Investigation of the Periodic Planar Oscillations of a Two-Body System in an Elliptic Orbit Using the Polynomial Algebra Methods

Gutnik S.A.

MGIMO University,

Moscow Institute of Physics and Technology

Polynomial Computer Algebra Conference 2024, April 15- 20, 2024 St. Petersburg

1. Equations of motion

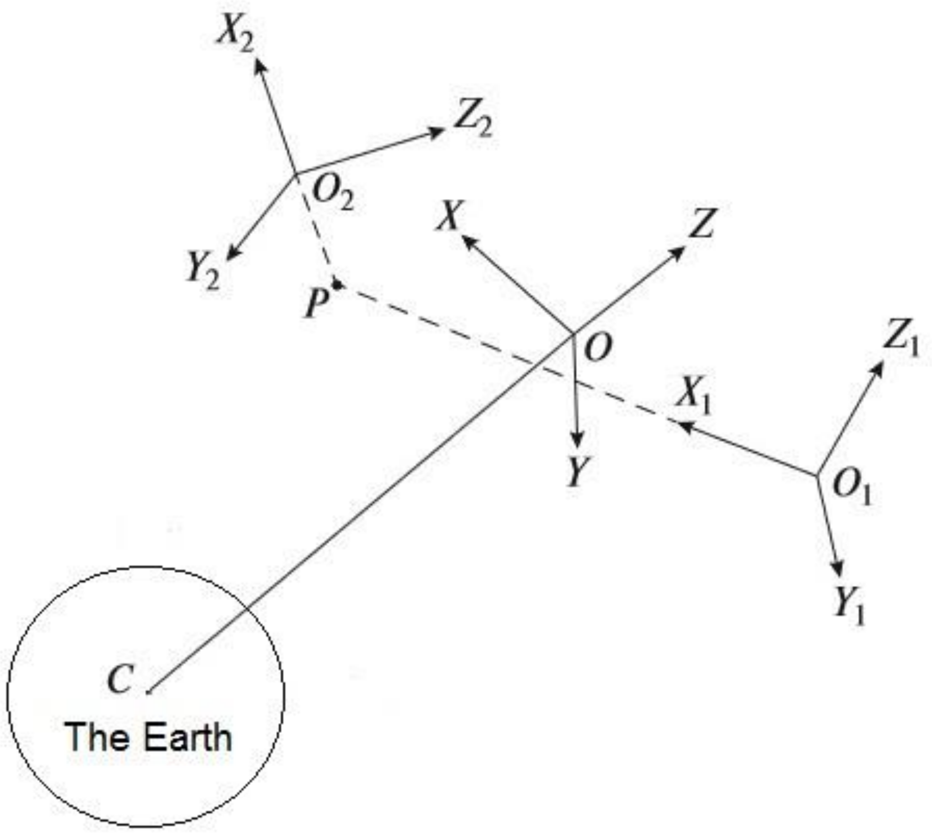

Fig. 1. Basic coordinate systems

$OXYZ$ - is the orbital coordinate system

(a_i, b_i, c_i) – are the coordinates of the spherical hinge P in the body coordinate system $O_i x_i y_i z_i$

