On parametrization of orthogonal symplectic matrices and its applications

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Talk outlook

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Abstract

The method of computing the parametric representation of an orthogonal symplectic matrix is considered. The dimension of the family of such matrices is calculated. The general structure of matrices of small even dimensions up to 8 is discussed in detail. A conjecture on the structure of a skew symmetric matrix generating a generic orthogonal symplectic transformation is formulated. The problem of constructing an orthogonal symplectic matrix of dimension 4 by a given vector is solved. The application of this transformation to the study of families of periodic solutions of an autonomous Hamiltonian system with two degrees of freedom is discussed.

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Intro (1)

While studying a phase flow of a non-integrable Hamiltonian system, it is usually assumed that there is some information about its invariant varieties: stationary points, periodic solutions or invariant tori of different dimensions. In this case, one can compute the normal form of the system near the corresponding variety and use it to obtain information on the stability of this variety, local integrability in its vicinity, the nature of bifurcations at small changes of parameters and, under certain conditions, asymptotically integrate the normalized system of equations.

For studying dynamics near invariant varieties of dimension greater than zero, the normalization technique is less well developed. Usually we need a special coordinate transformation to simplify studying of the phase flow.

Intro (2)

Autonomous Hamiltonian systems are characterized by the existence of “natural families” of periodic orbits that are parameterized by the value of the integral of energy. By obtaining information on at least one periodic solution of the family using the predictor-corrector method, it is possible to continue along the family.

Successful continuation along the family requires the computation of normal and tangent displacements. Previously, such displacements were computed by reducing the system to Birkhoff normal form, which implied additional computational cost. Then variants of the method appeared, when at each step of integration along the periodic solution an orthogonal-symplectic transformation was performed [Karimov, Sokol’skii, 1990], or integration was performed in the Fresné basis [Lara, Peláez, 2002]. Later, Kreisman [Kreisman, 2005] showed that it is sufficient apply such a transformation only to the monodromy matrix of the periodic solution.

Intro (3)

In presented talk we provide a general algorithm of computation of an generic ***orthogonal symplectic matrix*** (or simply ***OSM***) of any even dimension and gives more precise description of their structure dimension equals to 4.

All computations in this article are performed using `Maple 2023` and `Wolfram Mathematica` 13.3.

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General form of OSM (1)

Hereafter, any boldfaced capital symbol with sub index, e.g. $\mathbf{B}_n$ denotes a square real matrix of dimension $2n \times 2n$; $\mathbf{E}_n$ and $\mathbf{E}$ are unit matrices of dimension $n \times n$ and $2n \times 2n$, respectively. The sign $^\top$ denotes the transpose operation of a matrix or vector.

Statement 1.

A skew symmetric $2n \times 2n$ matrix $\mathbf{K}_n$ does not have non-zero real eigenvalues.

It follows that for any skew symmetric matrix $\mathbf{K}_n$ the matrices $\mathbf{E} \pm \mathbf{K}_n$ are nondegenerate and invertible.

General form of OSM (2)

According to Cayley's formula [Gantmacher, 2004, Chap. IX, S 14, n. 2] matrix $\mathbf{K}_n$ defines an orthogonal matrix $\mathbf{A}_n$:

$$\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)(\mathbf{E} - \mathbf{K}_n)^{-1} = 2(\mathbf{E} - \mathbf{K}_n)^{-1} - \mathbf{E}, \tag{1}$$

and $\mathbf{A}_n^\top \mathbf{A}_n = \mathbf{A}_n \mathbf{A}_n^\top = \mathbf{E}$.

The following theorem allows to parametrize an OSM of general form, i.e., a matrix $\mathbf{A}_n$ satisfying the condition of orthogonality and symplecticity simultaneously

$$\mathbf{A}_n^\top \mathbf{A}_n = \mathbf{E}, \quad \mathbf{A}_n^\top \mathbf{J} \mathbf{A}_n = \mathbf{J}.$$

General form of OSM (3)

Theorem 1 ([Petrov, 2020]).

