been maintained for a long time, and SageMath is actually not an independent system but a convenient front-end which provides access to a few tens of specialized systems for a qualified user. Two of them, Mathematica and Maple, are commercial systems. They are absolutely remarkable, great software products which, when created in 1980s, were an outstanding achievement in computer mathematics and de-facto became a standard for organization of such systems. On the other hand, certain critical decisions related to their general architecture, computing, data structures, etc. which were taken at that time could not be changed subsequently exactly due to the commercial nature of such systems and the necessity of securing back compatibility. In addition, alterations made in the last versions of those systems are more and more focused not on the aspects which are important from the point of view of mathematics itself, but on various purely marketing issues: different specific extra-mathematical applications, computer graphics, etc. As opposed to Axiom, the both systems has no simple and natural language features for describing mathematical structures in terms of axioms or properties. Some exclusively important mathematical constructions (symbolic polynomials, symbolic matrices, etc.) were included therein only post-factum, with algorithms which were not most efficient. For the last 30-40 years, however, a great progress has occurred in the understanding of principles of computer mathematics. Currently it has become conceptually and technically possible to create systems with a language which in its vocabulary and expressive power is far closer to the human mathematicians' language. Such a language shall make it possible to describe mathematical structures in the manner actually used in mathematical books (with somewhat stricter syntax). This would enable implementing of a front-end of such systems on any national language. In addition, more efficient computing algorithms and methods were proposed in many scenarios which make it possible to perform calculations faster and by using smaller resources. In particular, parallel algorithm have substantially been worked out which were not used in traditional CAS. For the recent decades, the difficulties of translating the results of symbolic computing into the language of traditional mathematics have been recognized a lot better and overcome to a significant extent. This gives us faith to the real possibility of creating an up-to-date Russian system of symbolic computing with a front-end in the Russian language. Such a

system could be vertically integrated and, on the one hand, available even for a schoolchild with respect to the requirements for the equipment and user qualification, and on the other hand, enabling quite sophisticated applications interesting for professional mathematics. It seems to us that none of the existing CAS, for all their undisputable advantages, satisfies such boundary conditions. The purpose of our project is to develop a theoretical framework of computer mathematics, to create a high-level Russian CAS with a full-fledged interface in the Russian and the English languages, to test the system on mathematical problems and to develop the basics for the use of the system in mathematical education. Apart from the merely scientific interest, the creation of such system would become the most important element of strategic security of scientific research and would be critical for mathematic education at very different levels. Such a system should preferably be an open-source system with clear separation of the kernel, the algorithm library supporting a variety of fields of modern pure and applied mathematics, with a developed data type system making it possible, on the language level, to build objects of new types by using the language structures most closely approximating the language of modern mathematics, as well as various interfaces allowing for modification of parts of the code by a qualified user. It is supposed to create a front-end software to ensure support of cloud computations, parallelization of algorithms, and interfaces for interaction with other computer algebra systems, e.g. Mathematica, Maple, Wolfram Alpha and others. We would keep in mind the availability of such a system for use at all levels of mathematical education in Russia and potentially in other countries, from secondary school to teaching professional mathematicians. The newest and the least technologically developed level, in a sense, would exactly be the medium level, i.e. teaching mathematics to non-mathematicians: both engineers, physicians, chemists, biologists and representatives of economic, social, and humanitarian disciplines. Historically, mathematics was extremely successful in a lot of applications, initially in astronomy and physics and then in other fields of natural science and engineering. Mathematics today could play the same role in all knowledge areas: biology, medicine, human science, social science, linguistics, cognitive science and others. If it is not so yet, this is only due to the fact that specialists in those fields are injured by the current modality of teaching mathematic starting from secondary school, do not know the mathematics they need, and which is worse, do not understand why they need such knowledge.