1. Equations of motion

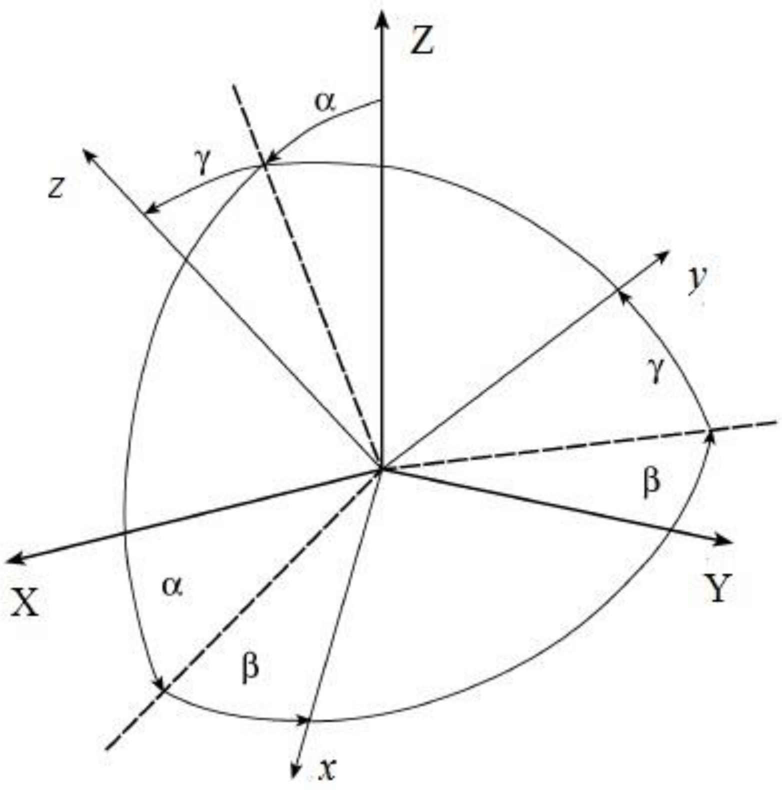

Fig. 2. Orientation of body–fixed axes with respect to the orbital coordinate system

1.1 Equations of motion in the orbital plane

Consider the motion of the two bodies system connected by a spherical hinge around its center of mass in the plane of an elliptic orbit when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, *two* aircraft angles $\beta_1 = \beta_2 = 0$, $\gamma_1 = \gamma_2 = 0$. The expressions for the force function, which determines the effect of the Earth gravitational field on the system of two bodies connected by a spherical hinge in the case $b_1 = b_2 = c_1 = c_2 = 0$ have the form:

$$U = \frac{3M\mu}{2\rho^3}(a_1 \sin\alpha_1 - a_2 \sin\alpha_2)^2 - \frac{3\mu}{2\rho^3}((A_1 - C_1)\sin^2\alpha_1 - (A_2 - C_2)\sin^2\alpha_2) + \frac{M\mu}{\rho^3}a_1 a_2 \cos(\alpha_1 - \alpha_2). \quad (1)$$

Here ρ is a radial distance between the center of mass of the Earth *C* and center of mass of the system *O*; $\mu = fM_0$, where f is a gravitational constant, and M_0 is the mass of the Earth; $\omega = d\vartheta/dt = \omega_0(1 + e\cos\vartheta)^2$; $\dot{\alpha}_1 = d\alpha_1/dt$; $\dot{\alpha}_2 = d\alpha_2/dt$; $\mu/\rho^3 = \omega_0^2(1 + e\cos\vartheta)^3$; ϑ is the true anomaly and e is the orbital eccentricity. On the circular orbit $\omega = \omega_0$, $\mu/\rho^3 = \omega_0^2$, $\vartheta = \omega_0 t$. A_i, B_i, C_i are the principal central moments, $M = M_1 M_2/(M_1 + M_2)$ and M_i is the mass of the i-th body; α_i, β_i, γ_i - are the angles of pitch, yaw and roll; $(a_i, 0, 0)$ - are the coordinates of the of the spherical hinge of the i-th body in reference frame.

1.2 Equations of motion

The expressions for the kinetic energy the system of two bodies connected by a hinge in the case when $b_1=b_2=c_1=c_2=0$ have the form

$$T=\frac{1}{2}(B_1+Ma_1^2)(\dot{\alpha}_1+\omega)^2+\frac{1}{2}(B_2+Ma_2^2)(\dot{\alpha}_2+\omega)^2- \qquad (2)$$
$$-Ma_1a_2\cos(\alpha_1-\alpha_2)(\dot{\alpha}_1+\omega)(\dot{\alpha}_2+\omega).$$

By using the kinetic energy expression (2) and the expression (1) for the force function, the equations of motion for this system can be written as Lagrange equations of the second kind by applying symbolic differentiation D in the *Wolfram Mathematica* 12.1

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}_i}-\frac{\partial T}{\partial \alpha_i}-\frac{\partial U}{\partial \alpha_i}=0, \quad i=1,2.$$

in the form of a system of second-order ordinary differential equations in two variables.

