New cases of integrability of the Euler–Poisson system

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Talk outlook

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Abstract

In the classical problem of motion of a rigid body around a fixed point described by the Euler-Poisson system, new cases of global integrability are found. For one of these cases, generalizing the Kovalevskaya case, a fourth global integral is proposed.

1. Introduction (1)

The Euler-Poisson equations (1750) are a real autonomous system of six ordinary differential equations (ODEs).

$$\begin{aligned} &Ap' + (C-B)qr = Mg\left(y_0\gamma_3 - z_0\gamma_2\right), \\ &Bq' + (A-C)pr = Mg\left(z_0\gamma_1 - x_0\gamma_3\right), \\ &Cr' + (B-A)pq = Mg\left(x_0\gamma_2 - y_0\gamma_1\right), \\ &\gamma_1' = r\gamma_2 - q\gamma_3, \quad \gamma_2' = p\gamma_3 - r\gamma_1, \quad \gamma_3' = q\gamma_1 - p\gamma_2, \end{aligned} \tag{1}$$

with dependent variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$ and parameters A, B, C, x_0, y_0, z_0, satisfying the triangle inequalities

$$0 < A \leqslant B + C, \quad 0 < B \leqslant A + C, \quad 0 < C \leqslant A + B. \tag{2}$$

Here, the prime indicates differentiation over independent variable time t, Mg is the weight of the body, A, B, C are the principal moments of inertia of the rigid body, x_0, y_0, z_0 are the coordinates of the center of gravity of the rigid body, $\gamma_1, \gamma_2, \gamma_3$ are the vertical directional cosines.

The system (1) describes the motion of a spinner around a fixed point [Golubev, 1960] and has three first integrals: energy, momentum, and geometric:

$$
\begin{aligned}
&I_1 \stackrel{\text{def}}{=} Ap^2 + Bq^2 + Cr^2 - 2Mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \text{const},\\
&I_2 \stackrel{\text{def}}{=} Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const},\\
&I_3 \stackrel{\text{def}}{=} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1.
\end{aligned} \tag{3}
$$

The system is integrable if there is a fourth general integral I_4. So far, 4 cases of integrability are known:

Case 1. Euler-Poinsot: $x_0 = y_0 = z_0 = 0$ and $I_4 \stackrel{\text{def}}{=} A^2p^2 + B^2q^2 + C^2r^2 = \text{const}.$

Case 2. Lagrange-Poisson $B = C$, $x_0 \neq 0$, $y_0 = z_0 = 0$, and $I_4 \stackrel{\text{def}}{=} p = \text{const}.$

Case 3. Kovalevskaya (1890): $A = B = 2C$, $x_0 \neq 0$, $y_0 = z_0 = 0$, and

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1\right)^2 + (2pq + c\gamma_2)^2 = \text{const}, \tag{4}$$

where $c = Mgx_0/C$.

Case 4. Kinematic symmetry: $A = B = C$ and $I_4 \stackrel{\text{def}}{=} x_0p + y_0q + z_0r = \text{const}.$ It is derived from case 2.

We found the following cases of integrability of the system (1).

Case 5: $A = B = 2C$, $x_0 \neq 0$, $y_0 \neq 0$, $z_0 = 0$. Then the fourth integral has the form

$$I_4 \stackrel{\mathrm{def}}{=} \left(p^2 - q^2 + c\gamma_1 - d\gamma_2\right)^2 + (2pq + d\gamma_1 + c\gamma_2)^2 = \mathrm{const}, \qquad (5)$$

where $c = Mgx_0/C$, $d = Mgy_0/C$. This is a generalization of Kovalevskaya's case 3 and her fourth integral (4). As for cases 1–4 the fourth integral (5) is independent of the integrals (3).

Case 6: $B = C$, $A^2\,(A - 2B)\,x_0^2 = B\,(2A - B)^2\,y_0^2$, $z_0 = 0$.

For case 6, the inequalities of triangle (2) are not satisfied. For case 6, the additional fourth integral I_4 was not written out, and local integrability was checked near the corresponding fixed points for third-order resonances. According to [Bruno, 2007, Section 5.3] the coefficients of the resonance terms of the normal form at $2:1$ resonance should be zero in integrable cases. In this case they are zero in some subcases and in other subcases are non zero.

3. Theory (1)

We have found some general property of integrable cases 1–4, which is formulated below as Hypothesis 2. So we have to compute all those values of parameters A, B, C, x_0, y_0, z_0 for which this property is satisfied. And then, by computing the resonance terms of the normal form of the system (1), to extract from them those values at which the system (1) is integrable.

Hypothesis 1 [Edneral, 2023]

If an autonomous polynomial ODE system is locally integrable in the neighborhood of all its stationary points, then it is globally integrable.

