Asymptotic expansions of a manifold near its curve of singular points

Alexander D. Bruno[1], Alijon A. Azimov[2]
abruno@keldysh.ru
Azimov_Alijon_Akhmadovich@mail.ru

[1]Keldysh Institute of Applied Mathematics of RAS
[2]Samarkand State University after Sh. Rashidov

PCA-2024, April 15-20, 2024

Abstract

In [1-3], parametric expansions near 5 singular points and 3 curves consisting of singular points were computed for a two-dimensional algebraic manifold Ω. Here we present general methods for computing the expansions of a manifold near its curve of singular points and their application to a single curve $\mathcal{F}$.

1.Indroduction

In [4-8] the study of the three-parameter family of special homogeneous spaces in terms of the normalized Ricci flow was started. Ricci flows give the evolution of Einstein metrics on a manifold. The equation of the normalized Ricci flow reduces to a system of two ordinary differential equations with three parameters a_1, a_2 and a_3:

$$\frac{dx_1}{dt} = \widetilde{f_1}(x_1, x_2, a_1, a_2, a_3),\ \frac{dx_2}{dt} = \widetilde{f_2}(x_1, x_2, a_1, a_2, a_3) \quad (1)$$

where $\widetilde{f_1}$ and $\widetilde{f_2}$ $-$are some concrete functions.

The special points of this system correspond to Einstein invariant metrics. At a special (fixed) point x_1^0, x_2^0 the system (1) has two eigenvalues λ_1 and λ_2. If at least one of them is equal to zero, the special point x_1^0, x_2^0 is called degenerate.

In [4-8] a theorem is proved that the set of Ω of values of parameters a_1, a_2, a_3, at which the system (1) has at least one degenerate special point is described by equation

$$
\begin{aligned}
&Q(s_1, s_2, s_3) \\
&\stackrel{\text{def}}{=} (2s_1 + 4s_3 - 1)\big(64s_1^5 - 64s_1^4 + 8s_1^3 + 240s_1^2 s_3 - 1536 s_1 s_3^2 - 4096\, s_3^3 + 12s_1^2 \\
&- 240 s_1 s_3 + 768 s_3^2 - 6s_1 + 60 s_3 + 1\big) \\
&- 8 s_1 s_2 (2s_1 + 4s_3 - 1)(2s_1 - 32 s_3 - 1)(10 s_1 + 32 s_3 - 5) \\
&-16 s_1^2 s_2^2 (52 s_1^2 + 640 s_1 s_3 + 1024 s_3^2 - 52 s_1 - 320 s_3 + 13) \\
&+ 64(2s_1 - 1) s_2^3 (2s_1 - 32 s_3 - 1) + 2048 s_1 (2s_1 - 1) s_2^4 = 0
\end{aligned}
$$

where $s_1, s_2, s_3 -$ are elementary symmetric polynomials equal to, respectively

$$s_1 = a_1 + a_2 + a_3, \qquad s_2 = a_1 a_2 + a_1 a_3 + a_2 a_3, \qquad s_3 = a_1 a_2 a_3.$$

In [9] for symmetry reasons, from coordinates $\boldsymbol{a} = (a_1, a_2, a_3)$ authors passed to the coordinates $\boldsymbol{A} = (A_1, A_2, A_3)$ by linear substitution

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = M \cdot \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}, \qquad M = \begin{pmatrix} \frac{1+\sqrt{3}}{6} & \frac{1-\sqrt{3}}{6} & \frac{1}{3} \\ \frac{1-\sqrt{3}}{6} & \frac{1+\sqrt{3}}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Definition 1. Let $\varphi(X)$ be some polynomial, $X = (x_1, \dots, x_n)$. Point $X = X^0$ of the set $\varphi(X) = 0$ is called a **singular point** k **−of order**, k if in this point all partial derivatives of the polynomial $\varphi(X)$ by $x_1, \dots, x_n$ go to zero up to $k - th$ order and at least one partial derivative of order $k + 1$ does not go to zero.

In [9] all singular points of the manifold Ω were found in coordinates $\boldsymbol{A} = (A_1, A_2, A_3)$. Five third-order points

Title	Coordinates **A**
$P_1^{(3)}$	$(0,0,3/4)$
$P_2^{(3)}$	$(0,0,-3/2)$
$P_3^{(3)}$	$\left(-\frac{1+\sqrt{3}}{2},\frac{\sqrt{3}-1}{2},\frac{1}{2}\right)$
$P_4^{(3)}$	$\left(\frac{\sqrt{3}-1}{2},-\frac{1+\sqrt{3}}{2},\frac{1}{2}\right)$
$P_5^{(3)}$	$(1,1,1/2)$

three second-order points

Title	Coordinates **A**
$P_1^{(2)}$	$\left(\frac{1+\sqrt{3}}{4},\frac{1-\sqrt{3}}{4},\frac{1}{2}\right)$
$P_2^{(2)}$	$\left(\frac{1-\sqrt{3}}{4},\frac{1+\sqrt{3}}{4},\frac{1}{2}\right)$
$P_3^{(2)}$	$(-1/2,-1/2,1/2)$

and three algebraic curves of singular points of the first order.

$$\mathcal{F} = \{a_1 = a_2, 16a_1^3 + 16a_1^2 a_3 - 4a_1 - 2a_3 + 1 = 0\},$$

$$\mathfrak{I} = \left\{A_1 + A_2 + 1 = 0; \quad A_3 = \frac{1}{2}\right\},$$

$$\mathcal{K} = \{A_1 = -\tfrac{9}{4} t h(t),\ A_2 = -\tfrac{9}{4} h(t),\ A_3 = \tfrac{3}{4},\ h(t) = \tfrac{t^2+1}{(t+1)(t^2-4t+1)}\}.$$

In this case, the points $P_3^{(3)}, P_4^{(3)}$ and $P_5^{(3)}$ are of the same type, they pass into each other at rotation around the origin of the plane A_1, A_2 by an angle $2\pi/3$, as well as all points $P_1^{(2)}, P_2^{(2)}, P_3^{(2)}$. Curves $\mathcal{F}, \mathfrak{I}$, $\mathcal{K}$ correspond to two more curves of the same type. Therefore, it is enough to study the manifold Ω in the neighborhoods of the points $P_1^{(3)}, P_2^{(3)}, P_5^{(3)}, P_3^{(2)}$ and curves $\mathcal{F}, \mathfrak{I}$, $\mathcal{K}$. Moreover, in [9] the cross sections of the manifold Ω by planes $A_3 = const$ were calculated, and it was shown that in a finite part of the space $\mathbb{R}^3 = \{A_1, A_2, A_3\}$ the manifold Ω consists of one-dimensional branches F_1, F_2, F_3, and two-dimensional branches G_1, G_2, G_3 which are broken into parts $F_i^{\pm}, G_i^{\pm}$ with boundaries on the plane $A_3 = 1/2$.