A matrix $\mathbf{A}_n$ is symplectic if and only if the matrix $\mathbf{\Psi}_n$

$$\mathbf{\Psi}_n = -2\mathbf{J}(\mathbf{E} + \mathbf{A}_n)^{-1}(\mathbf{A}_n - \mathbf{E}), \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{E}_n \\ -\mathbf{E}_n & \mathbf{0} \end{pmatrix} \tag{2}$$

is symmetric.

In fact, Theorem 1 allows us to constructively build a matrix $\mathbf{A}_n$ that is both orthogonal and symplectic. Such a class of matrices turns out to be very useful in the study of families of periodic solutions of Hamiltonian systems, and for critical solutions it allows to determine the type of bifurcation of the family (for details see [Batkhin, 2020; Kreisman, 2005]).

General form of OSM (4)

The computational scheme according to Theorem 1 can be organized as follows.

1. Define an arbitrary skew symmetric matrix $\mathbf{K}_n$ of size $2n\times 2n$, uniquely defined by $n(2n-1)$ elements.
2. Compute the orthogonal matrix $\mathbf{A}_n$ by Formula (1), which is always possible by virtue of Statement 1.
3. By Formula (2) we get the expression of the matrix $\mathbf{\Psi}_n$ through the matrix $\mathbf{K}_n$.
4. Using the symmetry condition $\mathbf{\Psi}_n^\top = \mathbf{\Psi}_n$, we obtain a system of relations between the elements of the matrix $\mathbf{K}_n$.
5. According to Theorem 1 the matrix $\mathbf{A}_n$ is symplectic.

General form of OSM (5)

Theorem 2.

The number of independent elements of the matrix $\mathbf{K}_n$, which defines by Cayley's formula (1) *the symplectic matrix $\mathbf{A}_n$, is equal to n^2. The number of relations between the elements of the matrix $\mathbf{K}_n$ is $n(n-1)$.*

Proof.

Since the group $Sp(2n,\mathbb{R}) \cap O(2n,\mathbb{R}) \cong U(n)$ (see [Kostrikin, Manin, 1989, Ch. 2, § 13] or [Fomenko, 1995, Ch. 1, § 2.4, Sect. 2]), then the matrix $\mathbf{A}_n$ is defined by n^2 independent elements of the matrix $\mathbf{K}_n$ with $n(2n-1)$ independent elements: $n^2 = n(2n-1) - n(n-1)$. $\square$

General form of OSM (6)

Remark 1.

In [Karimov, Sokol'skii, 1990] it is noted that the problem of building a matrix $\mathbf{A}_n$ is equivalent to the problem of building a nondegenerate continuous tangent vector field on the sphere S^{n-1}. As known from Theorem of hedgehog [Arnold, 1992, Chap. 5, Sect. 34, § 3], for spheres of dimensions 1, 3 and 7 (i.e. for $n = 2,4,8$) it can always be done, and for spheres of other dimensions only by puncturing at least one point on it. The latter means that such a transformation should have a singularity.

OSM for $n = 1$

It follows from Theorem 2 that any skew symmetric 2×2 matrix $\mathbf{K}_1 = \begin{pmatrix} 0 & k_1 \\ -k_1 & 0 \end{pmatrix}$ defines a symplectic matrix

$$\mathbf{A}_1 = \begin{pmatrix} \dfrac{1-k_1^2}{1+k_1^2} & \dfrac{2k_1}{1+k_1^2} \\ -\dfrac{2k_1}{1+k_1^2} & \dfrac{1-k_1^2}{1+k_1^2} \end{pmatrix} = \mathrm{Rot}(\alpha), \text{ where } \mathrm{Rot}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

when $k_1 = -\tan(\alpha/2)$.

It is a well known fact that any orthogonal 2×2 matrix is simultaneously symplectic, i.e., according to Theorem 2 there are no additional relations between the elements of the matrix $\mathbf{K}_1$.