Therefore, to find global integrability, we must first find all the stationary points of the ODE system, and then find out whether the system is locally integrable in their neighborhoods.

Let $X = (p, q, r, \gamma_1, \gamma_2, \gamma_3)$, the point $X = X^0$ is a stationary point in the system (1) and M is a matrix of the linear part of the system (1) near the point X^0. The characteristic polynomial $\chi(\lambda)$ of the matrix M is $\chi(\lambda) = \lambda^6 + a_4\lambda^4 + a_2\lambda^2$. Its discriminant

$$D_\lambda(\chi) = a_4^2 - 4a_2 \tag{6}$$

is a rational function $D = G/H$, where G and H are polynomials.

A stationary point is ***locally integrable*** [Bruno, 2007] if $a_2 < 0$ or $D_\lambda(\chi) < 0$. But this property is not satisfied for definite values of the system parameters (1).

The stationary points of the system (1) form one-dimensional and two-dimensional families $\mathcal{F}_j^l$ in $\mathbb{R}^6$.

Hypothesis 2

If near a stationary point X^0 of the family $\mathcal{F}_j^l$ the system (1) *is locally integrable, then at these parameter values the second discriminant $\Delta\left(\mathcal{F}_j^l\right)$ of the numerator G of the first discriminant $D_\lambda(\chi)$ on the parameter of the family $\mathcal{F}_j^l$ is zero.*

Considering Hypothesis 1, now the search for integrable cases consists of the following 5 steps.

Step 1 Fix the number l of non-zero parameters x_0, y_0, z_0 and find all families $\mathcal{F}_j^l$ of stationary points.

Step 2 Compute the discriminants (6) $D_\lambda(\chi)$ on the families $\mathcal{F}_j^l$.

Step 3 On families $\mathcal{F}_j^l$, compute the second discriminants $\Delta\left(\mathcal{F}_j^l\right)$ of the numerators G of the first discriminants D.

Step 4 Find the values of the parameters of the system (1) at which all $\Delta\left(\mathcal{F}_j^l\right) = 0$ at fixed l.

Step 5 Check the obtained parameter values for integrability by computing the normal forms of the system (1) near stationary points or by finding the fourth integral.

Organization of computations

All the computations described in Section "Theory" were implemented in the CAS Maple. Two different situations had to be implemented separately.

1. If for a family of stationary points it was possible to describe analytically the set of eigenvalues of the characteristic equation of the matrix M as a function of a parameter, then the normal form of the Euler-Poisson system (1) was computed analytically for the whole family.
2. If it was not possible to do so, then, after preliminary simplification of the obtained expressions, the parameter variation interval was set, the set of points on this interval was determined, other parameters of the system were calculated and then the numerical normalization procedure was performed. Such calculations were carried out using a high precision arithmetic with 30 decimal places. At each step, intermediate checks of significance of the obtained results were performed.

4.1. Case $l = 0 : x_0 = y_0 = z_0 = 0$

Then the system (1) has 3 families of stationary points:

$\mathcal{F}_1^0$: $\{q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0\}$, p is a parameter;
$\mathcal{F}_2^0$: $\{p = r = 0,\ \gamma_2 = q/k = \pm 1,\ \gamma_1 = \gamma_3 = 0\}$, q is a parameter;
$\mathcal{F}_3^0$: $\{p = q = 0,\ \gamma_3 = r/k = \pm 1,\ \gamma_1 = \gamma_2 = 0\}$, r is a parameter.

They all have $\Delta(\mathcal{F}_j^0) \equiv 0$ on them, so the system (1) is integrable (**Case 1**).

4.2 Case $l = 1 : x_0 \neq 0, y_0 = z_0 = 0$

Then the system (1) has 4 families of stationary points:

$\mathcal{F}_1^1$: $\{q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0,\ B \neq C\}$, p is a parameter;

$\mathcal{F}_2^1$: $\left\{p = \dfrac{x_0}{k(C-A)}, q = 0,\ \gamma_1 = p/k,\ \gamma_2 = 0, \gamma_3 = s/k,\ \gamma_1^2 + \gamma_3^2 = 1, A \neq C \neq B\right\}$,
r is a parameter;

$\mathcal{F}_3^1$: $\left\{p = \dfrac{x_0}{k(B-A)}, r = 0,\ \gamma_1 = p/k,\ \gamma_2 = q/k, \gamma_3 = 0,\ \gamma_1^2 + \gamma_2^2 = 1, A \neq B \neq C\right\}$,
q is a parameter;

$\mathcal{F}_4^1$: $\left\{p = \dfrac{x_0}{k(B-A)},\ \gamma_1 = p/k,\ \gamma_2 = q/k, \gamma_3 = r/k,\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, A \neq B = C\right\}$,
q, r are parameters.