Structure of the manifold Ω near singular points $P_i^{(3)}$ and $P_i^{(2)}$ was considered in [1,2]. Here we consider the structure of the manifold Ω near three algebraic curves $\mathfrak{I}, \mathcal{K}, \mathcal{F}$ of singular points of the first order [3]. For this study, we use an algorithm consisting of 8 steps.

2. Calculation scheme

Step 1. Introduce local coordinates $X = (x_1, x_2, x_3)$. If we consider a straight line consisting of singular points (as $\mathfrak{I}$), then one coordinate x_1 directed along the line and coordinates x_2, x_3 describe deviations from the line. If the curve is located on a plane, we introduce the coordinate x_3, normal to this plane, coordinates x_1, x_2 of the curve on the plane are parameterized $x_1 = b_1(t), x_2 = b_2(t)$ and a coordinate $y_2 = x_2 - b(t)$ of the deviation from this curve.

Step 2. The original polynomial $R(\boldsymbol{A})$, we write in local coordinates as

$$g(t, y_2, x_3) = \sum \varphi_{pq}(t) y_2^p x_3^q, \tag{2}$$

and compute its support $\boldsymbol{S} = \{(p, q) : \varphi_{pq} \not\equiv 0\}$. Let the support $\boldsymbol{S}$ consists of points $(p_i, q_i), i = 1, \ldots, k$.

Step 3. Newton's polygon $\Gamma(g)$ is calculated as a convex hull of the support $\boldsymbol{S}$:

$$\Gamma(g) = \left\{(p,q) = \sum_{i=1}^{k} \lambda_i (p_i, q_i), \lambda_i \geq 0, i = 1, \ldots, k, \sum_{i=1}^{k} \lambda_i = 1\right\}.$$

Boundary $\partial\Gamma$ of the polygon $\Gamma(g)$ consists of its vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$, which we call as generalized faces. Here j is the number of the generalized face $\Gamma_j^{(d)}$. Each face $\Gamma_j^{(d)}$ corresponds to its truncated polynomial

$$\hat{g}_j^{(d)}(Y) = \sum g_{(p,q)} y_2^p x_3^q \text{ over } (p,q) \in \boldsymbol{S} \cap \Gamma_j^{(d)}$$

and the normal cone $\mathbf{U}_j^{(d)}$, consisting of all normals to the face $\Gamma_j^{(d)}$, which are the external normals to the polygon Γ. For their computation we use PolyhedralSets of the computer algebra system (CAS) Maple package [10].

Step 4. Select the faces $\Gamma_j^{(1)}$ with normals $N_j \leq 0$ and corresponding truncated polynomials $\hat{g}_j^{(1)}(t, y_2, x_3)$.

Step 5. For each selected truncated polynomial $\hat{g}_j^{(1)}(t, y_2, x_3)$, we calculate the corresponding power transformations

$$(lny_2, lnx_3) = (lnz_1, lnz_3)\alpha, \tag{3}$$

where α is unimodular matrix 2×2, such that

$$\hat{g}_j^{(1)}(t, y_2, x_3) = h(z_1, t)z_3^l \tag{4}$$

with a multiplier z_3^l.

Step 6. We make the power transformation (3) in the polynomial (2) itself and write it in the following form

$$g(Y) = T(z_1, t, z_3) = z_3^l \sum_{k=0}^{m} T_k(z_1, t) z_3^k,$$

with some natural number m, polynomial $T_k(z_1, t)$ is calculated by the command coeff (T,z[k],m) in CAS Maple, and $T_0(z_1, t) = h(z_1, t)$ from equality (4).

Step 7. If $T_0(z_1, t) \not\equiv 0$, then we substitute in the polynomial $T(z_1, t, z_3)z_3^{-l}$

$$z_1 = b_1(t) + \varepsilon,\ \ z_2 = b_2(t) + \varepsilon \qquad (5)$$

and obtain the function $u(\varepsilon, t, z_3) = T(z_1, z_2, z_3)z_3^{-l}$. Then we apply to the equation $u(\varepsilon, t, z_3) = 0$ Theorem 1[1] on the generalized implicit function and obtain the parametric expansion

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t)\, z_3^k. \qquad (6)$$

Step 8. Calculate several terms of expansion (6) and substitute them into (5). The result is substituted into the power transformation (3) and we obtain the parametric expansion of Ω into a power series by z_3, with coefficients which are rational functions of the t.