OSM for case $n = 2$ (1)

Let a skew-symmetric matrix $\mathbf{K}_2$ be of the form

$$\mathbf{K}_2 = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & k_4 & k_5 \\ -k_2 & -k_4 & 0 & k_6 \\ -k_3 & -k_5 & -k_6 & 0 \end{pmatrix}.$$

Carrying out the computation of items 1–3, we obtain according to Theorem 2 that for the matrix $\mathbf{A}_2$ to be symplectic, two conditions should be fulfilled:

$$k_4 = k_3, \quad k_6 = k_1.$$

OSM for case $n = 2$ (2)

Hence, we obtain a matrix $\mathbf{K}_2$ of the form

$$\mathbf{K}_2 = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & k_3 & k_5 \\ -k_2 & -k_3 & 0 & k_1 \\ -k_3 & -k_5 & -k_1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{pmatrix}.$$

Here $\mathbf{B} = -\mathbf{B}^\top$ and $\mathbf{C} = \mathbf{C}^\top$.

According to Theorem 2 there is a four-parameter family of OSMs $\mathbf{A}_2$. Their structure can be described as follows.

Structure of OSM for $n = 2$ (1)

Let us denote by $\mathbf{A}_2^{(j)}$, $j = 1,\ldots,4$, the jth column of the matrix $\mathbf{A}_2$. Then the following conditions hold.

1. Each column of $\mathbf{A}_2^{(j)}$ is a unit vector.
2. All columns are pairwise orthogonal.
3. $\mathbf{A}_2^{(j)} = \mathbf{J}\mathbf{A}_2^{(j+2)}$ or $\mathbf{A}_2^{(j+2)} = -\mathbf{J}\mathbf{A}_2^{(j)}$ for $j = 1,2$.
4. Let us put the following notations:

$$Q \stackrel{\text{def}}{=} k_1^2 - k_2 k_5 + k_3^2, R^2 \stackrel{\text{def}}{=} 2k_1^2 + k_2^2 + 2k_3^2 + k_5^2,$$

Structure of OSM for $n = 2$ (2)

then the first two columns are

$$\mathbf{A}_2^{(1)} = \frac{1}{d^2}\begin{pmatrix} 1 + k_5^2 - k_2^2 - Q^2 \\ -2k_1(1+Q) - 2k_3(k_5+k_2) \\ -2k_2 + 2k_5 Q \\ -2k_3(1+Q) + 2k_1(k_5+k_2) \end{pmatrix}, \quad \mathbf{A}_2^{(2)} = \frac{1}{d^2}\begin{pmatrix} 2k_1(1+Q) - 2k_3(k_5+k_2) \\ 1 - k_5^2 + k_2^2 - Q^2 \\ -2k_3(1+Q) - 2k_1(k_5+k_2) \\ -2k_5 + 2k_2 Q \end{pmatrix} \tag{3}$$

where $d^2 = (k_2 + k_5)^2 + (1 + Q)^2 = R^2 + Q^2 + 1$. Here $d^2 = \det(\mathbf{E} - \mathbf{K}_2)$.

5. Columns $\mathbf{A}_2^{(3)}$ and $\mathbf{A}_2^{(4)}$ are obtained according to the property 3.

Since the denominator d^2 never goes to zero, the matrix $\mathbf{A}_2$ is nondegenerate, which agrees with the above reasoning in Remark 1.

Representation OSM as isoclinic rotations (1)

Since the matrix $\mathbf{A}_2 \in SO(4)$, it defines a rotation in $\mathbb{R}^4$. A special case of double rotations is ***isoclinic rotations*** with the same rotation angle $\alpha_1 = \pm\alpha_2$. These rotations can be left-isoclinic, when $\alpha_1 = \alpha_2$, or right-isoclinic, when $\alpha_1 = -\alpha_2$.

Isoclinic rotation properties

a) a composition of two right (left) isoclinic rotations is a right (left) isoclinic rotation;

b) the composition of right and left isoclinic rotations is commutative.

c) any 4-dimensional rotation can be decomposed into a composition of right and left isoclinic rotations.