It is clear that the creation of a convenient and available system of computer algebra which the technical aspect of the matter could be delegated to, while concentrating on the conceptual aspect, could substantially resolve the problem. However, on the main obstacles is the lack of experience exactly of creation, debugging, and testing of large system of that type. Russian programmers have a great experience and top achievements in the field of writing short programs, competitive programming, etc. It seems that it is time to start creating a full-fledged high-level Russian CAS. The circumstances necessitate such a development.

We would like to note another important aspect of our project. There are still no functional quantum computers, in spite of numerous declarations made on this subject. Nevertheless, we are convinced that even today it is necessary to develop quantum algorithms of computer algebra and teach specialists in this area. We would note that the symbolic computing speed-up program proposed here is absolutely knew for this sphere, as only problems of numerical calculation speed-up were usually discussed therein. We would say that the existence of fast quantum algorithm makes it possible to consider post quantum computer algebra as a separate relevant field of research.

References

[1] Vavilov N.A., Khalin V.G., Yurkov A.V. *Mathematica for non-mathematics: a textbook for universities.* Electronic edition Moscow: MCCME, 2021. 483 pp. ISBN 978-5-4439-3584-3. Available at `https://www.mccme.ru/free-books/mathematica.pdf`

[2] Khalin, V.; Vavilov, N.; Yurkov, A. *The skies are falling: Mathematics for non-mathematicians*, Polynomial Computer Algebra'2022, May 02-07, 2022. Euler International Mathematical Institute, St. Petersburg, RUSSIA p. 121-149. Available at `https://pca-pdmi.ru/2022/pca2022_book.pdf`

[3] Vavilov N.A., Khalin V.G. *Mathematica 5.* dlja nematematika. Vypusk 1-2, Uchebnoe posobie* [Mathematica 5.* for non-mathematics, Issue 1-2, Manual]. Saint Petersburg, 2005. 317 p.

[4] Vavilov N.A., Ivanov O.A., Lushnikova G.A., Khalin V.G. *Uroki matematiki pri pomoshhi Mathematica* [Mathematics lessons with Mathematica]. Saint Petersburg, 2008. 146 p.

[5] Volkov V.A., Khalin V.G., Chernyaev P.K. *Uchebnye i kontrol'nye zadanija po matematike. Matematicheskij analiz. Uchebnoe posobie. 3 izd. ispr. i dop.* [Educational and control tasks in mathematics. Mathematical analysis. Tutorial. 3rd ed. corrected and additional]. Saint Petersburg: SPbSU Faculty of Economics, 2010. 112 p.

[6] Vavilov N.A., Khalin V.G. *Dopolnitel'nye zadachi po kursu «Matematika i komp'juter». Uchebnoe izdanie* [Additional tasks for the course "Mathematics and Computer". Tutorial]. Saint Petersburg, 2008. 172 p.

[7] Vavilov N.A., Khalin V.G. *Zadachi po kursu «Matematika i komp'juter», Vyp. 1. Arifmetika i teorija chisel. Uchebnoe izdanie"* [Tasks for the course "Mathematics and Computer", Vol. 1. Arithmetic and number theory. Tutorial]. Saint Petersburg, 2006. 180 p.

[8] Vavilov N.A., Khalin V.G. *Zadachi po kursu «Matematika i komp'juter», Vyp. 1. Arifmetika i teorija chisel. Uchebnoe izdanie* [Tasks for the course "Mathematics and Computer", Vol. 2. Algebra of polynomials. Tutorial]. Saint Petersburg, 2007. 207p.

[9] Vavilov N.A., Khalin V.G. *Zadachi po kursu «Matematika i komp'juter», Vyp. 1. Arifmetika i teorija chisel. Uchebnoe izdanie* [Tasks for the course "Mathematics and Computer", Vol. 3. Algebra of polynomials. Tutorial]. Saint Petersburg, 2008. 204p.

[10] Vavilov N.A., Semyonov A.A., Khalin V.G. *Zadachi po algebre. Linejnaja algebra* [Problems in algebra. Linear algebra]. Saint Petersburg, 2003. 52 p.