3. Structure of the manifold Ω near the curve $\mathcal{F}$ of singular points

We take the polynomial $Q(\boldsymbol{s}) = Q(s_1, s_2, s_3)$, where $s_1 = a_1 + a_2 + a_3$, $s_2 = a_1 \cdot a_2 + a_1 \cdot a_3 + a_2 \cdot a_3$, $s_3 = a_1 \cdot a_2 \cdot a_3$ are elementary symmetric polynomials, and we make a substitution $a_1 = a_2$. Then the polynomial $Q(\boldsymbol{s})$ will take the form

$$\tilde{Q}(a_1, a_3) = -(1 + 2a_3)(8a_1a_3 + 8a_3^2 - 4a_1 - 4a_3 + 1) \cdot$$
$$\cdot (16a_1^3 + 16a_1^2a_3 - 4a_1 - 2a_3 + 1)^3.$$

Let's write the polynomial $16a_1^3 + 16a_1^2a_3 - 4a_1 - 2a_3 + 1$ in coordinates $\boldsymbol{A}$. Instead of a_1 and a_3 substitute

$$a_1 = \frac{1 + \sqrt{3}}{6} A_1 + \frac{1 - \sqrt{3}}{6} A_1 + \frac{1}{3} A_3,$$

$$a_3 = -\tfrac{1}{3} A_1 - \tfrac{1}{3} A_1 + \tfrac{1}{3} A_3 \text{ with } A_1 = A_2.$$

We get a polynomial $-\frac{1}{27}(16A_1^3 - 48A_1A_3^2 - 32A_3^3 + 54A_3 - 27)$. Let's put

$$\mathcal{F}(A_1, A_3) = 16A_1^3 - 48A_1A_3^2 - 32A_3^3 + 54A_3 - 27.$$

The curve $\mathcal{F} = 0$ consists of singular points, has genus 0, parameterization

$$[A_1, A_3] = [b_1(t) = -\frac{(5t+2)(t+4)^2}{6t(t^2-16t-8)}, \quad b_2(t) = \frac{11t^3-48t^2-48t-16}{6t(t^2-16t-8)}] \tag{7}$$

and is shown in Fig. 1

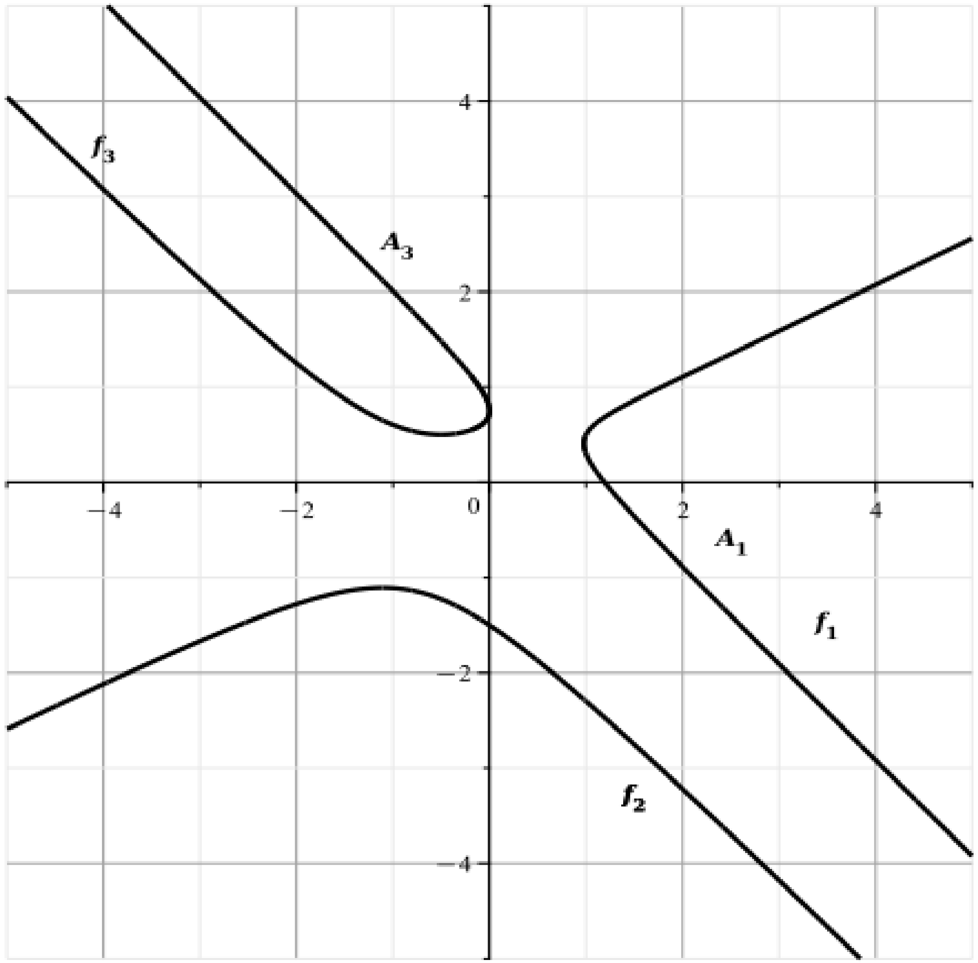

Figure 1. Curve $\mathcal{F}(A_1, A_3) = 0$.

In Fig. 3 [9], the components $f_1, f_2, f_3^{\pm}$ of this curve are shown in gray. There the scales on the axes are different. In the polynomial $R(\boldsymbol{A}) = Q(\boldsymbol{s})$ we substitute

$$A_1 = B_1, A_2 = B_1 + B_2, A_3 = B_3, \tag{8}$$

and obtain a polynomial depending on three variables,

$$K(B_1, B_2, B_3) = \sum_{l=0}^{12} K_l(B_1, B_3) B_2^l. \tag{9}$$

Factorize K_l for $l = 0,1,2$ since they are necessary for our calculations and we obtain

$$-531441 K_0(B_1, B_3)(-2B_3 - 3 + 4B_1)(16B_1^2 - 40B_1B_3 + 16B_3^2 + 12B_1 - 24B_3 + 9) \cdot (16B_1^3 - 48B_1B_3^2 - 32B_3^3 + 54B_3 - 27)^3 =$$

$$= (-2B_3 - 3 + 4B_1)(16B_1^2 - 40B_1B_3 + 16B_3^2 + 12B_1 - 24B_3 + 9)\mathcal{F}^3(B_1, B_3).$$

and

$$-177147 K_1(B_1, B_3) = 8B_1 \cdot (16B_1^3 - 48B_1B_3^2 - 32B_3^3 + 54B_3 - 27)^2 \cdot$$