1.3 Equations of motion

The Lagrange equations have the form

$$
\begin{aligned}
&(B_1 + Ma_1^2)\ddot{\alpha}_1 - Ma_1 a_2 \cos(\alpha_1 - \alpha_2)\ddot{\alpha}_2 - \\
&- Ma_1 a_2((\dot{\alpha}_2 + \omega)^2 - \mu/\rho^3)\sin(\alpha_1 - \alpha_2) \\
&+3\mu/\rho^3(((A_1 - C_1) - Ma_1^2)\sin\alpha_1 + Ma_1 a_2 \sin\alpha_2)\cos\alpha_1 + \\
&+((B_1 + Ma_1^2) - Ma_1 a_2 \cos(\alpha_1 - \alpha_2))\dot{\omega} = 0,
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
&-Ma_1 a_2 \cos(\alpha_1 - \alpha_2)\ddot{\alpha}_1 + (B_2 + Ma_2^2)\ddot{\alpha}_2 + \\
&+Ma_1 a_2((\dot{\alpha}_1 + \omega)^2 - \mu/\rho^3)\sin(\alpha_1 - \alpha_2) + \\
&+3\mu/\rho^3(((A_2 - C_2) - Ma_2^2)\sin\alpha_2 + Ma_1 a_2 \sin\alpha_1)\cos\alpha_2 + \\
&+((B_2 + Ma_2^2) - Ma_1 a_2 \cos(\alpha_1 - \alpha_2))\dot{\omega} = 0.
\end{aligned}
\tag{4}
$$

which determine the oscillations of the system in the plane of the elliptic orbit in the orbital coordinate system. In (3-4), the dot denotes differentiation with respect to time. This system has the stationary solution

$$\alpha_1 = \alpha_2 = 0. \tag{5}$$

2.1 Periodic solutions

Our goal is to obtain the periodic solution of the equations of motion (3-4) in the form of a power series in a small parameter e ($e \ll 1$) in the neighborhood of the stationary solution (5).

We assume that the oscillations are small and replace the sine and cosine in (3-4) by their expansions in power series.

Doing in (3-4) the substitution

$$dt = d\vartheta / \omega_0 (1 + e\cos\vartheta)^2,$$

we change the independent variable from t to ϑ and reduce the system to the form of two second order of differential equations.

The planar oscillations of a rigid body on an elliptic orbit are described by

$$(1 + e\cos\vartheta)\alpha'' + 2e\alpha'\sin\vartheta + 3(A - C)/B\sin\alpha\cos\alpha - 2e\sin\vartheta = 0. \quad (6)$$

In the non-resonant case $(m = 3(A - C)/B \neq 1)$ the solutions were found in the form of a power series in e. In the resonant case in the form of a power series in of $e^{1/3}$. It was shown that after the substitution $\alpha = z(1 + e\cos\vartheta)^{-1}$ (6) reduces to the inhomogeneous Hill equation

$$z'' + (m + e\cos\vartheta)(1 + e\cos\vartheta)^{-1} z - 2e\sin\vartheta = 0. \quad (7)$$

2.2 Periodic solutions

System (3-4) reduces to the system on independent variable ϑ

$$
\begin{aligned}
&-(1+e\cos\vartheta)\alpha_2'' + 2e\alpha_2'\sin\vartheta + 3(A_1 - C_1 - Ma_1^2)/Ma_1a_2 + 4) + \\
&+((1+e\cos\vartheta)\alpha_1'' - 2e\alpha_1'\sin\vartheta)(B_1 + Ma_1^2)/Ma_1a_2 - \\
&-e(1+e\cos\vartheta)(\alpha_2'+1)^2 + e(2\sin\vartheta(1-(B_1+Ma_1^2)/Ma_1a_2) = 0, \\
&-(1+e\cos\vartheta)\alpha_1'' + 2e\alpha_1'\sin\vartheta + 3(A_2 - C_2 - Ma_2^2)/Ma_1a_2 + 2) + \\
&+((1+e\cos\vartheta)\alpha_2'' - 2e\alpha_2'\sin\vartheta)(B_2 + Ma_2^2)/Ma_1a_2 + \\
&+e(1+e\cos\vartheta)(\alpha_1'+1)^2 + e(2\sin\vartheta(1-(B_2+Ma_2^2)/Ma_1a_2) = 0.
\end{aligned}
\tag{8}
$$

It is possible to check that a general solution of nonlinear system (8) cannot be found in analytic form.
To find a solution of (8) it is convenient to apply the Krylov-Bogolyubov perturbation techniques and symbolic computations.