[11] Vavilov N.A. *Ne sovsem naivnaja linejnaja algebra. II. Algebra matriks* [Not exactly naive linear algebra. II. Matrix Algebra.] Saint Petersburg, 2006. 232 p.

[12] Internet links to the reports of the authors at scientific conferences, videos of training courses and the electronic edition of the textbook *Mathematica for non-mathematics*:

- `https://www.mccme.ru/free-books/mathematica.pdf`
- `https://elibrary.ru/item.asp?id=38582557`
- `http://cam19.ipo.spb.ru/cam19/Conferenceproceedings/go/p`
- `https://www.wolfram.com/broadcast/video.php?v=2685`
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Accelerating modular arithmetic with special choice of moduli

Eugene V. Zima

Abstract. Several methods of selection of moduli in modular arithmetic are considered [1] – [6]. With the proposed choice of moduli both modular reduction of an integer and reconstruction from modular images are accelerated. Special attention is paid to the moduli of the forms $2^n \pm 1$ and $2^n \pm 2^k \pm 1$. Different schemes of choice of these types of moduli and algorithms for conversion of arbitrary precision integers into the modular representation and back are considered. Results of experimental implementation of the described algorithms in the GMP system are discussed.

References

[1] J. Doliskani, P. Giorgi, R. Lebreton, E. Schost Simultaneous Conversions with the Residue Number System Using Linear Algebra. *ACM Transactions on Mathematical Software*, Volume 44, Issue 3, Article No.: 27, pp. 1–21, 2018.

[2] H. Garner. The Residue Number System. *IRE Transactions, EC-8*. pp.140–147, 1959.

[3] K. O. Geddes, S. R. Czapor, and G. Labahn. *Algorithms for Computer Algebra* (6^{th} printing). Boston: Kluwer Academic, 1992.

[4] D. E. Knuth. *The Art of Computer Programming*, Vol 2.

[5] M. Křížek, F. Luca, and L. Somer. *17 Lectures on Fermat Numbers: From Number Theory to Geometry*. New York: Springer-Verlag, 2001.

[6] Stewart A.M., Zima E.V. Base-2 Cunningham numbers in modular arithmetic. Program. Comput. Software, 2007, vol. 33, pp.80 -86.

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Sparse Triangular Decomposition Based on Chordal Graphs

Chenqi Mou

Abstract. For an arbitrary multivariate polynomial set, its sparsity with respect to the variables can be described by its associated graph [1]. Based on the theoretical results that top-down triangular decomposition for solving polynomial systems is proved to preserve the chordal structures of input polynomial sets [3, 6], sparse triangular decomposition is proposed to make use of the variable sparsity [4]. In this talk, I will first briefly present the underlying ideas of sparse triangular decomposition based on chordal graphs. Then I will show how to exploit the variable sparsity of biological dynamic systems in computing their equilibria by using sparse triangular decomposition [2, 5].

References

[1] Diego Cifuentes and Parrilo Pablo. Chordal networks of polynomial ideals. *SIAM Journal on Applied Algebra and Geometry*, 1(1):73–110, 2017.

[2] Wenwen Ju and Chenqi Mou. Exploiting variable sparsity in computing equilibria of biological dynamical systems by triangular decomposition. In *Proceedings of AlCoB 2021*, pages 29–41. Springer, 2021.

[3] Chenqi Mou and Yang Bai. On the chordality of polynomial sets in triangular decomposition in top-down style. In *Proceedings of ISSAC 2018*, pages 287–294. ACM Press, 2018.

[4] Chenqi Mou, Yang Bai, and Jiahua Lai. Chordal graphs in triangular decomposition in top-down style. *Journal of Symbolic Computation*, 102:108–131, 2021.

[5] Chenqi Mou and Wenwen Ju. Sparse triangular decomposition for computing equilibria of biological dynamic systems based on chordal graphs. *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, 2022.

[6] Chenqi Mou and Jiahua Lai. On the chordality of simple decomposition in top-down style. In *Proceedings of MACIS 2019*, pages 138–152. Springer, 2020.

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