$$(256B_1^4 - 704B_1^3B_3 + 96B_1^2B_3^2 + 736B_1B_3^3 - 320B_3^4 + 378B_1B_3 - 432B_3^2 - 189B_1$$

$$+216B_3) = 8B_1(256B_1^4 - 704B_1^3B_3 + 96B_1^2B_3^2 + 736B_1B_3^3 - 320B_3^4 + 378B_1B_3$$

$$-432B_3^2 - 189B_1 + 216B_3)\mathcal{F}^2(B_1, B_3).$$

Next

$$K_2(B_1,B_3) = -\frac{45056}{729}B_1^7B_3 + \frac{1856}{27}B_1^4B_3 - \frac{131072}{19683}B_1^{10} - \frac{467}{27}B_1^4$$
$$+\frac{22528}{729}B_1^7 + \frac{1310720}{177147}B_3^{10} - \frac{32768}{2187}B_3^8 + \frac{16384}{2187}B_3^7 - \frac{128}{3}B_3^3$$
$$+\frac{2048}{81}B_3^4 + \frac{1024}{81}B_3^5 - \frac{1024}{81}B_3^6 + \frac{64}{3}B_3^2 - \frac{32}{9}B_3 - \frac{65536}{729}B_1^2B_3^5$$
$$+\frac{106496}{2187}B_1^3B_3^4 + \frac{136192}{729}B_1^4B_3^3 + \frac{512}{27}B_1^2B_3^2 + \frac{2048}{81}B_1^3B_3 - \frac{212992}{2187}B_1^3B_3^5$$
$$-\frac{40960}{729}B_1^5B_3^2 - \frac{177152}{2187}B_1^6B_3 + \frac{131072}{729}B_1^2B_3^6 + \frac{81920}{729}B_1^5B_3^3 - \frac{272384}{729}B_1^4B_3^4$$
$$+\frac{354304}{2187}B_1^6B_3^2 + \frac{3080192}{177147}B_1^9B_3 - \frac{1638400}{19683}B_1^7B_3^3 - \frac{5472256}{59049}B_1^6B_3^4$$
$$+\frac{2621440}{19683}B_1^5B_3^5 + \frac{2768896}{19683}B_1^4B_3^6 + \frac{671744}{19683}B_1^8B_3^2 - \frac{1441792}{19683}B_1^2B_3^8$$
$$-\frac{3407872}{59049}B_1^3B_3^7 + \frac{2048}{27}B_1^2B_3^4 + \frac{8192}{81}B_1^3B_3^3 - \frac{1856}{27}B_1^4B_3^2 - \frac{8192}{81}B_1^3B_3^2$$
$$-\frac{2048}{27}B_1^2B_3^3,$$

Multiplier $\mathcal{F}(B_1, B_3)$ enters to $K_0(B_1, B_3)$ in the third degree, to $K_1(B_1, B_3)$ in the second, in $K_2(B_1, B_3)$ and in $K_3(B_1, B_3)$ it does not enter. Then K_0 is divisible by $\mathcal{F}^3$, K_1 is divisible by $\mathcal{F}^2$, but K_2 is not divisible by $\mathcal{F}$. The curve $\mathcal{F}(B_1, B_3) = 0$ has genus 0, parametrization (7). The curve $\mathcal{F} = 0$ goes to infinity at

$$t_1 = 0, \quad t_2 = 2\left(4 + 3\sqrt{2}\right) \approx 16.48528137, \quad t_3 = 2\left(4 - 3\sqrt{2}\right) \approx -0.485281372.$$

In the polynomials $K_l(B_1, B_3)$ we make substitution

$$B_1 = b_1(t) + \varepsilon, \quad B_3 = b_2(t) \qquad (10)$$

according to (7). Then the polynomial (9) transforms to the polynomial

$$K(B_1, B_3, B_2) = u(\varepsilon, B_2) = \sum_{p,q \geq 0} u_{pq}(t) \varepsilon^p B_2^q, \quad u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p K_q}{\partial B_1^p}(b_1(t), b_2(t)), \qquad (11)$$

herewith

$$u_{pq} = \frac{1}{p!} \cdot \frac{\partial^p K_q}{\partial B_1^p}\big(b_1(t), b_2(t)\big),$$

where $B_1 = b_1(t)$, $B_3 = b_2(t)$ according to (7). In particular, we obtain

$$u_{00} = u_{10} = u_{01} \equiv 0,$$

$$u_{20}(t) = \frac{1}{2} \cdot \left(\frac{\partial^2 K_0}{\partial B_1^2}\right)\big(b_1(t), b_2(t)\big) = 0,$$

$$u_{11}(t) = \left(\frac{\partial K_1}{\partial B_1}\right)\big(b_1(t), b_2(t)\big) = 0,$$

$$u_{02}(t) = K_2\big(b_1(t), b_2(t)\big) = \frac{64(5t+2)^2(t+4)^4(t-2)^4(t+1)^4}{t^2(t^2-16t-8)^6}. \qquad (12)$$

$$u_{30}=\frac{1}{6}\cdot\frac{\partial^3 K_0}{\partial B_1^3}=\frac{40960}{81}B_1^2B_3^3+\frac{14417920}{59049}B_1^8B_3+\frac{10240}{81}B_1^2B_3-\frac{286720}{729}B_1^4B_3^2-\frac{57671680}{531441}B_1^9$$

$$+\frac{360448}{6561}B_3^7+\frac{4259840}{6561}B_1^3B_3^3+\frac{3670016}{531441}B_3^9+\frac{22528}{243}B_3^4-\frac{22528}{243}B_3^5-\frac{180224}{6561}B_3^6$$

$$+\frac{1835008}{2187}B_1^5B_3^2+\frac{102400}{243}B_1^3B_3+\frac{2621440}{6561}B_1^7B_3^2-\frac{2981888}{6561}B_1^6B_3-\frac{150470656}{177147}B_1^6B_3^3$$

$$-\frac{2883584}{19683}B_1B_3^8+\frac{4096}{27}B_1B_3^4-\frac{4096}{27}B_1B_3^3-\frac{131072}{729}B_1B_3^5+\frac{262144}{729}B_1B_3^6+\frac{573440}{729}B_1^4B_3^3$$

