New cases of integrability of the Euler–Poisson system

Alexander Bruno and Alexander Batkhin

Abstract. In the classical problem of motion of a rigid body around a fixed point described by the Euler–Poisson system, new cases of global integrability are found. For one of these cases, generalizing the Kovalevskaya case, a fourth global integral is proposed.

1. Introduction

The Euler-Poisson equations (1750) are a real autonomous system of six ordinary differential equations (ODEs).

$$\begin{aligned}
&Ap' + (C - B)qr = Mg\,(y_0\gamma_3 - z_0\gamma_2)\,,\\
&Bq' + (A - C)pr = Mg\,(z_0\gamma_1 - x_0\gamma_3)\,,\\
&Cr' + (B - A)pq = Mg\,(x_0\gamma_2 - y_0\gamma_1)\,,\\
&\gamma_1' = r\gamma_2 - q\gamma_3, \quad \gamma_2' = p\gamma_3 - r\gamma_1, \quad \gamma_3' = q\gamma_1 - p\gamma_2,
\end{aligned} \tag{1}$$

with dependent variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$ and parameters A, B, C, x_0, y_0, z_0, satisfying the triangle inequalities

$$0 < A \leqslant B + C, \quad 0 < B \leqslant A + C, \quad 0 < C \leqslant A + B. \tag{2}$$

Here, the prime sign $'$ indicates differentiation over independent variable time t, Mg is the weight of the body, A, B, C are the principal moments of inertia of the rigid body, x_0, y_0, z_0 are the coordinates of the center of gravity of the rigid body, $\gamma_1, \gamma_2, \gamma_3$ are the vertical directional cosines.

The system (1) describes the motion of a spinner around a fixed point (Golubev, [1]) and has three first integrals: energy, momentum, and geometric:

$$\begin{aligned} &I_1 \stackrel{\text{def}}{=} Ap^2 + Bq^2 + Cr^2 - 2Mg\left(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3\right) = h = \text{const},\\ &I_2 \stackrel{\text{def}}{=} Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const},\\ &I_3 \stackrel{\text{def}}{=} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{3}$$

The system is integrable if there is a fourth general integral I_4. So far, 4 cases of global integrability are known:

Case 1. Euler-Poinsot: $x_0 = y_0 = z_0 = 0$ and $I_4 \stackrel{\text{def}}{=} A^2p^2 + B^2q^2 + C^2r^2 = \text{const}$.

Case 2. Lagrange-Poisson: $B \neq C$, $x_0 =\neq 0$, $y_0 = z_0 = 0$, and $I_4 \stackrel{\text{def}}{=} p = \text{const}$.

Case 3. Kovalevskaya (1890): $A = B = 2C$, $x_0 \neq 0$, $y_0 = z_0 = 0$, and

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1\right)^2 + \left(2pq + c\gamma_2\right)^2 = \text{const}, \tag{4}$$

where $c = Mgx_0/C$.

Case 4. Kinematic symmetry: $A = B = C$ and $I_4 \stackrel{\text{def}}{=} x_0p + y_0q + z_0r = \text{const}$. It is derived from case 2.

2. Results

We found the following new cases of integrability of the system (1).

Case 5: $A = B = 2C$, $x_0 \neq 0$, $y_0 \neq 0$, $z_0 = 0$. Then the fourth integral I_4 has the form

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1 - d\gamma_2\right)^2 + \left(2pq + d\gamma_1 + c\gamma_2\right)^2 = \text{const}, \tag{5}$$

where $c = Mgx_0/C$, $d = Mgy_0/C$. This is a generalization of Kovalevskaya's case 3 and her fourth integral (4). As for cases 1–4 the fourth integral (5) is independent of the integrals (3).

Case 6: $B = C$, $A^2\left(A - 2B\right)x_0^2 = B\left(2A - B\right)^2 y_0^2$, $z_0 = 0$.

For case 6, the inequalities of triangle (2) are not satisfied. For case 6, the additional fourth integral I_4 was not written out, and local integrability was checked near the corresponding fixed points for third-order resonances. According to (Bruno, [2, Section 5.3]) the coefficients of the resonance terms of the normal form at 2 : 1 resonance should be zero in integrable cases. In this case they are zero.

3. Theory

We have found some general property of integrable cases 1–4, which is formulated below as Hypothesis 2. So we have to compute all those values of parameters A, B, C, x_0, y_0, z_0 for which this property is satisfied. And then, by computing the resonance terms of the normal form of the system (1), to extract from them those values at which the system (1) is integrable.

