On parametrization of orthogonal symplectic matrices and its applications

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Abstract. The method of computing the parametric representation of an orthogonal symplectic matrix is considered. The dimension of the family of such matrices is calculated. The general structure of matrices of small even dimensions up to 8 is discussed in detail. A conjecture on the structure of a skew symmetric matrix generating a generic orthogonal symplectic transformation is formulated. The problem of constructing an orthogonal symplectic matrix of dimension 4 by a given vector is solved. The application of this transformation to the study of families of periodic solutions of an autonomous Hamiltonian system with two degrees of freedom is discussed.

1. Introduction

While studying the phase flow of a non-integrable Hamiltonian system, it is usually assumed that there is some information about its invariant varieties: equilibrium positions, periodic solutions or invariant tori of different dimensions. In this case, one can compute the normal form of the system near the corresponding variety and use it to obtain information on the stability of this variety, local integrability in its vicinity, the nature of bifurcations at small changes of parameters and, under certain conditions, asymptotically integrate the normalized system of equations.

For studying dynamics near invariant varieties of dimension greater than zero, the normalization technique is less well developed. Here, either a Poincaré mapping reduces a continuous-time Hamiltonian system to a mapping that preserves the phase volume, or a special coordinate transformation is performed to simplify the study of phase flow. Successful continuation along the family requires the computation of normal and tangent displacements. Previously, such displacements were computed by reducing the system to Birkhoff normal form, which implied additional computational cost. Then variants of the method appeared, when at each

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step of integration along the periodic solution an orthogonal-symplectic transformation was performed [1, 2], or integration was performed in the Fresné basis [3]. Later, in a series of papers by Kreisman [4, 5], it was shown that it is sufficient to do such a transformation once after the monodromy matrix of the periodic solution has been computed. In presented paper we provide a general algorithm of computation of an generic *orthogonal symplectic matrix* (or simply OSM) of any even dimension and gives more precise description of their structure dimension equals to 4.

2. General form of orthogonal-symplectic matrix

Hereafter, $\mathbf{B}_n$ denotes a square real matrix of dimension $2n \times 2n$; $\mathbf{E}_n$ and $\mathbf{E}$ are unit matrices of dimension $n \times n$ and $2n \times 2n$, respectively. The sign $^\top$ denotes the transpose operation of a matrix or vector.

Let us state a theorem that allows to parametrize an OSM of a general form

$$\mathbf{A}_n^\top \mathbf{A}_n = \mathbf{E}, \quad \mathbf{A}_n^\top \mathbf{J} \mathbf{A}_n = \mathbf{J}.$$

Theorem 1 ([6]). *A matrix $\mathbf{A}_n$ is symplectic if and only if the matrix $\mathbf{\Psi}_n$*

$$\mathbf{\Psi}_n = -2\mathbf{J}(\mathbf{E} + \mathbf{A}_n)^{-1}(\mathbf{A}_n - \mathbf{E}), \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{E}_n \\ -\mathbf{E}_n & \mathbf{0} \end{pmatrix} \tag{1}$$

is symmetric.

In fact, Theorem 1 allows us to constructively build a matrix $\mathbf{A}_n$ that is both orthogonal and symplectic. Such a class of matrices turns out to be very useful in the study of families of periodic solutions of Hamiltonian systems, and for critical solutions it allows to determine the type of bifurcation of the family (for details see [5, 7]).

According to Theorem 1 the computations can organized as follows.

1. Define an arbitrary skew symmetric matrix $\mathbf{K}_n$ of size $2n \times 2n$, which is uniquely defined by $n(2n-1)$ elements.
2. Compute the orthogonal matrix $\mathbf{A}_n$ by the Cayley-like formula
$$\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)(\mathbf{E} - \mathbf{K}_n)^{-1}, \tag{2}$$
which is always possible due to the fact that a skew symmetric matrix $\mathbf{K}_n$ does not have non-zero real eigenvalues.
3. By (1) we get the expression of the matrix $\mathbf{\Psi}_n$ through the matrix $\mathbf{K}_n$.
4. Using the symmetry condition $\mathbf{\Psi}_n^\top = \mathbf{\Psi}_n$, we obtain a system of relations between the elements of the matrix $\mathbf{K}_n$.
5. According to Theorem 1 the matrix $\mathbf{A}_n$ is symplectic.