$$+\frac{1490944}{6561}B_1^6-\frac{224}{9}B_3-\frac{13696}{243}B_3^3+\frac{448}{9}B_3^2-\frac{25600}{243}B_1^3+\frac{112}{27}-\frac{917504}{2187}B_1^5B_3-\frac{102400}{243}B_1^3B_3^2$$

$$-\frac{17039360}{59049}B_1^2B_3^7+\frac{103546880}{177147}B_1^3B_3^6+\frac{1024}{27}B_1B_3^2-\frac{40370176}{59049}B_1^5B_3^4-\frac{1064960}{2187}B_1^2B_3^5$$

$$+\frac{18350080}{19683}B_1^4B_3^5+\frac{532480}{2187}B_1^2B_3^4-\frac{40960}{81}B_1^2B_3^2-\frac{8519680}{6561}B_1^3B_3^4=$$

$$=-\frac{8192(5t+2)(t+4)^2(t-2)^2(t+1)^3(8t^3-3t^2+24t+8)^3}{6561t^5(t^2-16t-8)^6}. \qquad (13)$$

From formulas (12) and (13) we can see that the Newton's polygon Γ of polynomial $u(\varepsilon, B_2)$ (11) in the plane p, q has an edge $\Gamma_1^{(1)}$, containing points (3,0), (0,2) (Fig.2) with the external normal $N_1 = (-2, -3)$.

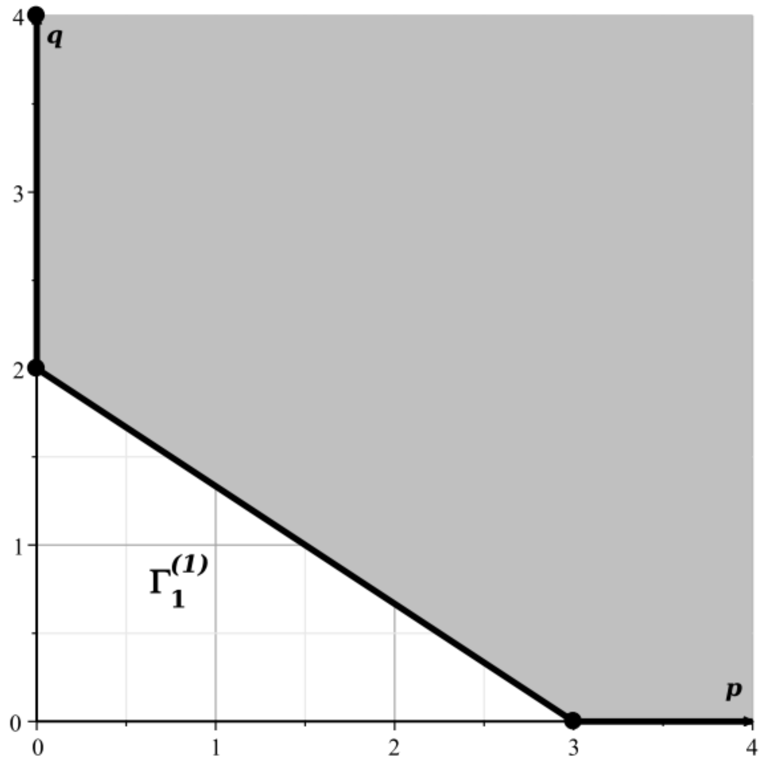

Figure 2. Lower left part of the polygon Γ.

This edge corresponds to the truncated polynomial

$$\varepsilon^3 u_{30}(t) + B_2^2 u_{02}(t) = 0. \tag{14}$$

We find the unimodular matrix

$$\alpha = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

for N_1 such that

$$N_1 \alpha = (0, -1).$$

Therefore, we need to make a power transformation

$$(ln\delta, lnD) = (ln\varepsilon, lnB_2) \cdot \alpha,$$

i.e

$$(ln\varepsilon, lnB_2) = (ln\delta, lnD) \cdot \alpha^{-1}$$

Since $\alpha^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, then

$$\varepsilon = \delta D^2, \;\; B_2 = \delta D^3.$$

From here we can write

$$K(B_1, B_3, B_2) = u(\varepsilon, B_2) =$$

$$\sum u_{pq}(t)\varepsilon^p B_2^q = \sum u_{pq}(t)\, \delta^{p+q} D^{2p+3q} = \delta^2 D^6 V(\delta, D).$$

Then the polynomial $u(\varepsilon, B_2)$ will turn into a polynomial

$$\delta^2 D^6 V(\delta, D) = \delta^2 D^6 \sum V_{rs}(t)\delta^r D^s = u(\varepsilon, B_2),$$

where

$$V_{r,s}(t) = V_{p+q,2p+3q}(t) = u_{p,q}(t).$$

In this case the polygon Γ of Fig.2. takes the form shown in Fig. 3. For polynomial $V(\delta, D)$ the polygon is shown in Fig.4. The shortened equation (14) takes the form

$$\delta^2 D^6\big(u_{30}(t)\delta + u_{02}(t)\big) = 0.$$

From where

$$\delta_0(t) = c_0(t) = -\frac{u_{02}(t)}{u_{30}(t)} = \frac{6561(5t+2)\cdot(t\ +\ 4)^2\cdot(t\ -\ 2)^2(t+1)t^3}{128\,(8t^3 - 3t^2 + 24t + 8)^3}.$$

The only real zero of the denominator

$$t_4 = -\frac{3\left(13+16\sqrt{2}\right)^{\frac{1}{3}}}{8} + \frac{21}{8\left(13+16\sqrt{2}\right)^{\frac{1}{3}}} + \frac{1}{8} \approx -0.3111842957. \quad (15)$$