2.3 Periodic solutions

We can seek for an approximate solution of (8) in the form of power series in the small parameter e

$$\alpha_1(\vartheta) = e\alpha_1^{(1)}(\vartheta) + e^2\alpha_1^{(2)}(\vartheta) + \dots,$$
$$\alpha_2(\vartheta) = e\alpha_2^{(1)}(\vartheta) + e^2\alpha_2^{(2)}(\vartheta) + \dots. \qquad (9)$$

Computation of unknown functions $\alpha_i^{(k)}(\vartheta)$ in (9) was done in accordance with the techniques proposed by Prokopenya, A.N.[8], which requires quite tedious symbolic computations.
In our work the symbolic computations are performed using *Wolfram Mathematica* functions:

Expand; TrigExpand, Series; Normal, Replace; DSolve; NDSolve

Substituting (9) into (8) and collecting coefficients of equal powers of e, we obtain the set of systems of linear differential equations which can be solved in succession.

2.4 Periodic solutions

For example, using in (9) only the first linear elements we obtain the corresponding periodic solutions in the form

$$\alpha_1^{(1)}(\vartheta) = \bar{a}_1 \sin\vartheta + \bar{b}_1 \cos\vartheta,$$
$$\alpha_2^{(1)}(\vartheta) = \bar{a}_2 \sin\vartheta + \bar{b}_2 \cos\vartheta, \quad (10)$$

where the coefficients $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2$ can be defined from the linear algebraic system.

The amplitudes of the oscillations of the first and the second bodies have the expressions

$$R_1^2 = (\bar{a}_1^2 + \bar{b}_1^2)e^2 = 4\frac{e^2 b^2}{d^2},$$

$$R_2^2 = (\bar{a}_2^2 + \bar{b}_2^2)e^2 = 4\frac{e^2 \bar{b}^2}{d^2}.$$

Here

$$b = (B_1 + Ma_1(a_1 - a_2))(3A_2 - 3C_2 - B_2) - 4Ma_2(a_1B_2 + a_2B_1),$$
$$\bar{b} = (B_2 + Ma_2(a_1 - a_2))(3A_1 - 3C_1 - B_1) - 4Ma_1(a_1B_2 + a_2B_1),$$
$$d = (3A_1 - 3C_1 - B_1)(3A_2 - 3C_2 - B_2) - 4Ma_1^2(3A_2 - 3C_2 - B_2) -$$
$$-4Ma_2^2(3A_1 - 3C_1 - B_1).$$

3. Conclusion

- We have considered the first approximation of the planar oscillations of a system of two bodies connected by a spherical hinge that moves along an elliptic orbit.
- We have found the expressions of the periodic motion of the system in the linear approximation
- At the next step we plan to obtain the quadratic and cubic approximation of the periodic solutions
- To find the periodic solutions, the polynomial algebra methods and computer algebra system *Wolfram Mathematica* were used.

4. References

- 1. *Sarychev V.A.* Problems of orientation of satellites. *Itogi Nauki I Tekhniki.* Ser. Space Research. VINITI, Moscow Vol.11(1978) (in Russian)
- 2. Gutnik, S.A., Sarychev V.A.: Symbolic investigation of the dynamics of a system of two connected bodies moving along a circular orbit. In: England, M., Koepf, W., Sadykov T.M., Seiler, W.M., Vorozhtsov, E.V. (eds.) CASC 2019. LNCS, vol. 11661, pp. 164 -178. Springer, Cham (2019)
- 3. Gutnik, S.A., Sarychev, V.A.: Computer Algebra Methods for Searching the Stationary Motions of the Connected Bodies System Moving in Gravitational Field. Math. Comput. Sci. 16 (2-3) P. 1-15 (2022)
- 4. Zlatoustov, V.A., Markeev, A.P.: Stability of planar oscillations of a satellite in an elliptic orbit. Celest. Mech. 7(1), 31-45 (1973)
- 5. Sarychev, V.A, Mirer, S.A., Sazonov, V.V.: Plane oscillations of a gravitational
- system satellite-stabilizer with maximal speed of response. Acta Astron. 3(9-10), 651-669 (1976)
- 6. Clark, J.P.C.: Response of a two-body gravity gradient system in a slightly eccentric orbit. J. Spacecraft and Rockets. 7(3), 294-298 (1970)
- 7. Wolfram, S.: The Mathematica Book, 5-th ed. Wolfram media, Inc. Cham (2003)
- 8. Prokopenya, A.N.: Construction of a periodic solution to the equations of motions of generalized Atwood's machine using computer algebra. *Programming and Computer Software* 46(2), 120-125 (2020)