Representation OSM as isoclinic rotations (2)

The characteristic polynomial $\chi_{\mathbf{A}_2}(\lambda)$ of the matrix $\mathbf{A}_2$ is a reciprocal polynomial due to its symplectic nature

$$\chi_{\mathbf{A}_2}(\lambda) = \lambda^4 + \frac{4(Q^2-1)}{d^2}\lambda^3 + \frac{2(3Q^2-R^2+3)}{d^2}\lambda^2 + \frac{4(Q^2-1)}{d^2}\lambda + 1$$

and it is factorized by two quadratic trinomials due to the substitution $z = (\lambda + 1/\lambda)/2$:

$$\chi_{\mathbf{A}_2}(\lambda) = \left(\lambda^2 - 2z_1\lambda + 1\right)\left(\lambda^2 - 2z_2\lambda + 1\right),$$

where $z_{1,2}$ are the roots of the quadratic equation.

$$z^2 + \frac{2(Q^2-1)}{d^2}z + \frac{Q^2-R^2+1}{d^2} = 0. \qquad (4)$$

Representation OSM as isoclinic rotations (3)

The discriminant D of the polynomial Equation (4) is non-negative and takes the value 0 only under the condition $k_5 = -k_2$:

$$D = 4\left(R^4 - 4Q^2\right) = 4(k_2 + k_5)^2 \left(4k_1^2 + 4k_3^2 + (k_2 + k_5)^2\right) \geqslant 0.$$

Since the absolute value of the eigenvalue of the orthogonal matrix is equal to 1, the roots $z_{1,2}$ of the equation (4) should belong to the interval $[-1; +1]$. Then $\lambda_j \in S^1$, and, for real matrix $\mathbf{A}_2$ they form complex-conjugate pairs $\lambda_j = \bar{\lambda}_{j+2} = \mathrm{e}^{i\alpha_j}$, $\alpha_j \in [0; 2\pi)$, $j = 1{,}2$. In generic case $\alpha_1 \neq \alpha_2$ and the matrix $\mathbf{A}_2$ can be reduced to the form

$$\mathbf{A}_2 = \begin{pmatrix} \mathrm{Rot}(\alpha_1) & 0 \\ 0 & \mathrm{Rot}(\alpha_2) \end{pmatrix}.$$

Representation OSM as isoclinic rotations (4)

An arbitrary matrix $\mathbf{A}_2 \in SO(4)$ can be represented as a composition of left-isoclinic and right-isoclinic rotations

$$\mathbf{A}_2 = \begin{pmatrix} a & -b & -c & -f \\ b & a & -f & c \\ c & f & a & -b \\ f & -c & b & a \end{pmatrix} \cdot \begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix},$$

where $a^2 + b^2 + c^2 + f^2 = p^2 + q^2 + r^2 + s^2 = 1$, by so-called *Cayley decomposition* [Thomas, 2014].

Representation OSM as isoclinic rotations (5)

According to this algorithm, the matrix $\mathbf{A}_2$ can be represented as $\mathbf{A}_2 = \mathbf{A}_L \cdot \mathbf{A}_R$, where the matrices $\mathbf{A}_L$ and $\mathbf{A}_R$ are the following

$$\mathbf{A}_L = \frac{1}{d}\begin{pmatrix} 1-Q & 2k_1 & k_2-k_5 & 2k_3 \\ -2k_1 & 1-Q & 2k_3 & -k_2+k_5 \\ -k_2+k_5 & -2k_3 & 1-Q & 2k_1 \\ -2k_3 & k_2-k_5 & -2k_1 & 1-Q \end{pmatrix},$$

$$\mathbf{A}_R = \frac{1}{d}\begin{pmatrix} Q+1 & 0 & k_2+k_5 & 0 \\ 0 & Q+1 & 0 & k_2+k_5 \\ -k_2-k_5 & 0 & Q+1 & 0 \\ 0 & -k_2-k_5 & 0 & Q+1 \end{pmatrix}.$$

General structure of an arbitrary OSM (1)

Generalizing the computations performed for cases $n = 1,2,3,4$, we can formulate the following

Conjecture 1.