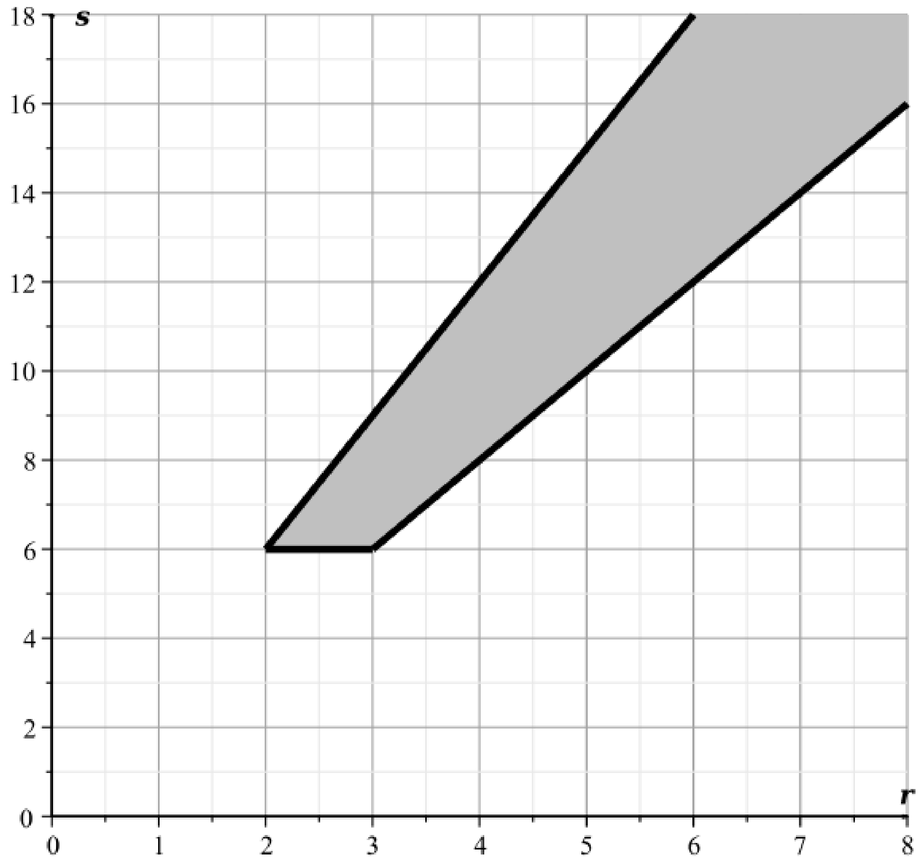

Figure 3. Newton's polygon of the polynomial $\delta^2 D^6 V(\delta, D)$.

In doing so.

$$V_{g,h}(t) = V_{p+q-2,2p+3q-6}(t) = u_{p,q}(t).$$

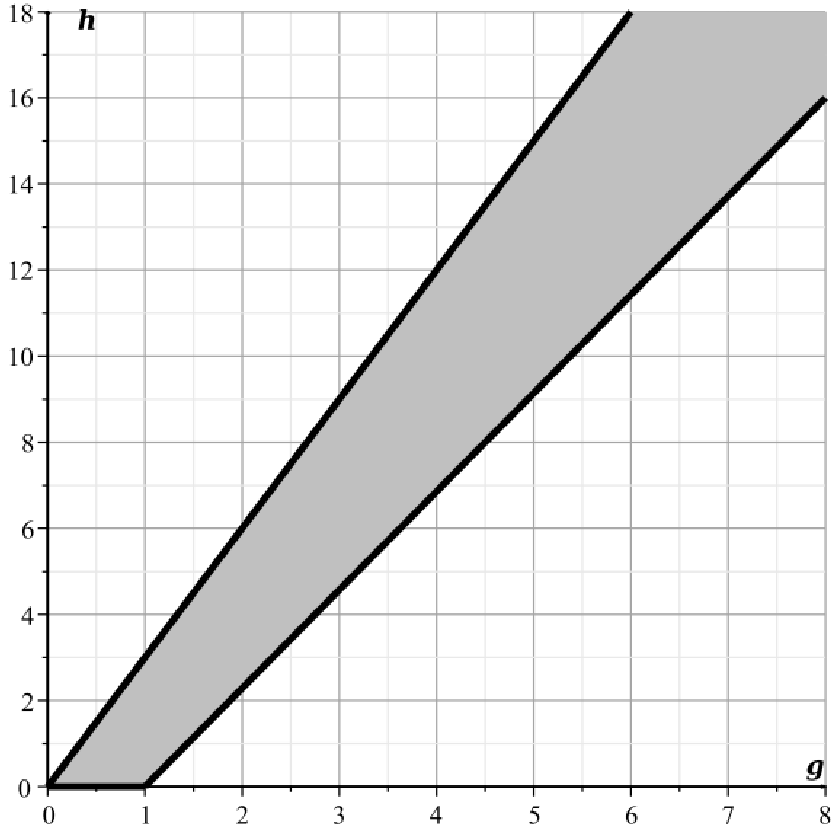

Figure 4. Newton's polygon of the polynomial $V(\delta, D)$.

We substitute into the polynomial $V(\delta, D)$

$$\delta = \delta_0(t) + \xi,$$

it turns out

$$W(\xi, D) = V(\delta_0(t) + \xi, D).$$

With $\xi = 0$ polynomial $W(0, D)$ is calculated using the command $coeff\ (M(0, D), 0)$[2]. The coefficient at D of zero degree is equal to zero. The coefficient at the first degree D is obtained

$$a(t) = coeff\ (M(0, D), D, 1) = \frac{1594323t^6(5t+2)^2(t+4)^4(t-2)^4(t+1)^4}{2}.$$