Hypothesis 1 (Edneral, [3]**).** *If an autonomous polynomial ODE system is locally integrable in the neighborhood of all its stationary points, then it is globally integrable.*

Therefore, to find global integrability, we must first find all the stationary points of the ODE system, and then find out whether the system is locally integrable in their neighborhoods.

Let $X = (p, q, r, \gamma_1, \gamma_2, \gamma_3)$, the point $X = X^0$ is a stationary point of the system (1) and M is a matrix of the linear part of the system (1) near the point X^0. The characteristic polynomial $\chi(\lambda)$ of the matrix M is $\chi(\lambda) = \lambda^6 + a_4\lambda^4 + a_2\lambda^2$. Its discriminant

$$D_\lambda(\chi) = a_4^2 - 4a_2 \tag{6}$$

is a rational function $D = G/H$, where G and H are polynomials.

A stationary point is locally integrable (Bruno, [2]) if $a_2 < 0$ or $D_\lambda(\chi) < 0$. But this property is not satisfied for definite values of the system parameters (1).

The stationary points of the system (1) form one-dimensional and two-dimensional families $\mathcal{F}_j^l$ in $\mathbb{R}^6$.

Hypothesis 2. *If near a stationary point X^0 of the family $\mathcal{F}_j^l$ the system* (1) *is locally integrable, then at these parameter values the second discriminant $\Delta\left(\mathcal{F}_j^l\right)$ of the numerator G of the first discriminant $D_\lambda(\chi)$ on the parameter of the family $\mathcal{F}_j^l$ is zero.*

Considering Hypothesis 1, now the search for integrable cases consists of the following 5 steps.

- **Step 1:** Fix the number l of non-zero parameters x_0, y_0, z_0 and find all families $\mathcal{F}_j^l$ of stationary points.
- **Step 2:** Compute the discriminants (6) $D_\lambda(\chi)$ on the families $\mathcal{F}_j^l$.
- **Step 3:** On families $\mathcal{F}_j^l$, compute the second discriminants $\Delta\left(\mathcal{F}_j^l\right)$ of the numerators G of the first discriminants D.
- **Step 4:** Find the values of the parameters of the system (1) at which all $\Delta\left(\mathcal{F}_j^l\right) = 0$ at fixed l.
- **Step 5:** Check the obtained parameter values for integrability by computing the normal forms of the system (1) near stationary points or by finding the fourth integral.

4. Computations

4.1. Case $l = 0 : x_0 = y_0 = z_0 = 0$

Then the system (1) has 3 families of stationary points:

$\mathcal{F}_1^0$: $\{q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0\}$, p is a parameter;

$\mathcal{F}_2^0$: $\{p = r = 0,\ \gamma_2 = q/k = \pm 1,\ \gamma_1 = \gamma_3 = 0\}$, q is a parameter;
$\mathcal{F}_3^0$: $\{p = q = 0,\ \gamma_3 = r/k = \pm 1,\ \gamma_1 = \gamma_2 = 0\}$, r is a parameter.

They all have $\Delta\left(\mathcal{F}_j^0\right) \equiv 0$ on them, so the system (1) is integrable (**Case 1**).

4.2. Case $l = 1 : x_0 \neq 0, y_0 = z_0 = 0$

Then the system (1) has 4 families of stationary points:

$\mathcal{F}_1^1$: $\{q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0,\ B \neq C\}$, p is a parameter;

$\mathcal{F}_2^1$: $\left\{p = \dfrac{x_0}{k(C-A)}, q = 0,\ \gamma_1 = \dfrac{p}{k}, \gamma_2 = 0, \gamma_3 = \dfrac{s}{k}, \gamma_1^2 + \gamma_3^2 = 1, A \neq C \neq B\right\}$,
r is a parameter;

$\mathcal{F}_3^1$: $\left\{p = \dfrac{x_0}{k(B-A)}, r = 0,\ \gamma_1 = \dfrac{p}{k}, \gamma_2 = \dfrac{q}{k}, \gamma_3 = 0, \gamma_1^2 + \gamma_2^2 = 1, A \neq B \neq C\right\}$,
q is a parameter;

$\mathcal{F}_4^1$: $\left\{p = \dfrac{x_0}{k(B-A)},\ \gamma_1 = \dfrac{p}{k}, \gamma_2 = \dfrac{q}{k}, \gamma_3 = \dfrac{r}{k}, \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, A \neq B = C\right\}$,
q, r are parameters.