If a skew symmetric matrix $\mathbf{K}_n$ has the form

$$\mathbf{K}_n = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C}^\top & \mathbf{D} \end{pmatrix}, \quad \mathbf{B}^\top = -\mathbf{B}, \quad \mathbf{D}^\top = -\mathbf{D},$$

then the matrix $\mathbf{\Psi}_n = -2\mathbf{J}(\mathbf{E}+\mathbf{A}_n)^{-1}(\mathbf{E}-\mathbf{A}_n)$ has the following form $\mathbf{\Psi}_n = -2\begin{pmatrix} \mathbf{C}^\top & \mathbf{D}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$, where $\mathbf{A}_n = (\mathbf{E}+\mathbf{K}_n)(\mathbf{E}-\mathbf{K}_n)^{-1}$.

General structure of an arbitrary OSM (2)

It follows from Conjecture 1 that in order for the matrix $\mathbf{\Psi}_n$ to be symmetric, the conditions $\mathbf{C} = \mathbf{C}^\top$ and $\mathbf{B} = \mathbf{D}$ should be satisfied. Thus, according to Theorem 1, the following statement is obtained:

Statement 2.

*If a skew-symmetric matrix $\mathbf{K}_n$ is a block matrix $\mathbf{K}_n = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{pmatrix}$, with $\mathbf{B}$ is a skew symmetric $n \times n$ matrix of $n(n-1)/2$ independent elements and $\mathbf{C}$ is a symmetric $n \times n$ matrix of $n(n+1)/2$ independent elements, then the matrix $\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)(\mathbf{E} - \mathbf{K}_n)^{-1}$ is a generic **orthogonal symplectic matrix** of n^2 independent elements.*

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Some application of parametric representation (1)

In applications, it is usually required to use the computed symplectic matrix $\mathbf{M}_n$ of some periodic solution to a Hamiltonian system to construct an OSM matrix $\mathbf{A}_n$ that simplifies $\mathbf{M}_n$.

Problem 1.

Let a monodromy matrix $\mathbf{M}_2$ of a periodic solution $\mathbf{z}(t, \mathbf{z}^0)$ with period T of an autonomous Hamiltonian system with two degrees of freedom be known. Find such a transformation with an OSM $\tilde{\mathbf{A}}_2$ that reduces the matrix $\mathbf{M}_2$ to a simpler form.

For autonomous system the column $\tilde{\mathbf{A}}_2^{(1)}$ is the normalized phase velocity vector $\mathbf{v}^0$. To complete the construction of the desired matrix, we need to be able to express the elements of the second column $\tilde{\mathbf{A}}_2^{(2)}$ through the elements of the first column.

Some application of parametric representation (2)

Steps of solution

1. The numerators of the elements of the vector $\tilde{\mathbf{A}}_2^{(1)}$ are used to compose an ideal and its Gröbner basis $\mathfrak{J}$ with some monomial order.
2. The reminders of the numerators of the second column of $\tilde{\mathbf{A}}_2^{(2)}$ modulo ideal $\mathfrak{J}$ are computed and they form a new polynomial system.
3. Solving the system obtained at the previous step one get additional conditions on the parameters k_1, k_2, k_3, k_5, which guarantee that the column $\tilde{\mathbf{A}}_2^{(2)}$ can be expressed through the components of the column $\tilde{\mathbf{A}}_2^{(1)}$.

The computations performed using the above scheme lead to two sets of conditions. **Condition 1** leads to a 3-parameter family of OSMs, and **Condition 2** leads to a one-parameter family of OSMs.

