Investigation of the Periodic Planar Oscillations of a Two-Body System in an Elliptic Orbit Using the Polynomial Algebra Methods

Sergey A. Gutnik

Abstract. Computer algebra methods are used to investigate the planar oscillations of a system of two bodies connected by a spherical hinge that moves along an elliptic orbit under the action of gravitational torque in the plane of the orbit. The two-body system motion on an elliptic orbit is described by the second order system of differential equations with the periodic coefficients. Applying the perturbation techniques the periodic solution of the equations of motion is constructed in the form of power series in a small parameter. Using the proposed approach it is shown that the motion of the two-body system is described by periodic oscillations in the plane of an elliptic orbit. All the relevant symbolic computations are performed with the help of computer algebra systems.

Introduction

This work presents the results of investigation the dynamics of a two-body system (satellite and stabilizer) connected by a spherical hinge that moves in gravitational field in the plane of an elliptical orbit using polynomial algebra methods. The dynamics of various schemes for satellite-stabilizer gravitational orientation systems on a circular orbit was discussed in many papers, some review of them can be found in papers [1, 2, 3].

Since the problem is very complicated, in the previous works we studied the equilibrium orientations of the system on a circular orbit only in the simplest cases when the spherical hinge is located at the intersection of the satellite and stabilizer principal central axis of inertia and in the case where the spherical hinge is positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer [3, 4, 5, 6, 7]. The application of computer algebra makes it possible to find the solutions of this problem.

On a circular orbit, there are spatial oscillations of a system of two connected bodies at the vicinity of equilibria. In paper [8], the eigenoscillations of a system of two bodies were studied and the parameters of the system, optimal in terms of speed, were found for the transition of the system to equilibrium. A detailed study of the oscillations of a satellite (a rigid body) in the plane of an elliptical orbit and the conditions for their stability were carried out in [9].

The works devoted to the study of planar oscillations of a system of two coupled bodies on an elliptic orbit were carried out only for simple cases, when the centers of mass of the first and second bodies coincide [10], [11]. Here, we consider the planar oscillations of a two-body system on an elliptic orbit in case when the spherical hinge is located at the intersection of the first and second body principal central axis of inertia. Applying the perturbation techniques and appropriate symbolic computations with the help of computer algebra system Wolfram Mathematica [12], we construct the periodic solution in the form of a power series in a small parameter.

1. Equations of Motion

We consider the problem of two bodies connected by a spherical hinge that move on an elliptic orbit. To write the equations of motion of two-body system, we introduce the following right-handed Cartesian coordinate systems: $OXYZ$ is the orbital coordinate system, the OZ axis is directed along the radius vector connecting the Earth center of mass C and the center of mass O of the two-body system, the OX axis is directed along the linear velocity vector of the center of mass O, and the OY axis coincides with the normal to the orbital plane. The axes of coordinate systems $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$, are directed along the principal central axes of inertia of the first and the second body, respectively. The orientation of the coordinate system $O_ix_iy_iz_i$ with respect to the orbital coordinate system is determined by the aircraft angles α_i (pitch), β_i (yaw), and γ_i (roll) (see [3]).

Suppose that (a_i, b_i, c_i) are the coordinates of the spherical hinge P in the body coordinate system $Ox_iy_iz_i$, A_i, B_i, C_i are principal central moments of inertia; $M_1M_2/(M_1+M_2) = M$; M_i is the mass of the ith body; ω is the angular velocity for the center of mass of the two-body system moving along an elliptic orbit. Then we use the expressions for kinetic energy of the system in the case when $b_1 = b_2 = c_1 = c_2 = 0$ and the coordinates of the spherical hinge P in the body coordinate systems are $(a_i, 0, 0)$ and when the motions of the two-body system are located in the plane of the elliptic orbit ($\alpha_1 \neq 0, \alpha_2 \neq 0, \beta_1 = \beta_1 = 0$, $\gamma_1 = \gamma_2 = 0$, $\dot{\alpha_1} = d\alpha_1/dt$, $\dot{\alpha_2} = d\alpha_2/dt$, where t is time) in the form [1]

$$\begin{aligned} T &= 1/2\big(B_1 + Ma_1^2\big)(\dot{\alpha_1} + \omega)^2 + 1/2\big(B_2 + Ma_2^2\big)(\dot{\alpha_2} + \omega)^2 \\ &- Ma_1a_2\cos(\alpha_1 - \alpha_2)(\dot{\alpha_1} + \omega)(\dot{\alpha_2} + \omega). \end{aligned} \tag{1}$$