Theorem 2. *The number of independent elements of the matrix $\mathbf{K}_n$, which defines by Cayley's formula (2) the symplectic matrix $\mathbf{A}_n$, is equal to n^2. The number of relations between the elements of the matrix $\mathbf{K}_n$ is $n(n-1)$.*

3. Representation of OSMs for $n = 2$

Let the matrix $\mathbf{K}_2$ be of the form $\mathbf{K}_2 = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & k_4 & k_5 \\ -k_2 & -k_4 & 0 & k_6 \\ -k_3 & -k_5 & -k_6 & 0 \end{pmatrix}$. Carrying out the calculations of items 1–3, we obtain that for the matrix $\mathbf{A}_2$ to be symplectic, two conditions should be fulfilled:

$$k_4 = k_3, \quad k_6 = k_1. \tag{3}$$

Hence, we obtain a matrix $\mathbf{K}_2$ of the form

$$\mathbf{K}_2 = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & k_3 & k_5 \\ -k_2 & -k_3 & 0 & k_1 \\ -k_3 & -k_5 & -k_1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{pmatrix}. \tag{4}$$

Here $\mathbf{B}$ is a skew symmetric 2×2 matrix and $\mathbf{C}$ is a symmetric 2×2 matrix.

According to Theorem 2 there is a four-parameter family of OSMs $\mathbf{A}_2$. Their structure can be described as follows. Let us denote by $\mathbf{A}_2^{(j)}$, $j = 1, \ldots, 4$, the jth column of the matrix $\mathbf{A}_2$. Then the following conditions hold.

1. Each column of $\mathbf{A}_2^{(j)}$ is a unit vector.
2. All columns are pairwise orthogonal.
3. $\mathbf{A}_2^{(j)} = \mathbf{J}\mathbf{A}_2^{(j+2)}$ or $\mathbf{A}_2^{(j+2)} = -\mathbf{J}\mathbf{A}_2^{(j)}$ for $j = 1, 2$.
4. Let us put the following notations: $Q \overset{\text{def}}{=} k_1^2 - k_2 k_5 + k_3^2$, $R^2 \overset{\text{def}}{=} 2k_1^2 + k_2^2 + 2k_3^2 + k_5^2$, then the first two columns are

$$\begin{aligned} \mathbf{A}_2^{(1)} &= \frac{1}{d^2} \begin{pmatrix} 1 + k_5^2 - k_2^2 - Q^2 \\ -2k_1(1+Q) - 2k_3(k_5 + k_2) \\ -2k_2 + 2k_5 Q \\ -2k_3(1+Q) + 2k_1(k_5 + k_2) \end{pmatrix}, \\ \mathbf{A}_2^{(2)} &= \frac{1}{d^2} \begin{pmatrix} 2k_1(1+Q) - 2k_3(k_5 + k_2) \\ 1 - k_5^2 + k_2^2 - Q^2 \\ -2k_3(1+Q) - 2k_1(k_5 + k_2) \\ -2k_5 + 2k_2 Q \end{pmatrix} \end{aligned} \tag{5}$$

   where $d^2 = (k_2 + k_5)^2 + (1 + Q)^2 = R^2 + Q^2 + 1$. Here $d^2 = \det(\mathbf{E} - \mathbf{K}_2)$.
5. Columns $\mathbf{A}_2^{(3)}$ and $\mathbf{A}_2^{(4)}$ are obtained according to the property 3.

Consider the following particular problem.

Problem. *Let a monodromy matrix $\mathbf{M}_2$ of a periodic solution $\mathbf{z}(t, \mathbf{z}^0)$ with period T of an autonomous Hamiltonian system with two degrees of freedom be known. Find such an orthogonal-symplectic transformation given by the matrix $\tilde{\mathbf{A}}_2$ that reduces the matrix $\mathbf{M}_2$ to a simpler form.*

Applying Gröbner basis technique one can deduce from (5) that the first two columns of the matrix $\tilde{\mathbf{A}}_2$ take the form of

$$\tilde{\mathbf{A}}_2^{(1)} = \frac{1}{\tilde{d}}\begin{pmatrix}1-\tilde{Q}\\ -2k_1\\ -2k_2\\ -2k_3\end{pmatrix}, \quad \tilde{\mathbf{A}}_2^{(2)} = \frac{1}{\tilde{d}}\begin{pmatrix}2k_1\\ 1-\tilde{Q}\\ -2k_3\\ 2k_2\end{pmatrix}, \quad \tilde{d} = 1+\tilde{Q}. \tag{6}$$