Therefore, the equation $W(\xi, D) = 0$ satisfy to conditions of Theorem 1, [1] according to which it has a solution

$$\xi = \sum_{k=1}^{\infty} \mathrm{c}_k(t) D^k. \tag{16}$$

We get the truncated equation $u_{30}(t) \cdot \xi + a(t) \cdot D = 0$. It follows

$$\xi = -\frac{a(t)}{u_{30}(t)} \cdot D =$$

$$= \frac{10460353203t^{11}(t+1)(5t+2)(t+4)^2(t-2)^2(t^2-16t-8)^6}{16384(8t^3-3t^2+24t+8)^3} \cdot D$$

$$= c_1(t)D.$$

Now let's go back and obtain approximation

$$\varepsilon = \delta D^2 = (\delta_0(t) + c_1(t)D)D^2 \approx \delta_0(t)D^2 + c_1(t)D^3, \quad (17)$$

$$B_2 = (\delta_0(t) + c_1(t)D)D^3 \approx \delta_0(t)D^3 + c_1(t)D^4. \quad (18)$$

Consequently, from formula (8) we obtain

$$A_1 = B_1 = b_1(t) + \delta_0(t)D^2 + c_1(t)D^3, \quad (19)$$

$$A_2 = B_1 + B_2 = b_1(t) + \delta_0(t)D^2 + (c_1(t) + \delta_0(t))D^3 + c_1(t)D^4,$$

$$A_3 = B_3 = b_2(t) = \frac{11t^3-48t^2-48t-16}{6t(t^2-16t-8)}. \quad (20)$$

Curves (19), (20) for $D = \pm 0{,}1$ are shown in Figs. 5 and 6, respectively. The discontinuities on these curves are the neighborhoods of the point t_4 from (15). They can be filled if instead of substitution (10) we make

$$B_1 = b_1(t) + \varepsilon, \;\; B_3 = b_2(t) + \varepsilon.$$

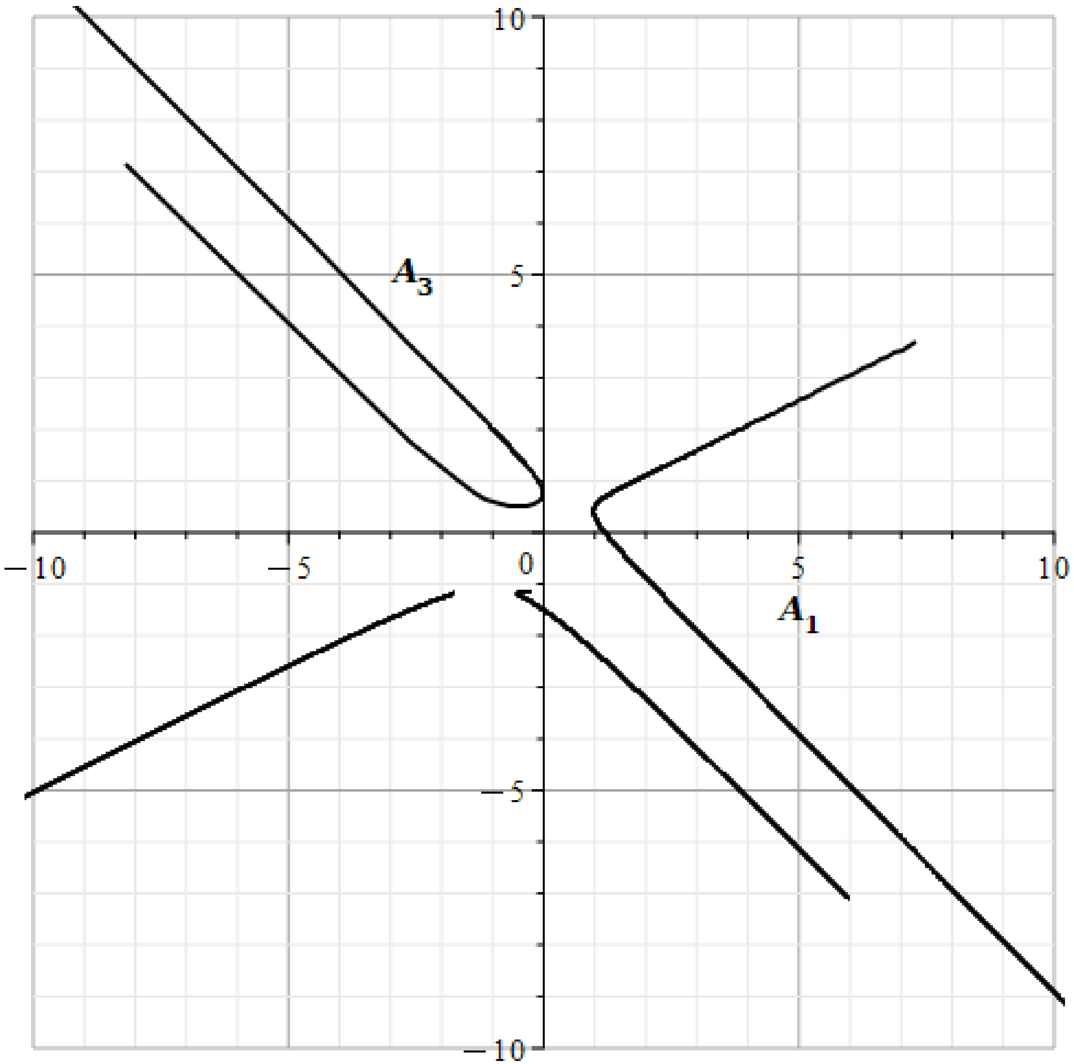

Figure 5. Curve (19) and (20) at $D = 1/10$.