The force function, which determines the effect of the Earth gravitational field on the system of two connected by a hinge bodies, is given by [1]

$$\begin{aligned} U &= -3\mu/(2\rho^3)\big((A_1 - C_1)\sin^2\alpha_1 + (A_2 - C_2)\sin^2\alpha_2\big) \\ &+ 3/2M\mu/\rho^3\big((a_1\sin\alpha_1 - a_2\sin\alpha_2\big)^2 + M\mu/\rho^3 a_1 a_2\cos(\alpha_1-\alpha_2). \end{aligned} \quad (2)$$

Here ρ is a radial distance between the center of mass of the Earth C and center of mass of the system O; $\mu = fM_0$, where f is a gravitational constant, and M_0 is the mass of the Earth; $\omega = \frac{d\vartheta}{dt} = \omega_0(1+e\cos\vartheta)^2$; $\frac{\mu}{\rho^3} = \omega_0^2(1+e\cos\vartheta)^3$; ϑ is the true anomaly and e is the orbital eccentricity. On the circular orbit $\omega = \omega_0$, $\frac{\mu}{\rho^3} = \omega_0^2$, $\vartheta = \omega_0 t$.

By using the kinetic energy expression (1) and the expression (2) for the force function, the equations of motion for this system can be written as Lagrange equations of the second kind by applying symbolic differentiation in the Wolfram Mathematica system [12], [13]

$$\frac{d}{dt}\frac{\partial T}{\partial\dot\alpha_i} - \frac{\partial T}{\partial\alpha_i} - \frac{\partial U}{\partial\alpha_i} = 0, \quad i = \overline{1,2}, \quad (3)$$

in the form of a system of second-order ordinary differential equations in variables α_1 and α_2 [1]

$$\begin{aligned} & (B_1 + Ma_1^2)(\ddot\alpha_1 + \dot\omega) - Ma_1a_2(\ddot\alpha_2 + \dot\omega)\cos(\alpha_1-\alpha_2) \\ - & Ma_1a_2\big((\dot\alpha_2+\omega)^2 - \mu/\rho^3\big)\sin(\alpha_1-\alpha_2) \\ + & 3\mu/\rho^3\big((A_1 - C_1 - Ma_1^2)\sin\alpha_1 + Ma_1a_2\sin\alpha_2\big)\cos\alpha_1 = 0, \\ - & Ma_1a_2(\ddot\alpha_1+\dot\omega)\cos(\alpha_1-\alpha_2) + (B_1 + Ma_1^2)(\ddot\alpha_2+\dot\omega) \\ + & Ma_1a_2\big((\dot\alpha_1+\omega)^2 - \mu/\rho^3\big)\sin(\alpha_1-\alpha_2) \\ + & 3\mu/\rho^3\big((A_2 - C_2 - Ma_2^2)\sin\alpha_2 + Ma_1a_2\sin\alpha_1\big)\cos\alpha_2 = 0, \end{aligned} \quad (4)$$

which determine the oscillations of the system in the plane of the elliptic orbit in the orbital coordinate system. In (4), the dot denotes differentiation with respect to time t. One can easily check that the system (4) has the stationary solution

$$\alpha_1 = \alpha_2 = 0. \quad (5)$$

Our goal is to obtain the periodic solution of the equations of motion (4) in the form of a power series in a small parameter e ($e \ll 1$) in the neighborhood of the stationary solution (5) with the help of computer algebra system.

2. Periodic solutions

To perform the calculations we assume that the oscillations are small and replace the sine and cosine in (4) by their expansions in power series. Doing the substitution $dt = d\vartheta/(\omega_0(1+e\cos\vartheta)^2)$ in (4) we change the independent variable from t to ϑ and reduce the system to the form

$$
\begin{aligned}
- &\quad (1+e\cos\vartheta)\alpha_2'' + 2e\alpha_2'\sin\vartheta + (B_1+Ma_1^2)/(Ma_1a_2)\big((1+e\cos\vartheta)\alpha_1{}''\\
- &\quad 2e\alpha_1'\sin\vartheta\big) - e(1+e\cos\vartheta)(\alpha_2'+1)^2 + e\big(2\sin\vartheta(1-(B_1+Ma_1^2)/Ma_1a_2)\\
+ &\quad (4+3\big((A_1-C_1)-Ma_1^2\big)/(Ma_1a_2))\big) = 0, \qquad (6)\\
- &\quad (1+e\cos\vartheta)\alpha_1'' + 2e\alpha_1'\sin\vartheta + (B_2+Ma_2^2)/(Ma_1a_2)\big((1+e\cos\vartheta)\alpha_2{}''\\
- &\quad 2e\alpha_1'\sin\vartheta\big) + e(1+e\cos\vartheta)(\alpha_1'+1)^2 + e\big(2\sin\vartheta(1-(B_2+Ma_2^2)/Ma_1a_2)\\
+ &\quad (2+3\big((A_2-C_2)-Ma_2^2\big)/(Ma_1a_2))\big) = 0.
\end{aligned}
$$