4.2 Case $l = 1 : x_0 \neq 0, y_0 = z_0 = 0$

For the family $\mathcal{F}_1^1$ the second discriminant is

$$\Delta(\mathcal{F}_1^1) = 4096\,(B - C)^2\,(A - 2C)^2\,(A - 2B)^2\,(A - B - C)^6\,C^2 B^2 A^2 x_0^6.$$

For the family $\mathcal{F}_2^1$, the second discriminant is

$$\Delta(\mathcal{F}_2^1) = 4294967296\,(B-C)^4\,(A-2C)^8\,(A-C)^{42}\,(A+B-C)^{14}$$
$$\big(A^5 - 4A^4B - 4A^4C + 4A^3B^2 + 10A^3BC + 8A^3C^2 - 16A^2B^2C +$$
$$2A^2B\,C^2 - 10A^2C^3 + 21A\,B^2C^2 - 20AB\,C^3 + 7A\,C^4 - 10B^2C^3 + 12B\,C^4 - 2C^5\big)^8$$
$$\big(A^4 - 4A^3B - 2A^3C + 4A^2B^2 + 2A^2BC + 3A^2C^2 - 8A\,B^2C +$$
$$10AB\,C^2 - 2A\,C^3 + B^2C^2 - 2B\,C^3 + C^4\big)^3$$
$$\big(A^2 - 2AB - 2AC + 3BC + C^2\big)^{12}\,A^{28}B^4C^{20}x_0^{28}.$$

4.2. Case $l = 1 : x_0 \neq 0, y_0 = z_0 = 0$

For the family $\mathcal{F}_3^1$, the second discriminant is

$$\begin{aligned}\Delta(\mathcal{F}_3^1) = 4294967296\,(B-C)^4\,(A-2B)^8\,(A-B)^{42}\,(A-B+C)^{14}\\ \big(A^5 - 4A^4B - 4A^4C + 8A^3B^2 + 10A^3BC + 4A^3C^2 - 10A^2B^3 + 2A^2B^2C\\ -16A^2B\,C^2 + 7A\,B^4 - 20A\,B^3C + 21A\,B^2C^2 - 2B^5 + 12B^4C - 10B^3C^2\big)^8\\ \big(A^4 - 2A^3B - 4A^3C + 3A^2B^2 + 2A^2BC + 4A^2C^2 - 2A\,B^3 +\\ 10A\,B^2C - 8AB\,C^2 + B^4 - 2B^3C + B^2C^2\big)^3\\ \big(A^2 - 2AB - 2AC + B^2 + 3BC\big)^{12}\,A^{28}B^{20}C^4x_0^{28}.\end{aligned}$$

For the family $\mathcal{F}_4^1$, the second discriminant is $\Delta(\mathcal{F}_4^1) \equiv 0$.

4.2 Case $l = 1 : x_0 \neq 0, y_0 = z_0 = 0$

All these second discriminants are 0 when:

1. $B = C$ – **Case 2**.
2. $A = B = 2C$ – **Case 3**.
3. $A = 2C$, $B = 3C$.
4. $A = 2C$, $B = \delta C$, where δ is the root of the equation $\delta^3 - 12\delta^2 + 33\delta - 24 = 0$, i.e., $\delta_1 \approx 1{,}194$, $\delta_2 \approx 2{,}387$, $\delta_3 \approx 8{,}419$.

But the check shows that there is no local integrability in items 3 and 4.

4.3 Case $l = 2 : x_0 \neq 0, y_0 \neq 0, z_0 = 0$

Then the system (1) has 2 families of stationary points:

$$\mathcal{F}_1^2 : \left\{ p = \frac{x_0}{k(C-A)}, q = \frac{y_0}{k(C-B)},\ \gamma_1 = p/k,\ \gamma_2 = q/k, \gamma_3 = r/k, \right.$$
$$\left. \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, A \neq C \neq B \right\}, r \text{ is a parameter;}$$

$$\mathcal{F}_2^2 : \left\{ p = -\frac{x_0}{k(A+T)}, q = -\frac{y_0}{k(B+T)}, r = 0,\ \gamma_1 = p/k,\ \gamma_2 = q/k, \gamma_3 = 0, \right.$$
$$\left. \gamma_1^2 + \gamma_2^2 = 1 \right\}, T \text{ is a parameter;}$$

4.3 Case $l = 2 : x_0 \neq 0, y_0 \neq 0, z_0 = 0$

For the family $\mathcal{F}_1^2$, the second discriminant is

$$\begin{aligned}\Delta(\mathcal{F}_1^2) &= 4096\,(B-C)^{16}\,(A-C)^{16}\,(A+B-C)^6\,A^2B^2C^8 \\ &\left(A\,(B-C)\,x_0^2 + B\,(A-C)\,y_0^2\right)^2 \left((B-C)^2\,(A-2C)^2\,x_0^2 + (B-2C)^2\,(A-C)^2\,y_0^2\right)^2 \\ &\Big(A\,(B-C)^2\left(A^2-2AB-2CA+3BC+C^2\right)x_0^2 - \\ &\qquad B\,(A-C)^2\left(2AB-3CA-B^2+2BC-C^2\right)y_0^2\Big)^2.\end{aligned}$$

The second discriminant $\Delta(\mathcal{F}_2^2)$ was obtained during the computation, but we could not factorize it because it contains several hundred thousand monomials. Therefore, we computed $\Delta(\mathcal{F}_2^2)$ on the zeros of $\Delta(\mathcal{F}_1^2)$. We get the following results.