All the second discriminants for these families are zero when:

1. $B = C$ – **Case 2**;
2. $A = B = 2C$ – **Case 3**;
3. $A = 2C$, $B = 3C$;
4. $A = 2C$, $B = \delta C$, where δ is the root of the equation $\delta^3 - 12\delta^2 + 33\delta - 24 = 0$, i.e., $\delta_1 \approx 1.194$, $\delta_2 \approx 2.387$, $\delta_3 \approx 8.419$.

But the check shows that there is no local integrability in items 3 and 4.

4.3. Case $l = 2 : x_0 \neq 0, y_0 \neq 0, z_0 = 0$

Then the system (1) has 2 families of stationary points:

$$\mathcal{F}_1^2 : \left\{p = \frac{x_0}{k(C-A)}, q = \frac{y_0}{k(C-B)},\ \gamma_1 = \frac{p}{k}, \gamma_2 = \frac{q}{k}, \gamma_3 = \frac{r}{k},\right.$$
$$\left.\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, A \neq C \neq B\right\}, r \text{ is a parameter;}$$

$$\mathcal{F}_2^2 : \left\{p = -\frac{x_0}{k(A+T)}, q = -\frac{y_0}{k(B+T)}, r = 0,\ \gamma_1 = \frac{p}{k}, \gamma_2 = \frac{q}{k}, \gamma_3 = 0,\right.$$
$$\left.\gamma_1^2 + \gamma_2^2 = 1\right\}, T \text{ is a parameter;}$$

The second discriminants $\Delta(\mathcal{F}_1^2)$ and $\Delta(\mathcal{F}_2^2)$ were obtained during the computation, but we could not factorize $\Delta(\mathcal{F}_2^2)$ because it contains several hundred thousand monomials. Therefore, we computed $\Delta(\mathcal{F}_2^2)$ on the zeros of $\Delta(\mathcal{F}_1^2)$. We get the following results.

With $B = C$ both Δ are zero at

- $A = B = C$ (**Case 4**),
- $B = C$ and $A^2(A - 2B)x_0^2 = B(2A - B)^2 y_0^2$ (new **Case 6**),
- $A = 2B = 2C$.

Checking shows that the last case is non-integrable.

With $A = B$ we get that $\Delta(\mathcal{F}_2^2) = -4A^2C^3(A - 2C)$. Therefore, when $A = B = 2C$, both Δ are zero. This is the *new* **Case 5**.

4.4. Case $l = 3 : x_0 \neq 0, y_0 \neq 0, z_0 \neq 0$

Then the system (1) has one family of stationary points:

$$\mathcal{F}_1^3 : \Big\{ p = -\frac{x_0}{k(A+T)}, q = -\frac{y_0}{k(B+T)}, r = -\frac{z_0}{k(C+T)}, \\ \gamma_1 = \frac{p}{k}, \gamma_2 = \frac{q}{k}, \gamma_3 = \frac{r}{k}, \gamma_1^2 + \gamma_2^2 = 1 \Big\},$$

T is a parameter;

The first discriminant of $D_\lambda(\chi)$ is a 10th degree polynomial of T. It is impossible to compute its discriminant on T in the generic case, but when $A = B = C$ (**Case 4**) it is zero. When $A = B = 2C$, the second discriminant $\Delta(\mathcal{F}_1^3) = 384A^2\left(x_0^2 + y_0^2\right)^4 \neq 0$. According to Hypothesis 2, this is a non-integrable case.

So here also as for the family $\mathcal{F}_2^2$ we should look for other methods of computing discriminants or more powerful computers.

References

[1] V. V. Golubev. *Lectures on Integration of the Equations of Motion of a Rigid Body about a Fixed Point*. Moscow State Publishing House Of Theoretical ..., Moscow, 1960.

[2] A. D. Bruno. Analysis of the Euler–Poisson equations by methods of power geometry and normal form. *Journal of Applied Mathematics and Mechanics*, 71(2):168–199, 2007. `https://doi.org/10.1016/j.jappmathmech.2007.06.002` `doi:10.1016/j.jappmathmech.2007.06.002`.

[3] V. F. Edneral. Integrable cases of the polynomial Liénard-type equation with resonance in the linear part. *Mathematics in Computer Science*, 17(3–4), July 2023. `https://doi.org/10.1007/s11786-023-00567-6` `doi:10.1007/s11786-023-00567-6`.

Alexander Bruno
Singular Problems Department
Keldysh Institute of Applied Mathematics of RAS
Moscow, Russia
e-mail: `abruno@keldysh.ru`

Alexander Batkhin
Department of Aerospace Engineering
Techion – Israel Institute of Technology
Haifa, Israel
e-mail: `batkhin@technion.ac.il`