Let some non degenerate periodic solution $\mathbf{z}(t,\mathbf{z}^0)$ of a family with initial condition $\mathbf{z}^0$ and period T be known. In a generic case, there is a periodic solution $\mathbf{z}(t)+\delta\mathbf{z}(t)$ with period $T+\delta T$ near the generic case. Decomposing the left-hand side of the periodicity condition $\mathbf{z}(T+\delta T,\mathbf{z}^0+\delta\mathbf{z}) = \mathbf{z}^0+\delta\mathbf{z}$ into a Taylor series, leaving in the expansion terms not higher than the first order of smallness for $\delta\mathbf{z}$ and δT, we obtain that small additives of $\delta\mathbf{z}$ and δT should satisfy a linear homogeneous system

$$(\mathbf{M}_2-\mathbf{E})\delta\mathbf{z}(T)+\mathbf{v}^0\delta T = 0, \tag{7}$$

where $\mathbf{v}^0 = \mathbf{J}\,\mathrm{grad}\,H(\mathbf{z}(T,\mathbf{z}^0))$. The set of solutions to this system is determined by the structure of the monodromy matrix $\mathbf{M}$.

The vectors $\mathbf{v}^0$ and $-\mathbf{J}\mathbf{v}^0$ are, respectively, the right and left eigenvectors of the matrix $\mathbf{M}_2$, then the transformation $\mathbf{M}_2 \to \tilde{\mathbf{A}}_2^\top\mathbf{M}_2\tilde{\mathbf{A}}_2$ reduces [5] the matrix $\mathbf{M}$ to the symplectic matrix $\mathbf{N}_2 = \begin{pmatrix}1 & n_{12} & n_{13} & n_{14}\\ 0 & n_{22} & n_{23} & n_{24}\\ 0 & 0 & 1 & 0\\ 0 & n_{42} & n_{43} & n_{44}\end{pmatrix}$. If we now substitute the variables $\delta\boldsymbol{\zeta} = \tilde{\mathbf{A}}_2\delta\mathbf{z}$, then the system (7) is written as $(\mathbf{N}_2-\mathbf{E})\delta\boldsymbol{\zeta}(T)+v^0\delta T\tilde{\mathbf{A}}_2^{(1)} = 0$, where v^0 is the magnitude of the phase velocity at the initial point $\mathbf{z}^0$ of the periodic solution. This system has a general solution in the form $\delta\boldsymbol{\zeta} = c_1\tilde{\mathbf{A}}_2^{(1)}+\delta\boldsymbol{\zeta}'$, $\delta T = -\frac{1}{v}\sum_{j=2}^4 n_{1j}\delta\zeta'_j$, and the vector $\delta\boldsymbol{\zeta}'$ is orthogonal to the vector $\tilde{\mathbf{A}}_2^{(1)}$. So $\delta\boldsymbol{\zeta}'$ specifies the displacement along the family of periodic orbits and $\tilde{\mathbf{A}}_2^{(1)}$ specifies the displacement along the periodic solution.

4. General structure of an arbitrary OSM

Generalizing the computations performed for cases $n = 1,2,3,4$, we can formulate the following conjecture.

Conjecture. *If a skew symmetric matrix $\mathbf{K}_n$ has the form*

$$\mathbf{K}_n = \begin{pmatrix}\mathbf{B} & \mathbf{C}\\ -\mathbf{C}^\top & \mathbf{D}\end{pmatrix}, \quad \mathbf{B}^\top = -\mathbf{B}, \quad \mathbf{D}^\top = -\mathbf{D},$$

then the matrix $\boldsymbol{\Psi}_n = -2\mathbf{J}(\mathbf{E}+\mathbf{A}_n)^{-1}(\mathbf{E}-\mathbf{A}_n)$ has the following form $\boldsymbol{\Psi}_n = -2\begin{pmatrix}\mathbf{C}^\top & \mathbf{D}^\top\\ \mathbf{B} & \mathbf{C}\end{pmatrix}$, where $\mathbf{A}_n = (\mathbf{E}+\mathbf{K}_n)(\mathbf{E}-\mathbf{K}_n)^{-1}$.

It follows from the conjecture that in order for the matrix $\boldsymbol{\Psi}_n$ to be symmetric, the conditions $\mathbf{C} = \mathbf{C}^\top$ and $\mathbf{B} = \mathbf{D}$ should be satisfied. Thus, according to Theorem 1, the following statement is obtained:

Statement. *If the matrix* $\mathbf{K}_n$ *is a block matrix* $\mathbf{K}_n = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{pmatrix}$*, with* $\mathbf{B}$ *is a skew symmetric* $n \times n$ *matrix of* $n(n-1)/2$ *independent elements and* $\mathbf{C}$ *is a symmetric* $n \times n$ *matrix of* $n(n+1)/2$ *independent elements, then the matrix* $\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)(\mathbf{E} - \mathbf{K}_n)^{-1}$ *is a generic* ***orthogonal symplectic matrix*** *of* n^2 *independent elements.*

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