Some application of parametric representation (3)

Condition 1 is $k_5 = -k_2$, and the other variables are free. Then the values of Q and R take the values $\tilde{Q} = k_1^2 + k_2^2 + k_3^2$, $\tilde{R}^2 = 2\tilde{Q}$, and Formulas (3) for the first two columns of the matrix $\tilde{\mathbf{A}}_2$ take the form of

$$\tilde{\mathbf{A}}_2^{(1)} = \frac{1}{\tilde{d}} \begin{pmatrix} 1 - \tilde{Q} \\ -2k_1 \\ -2k_2 \\ -2k_3 \end{pmatrix}, \quad \tilde{\mathbf{A}}_2^{(2)} = \frac{1}{\tilde{d}} \begin{pmatrix} 2k_1 \\ 1 - \tilde{Q} \\ -2k_3 \\ 2k_2 \end{pmatrix}, \quad \tilde{d} = 1 + \tilde{Q}.$$

Some application of parametric representation (4)

Earlier, the matrix $\tilde{\mathbf{A}}_2$ was proposed by Kreisman in a series of papers devoted to orbit design for the "Radioastron" project:

$$\tilde{\mathbf{A}}_2 = \begin{pmatrix} H_3 & -H_4 & H_1 & H_2 \\ H_4 & H_3 & H_2 & H_1 \\ -H_1 & -H_2 & H_3 & -H_4 \\ -H_2 & H_1 & H_4 & H_3 \end{pmatrix},$$

where $\tilde{\mathbf{A}}_2^{(3)} = (H_1, H_2, H_3, H_4)$ is the normalized vector $\operatorname{grad} H(\mathbf{z}_0)$.

Solution to continuation equation (1)

Let us show how the transformation with the matrix $\tilde{\mathbf{A}}_2$ simplifies the monodromy matrix $\mathbf{M}_2$ of a periodic solution.

Let some nondegenerate periodic solution $\mathbf{z}(t, \mathbf{z}^0)$ of a family with initial condition $\mathbf{z}^0$ and period T be known. If the parameters of the periodic solution vary smoothly along the family, in a generic case, there is a periodic solution $\mathbf{z}(t) + \delta\mathbf{z}(t)$ with period $T + \delta T$ near the generic case.

$$(\mathbf{M}_2 - \mathbf{E})\delta\mathbf{z}(T) + \mathbf{v}^0\delta T = 0, \tag{5}$$

where $\mathbf{v}^0 = \mathbf{J}\,\mathrm{grad}\,H(\mathbf{z}(T, \mathbf{z}^0))$.

Solution to continuation equation (2)

The transformation $\mathbf{M}_2 \to \tilde{\mathbf{A}}_2^\top \mathbf{M} \tilde{\mathbf{A}}_2$ reduces [Kreisman, 2005] the matrix $\mathbf{M}$ to a symplectic matrix $\mathbf{N}_2$ of the form

$$\mathbf{N}_2 = \begin{pmatrix} 1 & n_{12} & n_{13} & n_{14} \\ 0 & n_{22} & n_{23} & n_{24} \\ 0 & 0 & 1 & 0 \\ 0 & n_{42} & n_{43} & n_{44} \end{pmatrix}.$$

The stability index S of the periodic solution is $S = (n_{22} + n_{44})/2$.

Solution to continuation equation (3)

Substitution $\delta\zeta = \tilde{\mathbf{A}}_2 \delta\mathbf{z}$ reduces System (5) into

$$(\mathbf{N}_2 - \mathbf{E})\delta\boldsymbol{\zeta}(T) + v^0 \delta T \tilde{\mathbf{A}}_2^{(1)} = 0,$$

which has a general solution

$$\delta\boldsymbol{\zeta} = c_1 \tilde{\mathbf{A}}_2^{(1)} + \delta\boldsymbol{\zeta}', \quad \delta T = -\frac{1}{v} \sum_{j=2}^{4} n_{1j} \delta\zeta_j',$$

and the vector $\delta\zeta'$ is orthogonal to the vector $\tilde{\mathbf{A}}_2^{(1)}$.

So $\delta\zeta'$ specifies the displacement along the family of periodic orbits and $\tilde{\mathbf{A}}_2^{(1)}$ specifies the displacement along the periodic solution.

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Thanks for your attention!