When $B = C$

$$\Delta(\mathcal{F}_2^2) = 4096\,(A-B)^{18}\,(A-2B)^{6}\,A^{10}B^{14}x_0^{12}y_0^{14}f_1(T)f_2(T),$$

where

$$f_1(T) = \left(A^2\,(A-2B)\,x_0^2 - B\,(2A-B)^2\,y_0^2\right)^2,$$

$$f_2(T) = 16A^6\,(A-2B)^{10}\,x_0^{10} - \big(55A^8 - 446A^7B + 901A^6B^2 - 1740A^5B^3 + 2521A^4B^4 - 1782A^3B^5 + 875A^2B^6 - 592A\,B^7 + 128B^8\big)A^4\,(A-2B)^4\,x_0^8y_0^2 - \big(69A^{11} - 1472A^{10}B + 11813A^9B^2 - 35619A^8B^3 + 65349A^7B^4 - 82131A^6B^5 + 68141A^5B^6 - 35361A^4B^7 + 11266A^3B^8 - 2097A^2B^9 + 194A\,B^{10} + 8B^{11}\big)A^2\,(A-2B)^3\,x_0^6y_0^4 + \big(2A^{13} + 583A^{12}B - 11264A^{11}B^2 + 70104A^{10}B^3 - 198026A^9B^4 + 340960A^8B^5 - 413616A^7B^6 + 368453A^6B^7 - 235180A^5B^8 + 101982A^4B^9 - 27652A^3B^{10} + 4011A^2B^{11} - 184A\,B^{12} - 13B^{13}\big)\times A\,(A-2B)^2\,x_0^4y_0^6 - \big(2A^{14} + 159A^{13}B - 3147A^{12}B^2 + 18138A^{11}B^3 - 48400A^{10}B^4 + 73601A^9B^5 - 69996A^8B^6 + 43080A^7B^7 - 17055A^6B^8 + 4188A^5B^9 - 604A^4B^{10} + 29A^3B^{11} + 26A^2B^{12} - 13A\,B^{13} + 2B^{14}\big)\times 8B\,(A-2B)\,x_0^2y_0^8 + 16B^2A\,(2A-B)\,\big(A^2 + AB - B^2\big)^6\,y_0^{10}.$$

4.3. Case $l = 2 : x_0 \neq 0, y_0 \neq 0, z_0 = 0$

Therefore, both Δ are zero at

- $A = B = C$ (**Case 4**),
- $B = C$ and $A^2(A - 2B)x_0^2 = B(2A - B)^2 y_0^2$ (new **Case 6**),
- $A = 2B = 2C$.

Checking shows that the last case is non-integrable.

In the last moment according to new more accurate computations the integrability in Case 6 could not be approved.

With $A = B$

$$\Delta(\mathcal{F}_2^2) = -4A^2C^3(A - 2C).$$

Therefore, when $A = B = 2C$, both Δ are zero. This is *new* **Case 5**.

4.4 Case $l = 3 : x_0 \neq 0, y_0 \neq 0, z_0 \neq 0$

Then the system (1) has one family of stationary points:

$$\mathcal{F}_1^3 : \left\{ p = -\frac{x_0}{k(A+T)}, q = -\frac{y_0}{k(B+T)}, r = -\frac{z_0}{k(C+T)}, \right.$$
$$\left. \gamma_1 = p/k,\ \gamma_2 = q/k, \gamma_3 = r/k,\ \gamma_1^2 + \gamma_2^2 = 1 \right\},$$

T is a parameter;

The first discriminant $D_\lambda(\chi)$ is a 10th degree polynomial of T. To compute its discriminant on T in the generic case it is impossible (discriminant contains ***133881*** monomials). But when $A = B = C$ (**Case 4**) it is zero. When $A = B = 2C$, the second discriminant $\Delta(\mathcal{F}_1^3) = 384A^2 \left(x_0^2 + y_0^2\right)^4 \neq 0$. According to Conjecture 2, this is the non-integrable case.

So here as for the case of $\mathcal{F}_2^2$ we should look for other methods of computing discriminants or more powerful computers as well.

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