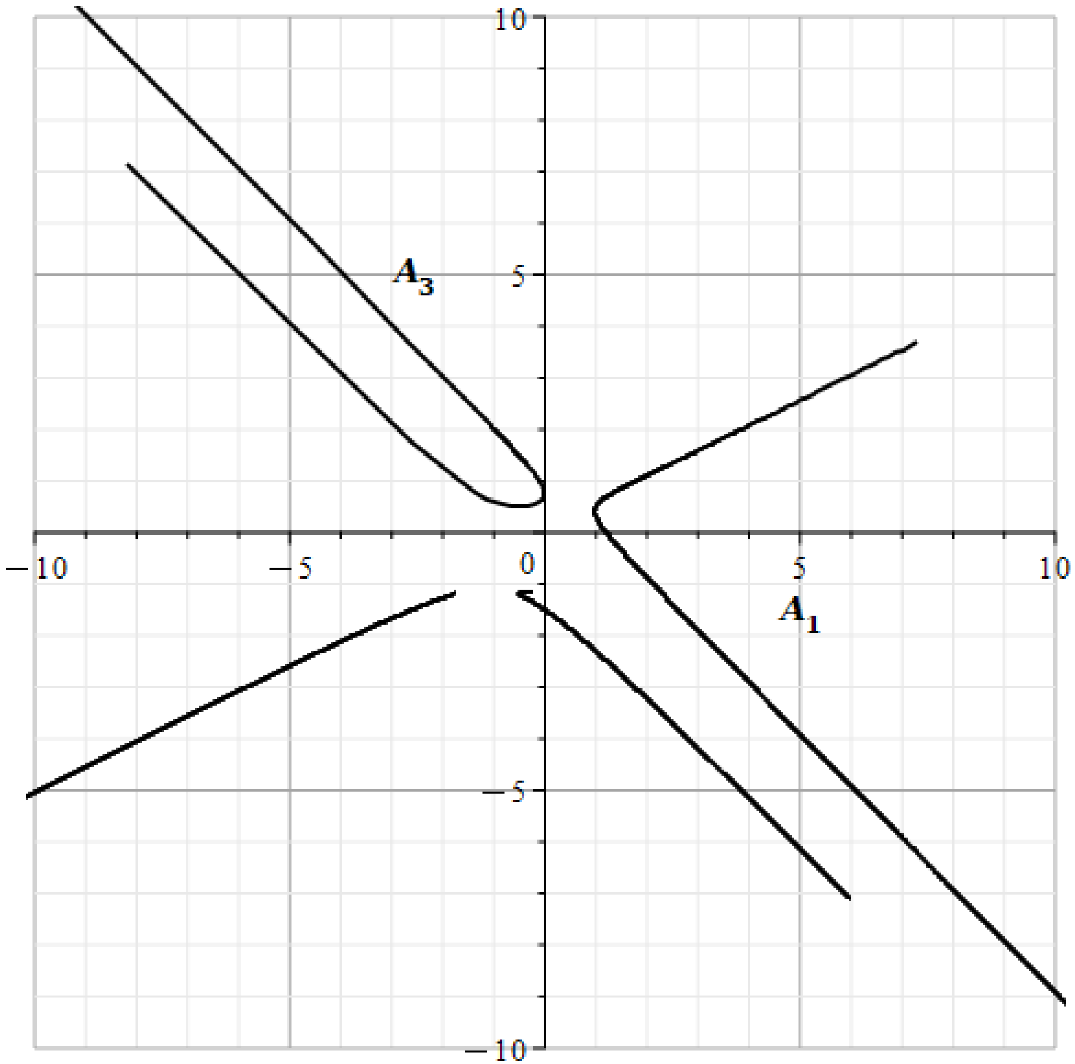

Figure 6. Curve (19) and (20) at $D = -1/10$.

The closeness of these curves to the curve of Fig. 1 confirms the correctness of the found parametric expansion (16) of the manifold Ω near the curve of special points. According to (18),(19) branches G_i intersect the curve F with a singularity of the type

$$\sqrt{\frac{A_1 - b_1(t)}{\delta_0(t)}} \approx \left(\frac{B_2}{\delta_0(t)}\right)^{\frac{1}{3}}.$$

So, we obtain

Theorem 1. *The curve* $\mathcal{F}$ *consists of branches* $F_1^{\pm}, F_2^{\pm}, F_3^{\pm}$. *On them two-dimensional branches* $G_1^{\pm}, G_2^{\pm}, G_3^{\pm}$ *of the manifold Ω meet (but do not intersect)* . *Their parametric expansions are (16)-(19).*

References

1. Bruno A.D., Azimov A.A. Parametric expansions of an algebraic variety near its singularities.\\ Axioms 2023, p. 469. https://doi.org/10.3390/axioms12050469.

2. Alexander Bruno, Alijon Azimov. Parametric expansions of an algebraic variety near its singularities. International Conference on Polynomial Computer Algebra St.Petersburg, April 17-22, 2023. page 22-26. ISSBN 978-5-9651-1473-3.

3. Bruno A.D., Azimov A.A. Parametric expansions of an algebraic variety near its singularities II. \\ Axioms 2024, *13*(2), 106; https://doi.org/10.3390/axioms13020106

4. Besse A.L. Einstein Manifolds; Springer: Berlin, Germany, 1987.

5. Abiev, N.A.; Arvanitoyeorgos, A.; Nikonorov, Y.G.; Siasos, P. The dynamics of the Ricci flow on generalized Wallach spaces. Differential Geometry and its Applications 2014, 35, 26–43. https://doi.org/10.1016/j.difgeo.2014.02.002.

6. Abiev, N.A.; Arvanitoyeorgos, A.; Nikonorov, Y.G.; Siasos, P. The normalized Ricci flow on generalized Wallach spaces. In Mat. Forum; Yuzhnii Matematicheskii Institut, Vladikavkazskii Nauchnii Tsentr Ross. Akad. Nauk: Vladikavkaz, 2014; Vol. 8, Studies in Mathematical Analysis, pp. 25–42. (in Russian).

7. Abiev, N.A.; Nikonorov, Y.G. The evolution of positively curved invariant Riemannian metrics on the Wallach spaces under the Ricci flow. Annals of Global Analysis and Geometry 2016, 50, 65–84. https://doi.org/10.1007/s10455-016-9502-8.

8. Jablonski, M. Homogeneous Einstein manifolds. Revista de la Unión Matemática Argentina 2023, 64, 461–485. https://doi.org/10.33044/revuma.3588.

9. Bruno, A.D.; Batkhin, A.B. Investigation of a real algebraic surface. Program. Comput. Softw. 2015, 41, 74–82. https://link.springer.com/article/10.1134/S0361768815020036

10. Thompson I., Understanding Maple, Cambridge Univ. Press, 2016.