The prime in (6) denotes differentiation with respect to ϑ. It is possible to check that a general solution of nonlinear system (6) cannot be found in analytic form. It is convenient for application of the perturbation techniques [14] and symbolic algorithms proposed in paper [15]. However, we can seek for an approximate solution in the form of power series in the small parameter e:

$$\alpha_i(\vartheta) = e\alpha_i{}^{(1)}(\vartheta) + e^2\alpha_i{}^{(2)}(\vartheta) + ..., \qquad (7)$$

Computation of unknown functions $\alpha_i(\vartheta)$ in (7) is done in accordance with the techniques proposed in [14] and [15] requires quite tedious symbolic computations. In this paper symbolic computations are performed using Wolfram Mathematica functions: Expand, TrigExpand, Series, Normal, Replace, DSolve, NDSolve.

Substituting (7) into (6) and collecting coefficients of equal powers of e, we obtain the set of systems of linear differential equations which can be solved in succession. For example, using in (7) only the first linear elements we obtain the corresponding periodic solutions in the form

$$\alpha_1^{(1)}(\vartheta) = \bar{a}_1\sin(\vartheta) + \bar{b}_1\cos(\vartheta), \quad \alpha_2^{(1)}(\vartheta) = \bar{a}_2\sin(\vartheta) + \bar{b}_2\cos(\vartheta), \qquad (8)$$

where the coefficients $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2$ can be defined from the linear algebraic system. The amplitudes of the oscillations of the first and the second bodies have the expressions

$$R_1^2 = (\bar{a}_1^2 + \bar{b}_1^2)e^2 = 4\frac{e^2b^2}{d^2}, \quad R_2^2 = (\bar{a}_2^2 + \bar{b}_2^2)e^2 = 4\frac{e^2\bar{b}^2}{d^2}, \qquad (9)$$

where

$$
\begin{aligned}
b &= (B_1 + Ma_1(a_1 - a2))(3(A_2 - C_2) - B_2) - 4Ma_2(a_1B_2 + a_2B_1),\\
\bar{b} &= (B_2 + Ma_2(a_1 - a2))(3(A_1 - C_1) - B_1) - 4Ma_1(a_1B_2 + a_2B_1), \quad (10)\\
d &= (3(A_1 - C_1) - B_1)(3(A_2 - C_2) - B_2) - 4Ma_1^2(3(A_2 - C_2) - B_2)\\
&- 4Ma_2^2(3(A_1 - C_1) - B_1).
\end{aligned}
$$

In the present work, we have considered the first approximation of the planar oscillations of a system of two bodies connected by a spherical hinge that moves along an elliptic orbit. We have found the expressions of the periodic motion of the system in the linear approximation. All the relevant computations in this work are performed with the computer algebra system Wolfram Mathematica. At the next step we will do the quadratic and cubic approximation of the periodic solutions which have very cumbersome expressions.

References

[1] Sarychev, V.A.: Problems of orientation of satellites, Itogi Nauki i Tekhniki. Ser. Space Research, Vol. 11. VINITI, Moscow (1978) (in Russian)

[2] Rauschenbakh, B.V., Ovchinnikov, M.Yu., McKenna–Lawlor, S.: Essential Spaceflight Dynamics and Magnetospherics. Kluwer Academic Publishers (2003)

[3] Gutnik, S.A., Sarychev, V.A.: Symbolic computations of the equilibrium orientations of a system of two connected bodies moving on a circular orbit around the Earth. Math. Comput. Sci. **15**(2), 407–417 (2021)

[4] Sarychev, V.A.: Relative equilibrium orientations of two bodies connected by a spherical hinge on a circular orbit. Cosmic Research **5**, 360–364 (1967)

[5] Gutnik, S.A., Sarychev V.A.: Symbolic investigation of the dynamics of a system of two connected bodies moving along a circular orbit. In: England, M., Koepf, W., Sadykov, T.M., Seiler, W.M., Vorozhtsov, E.V. (eds.) CASC 2019. LNCS, vol. 11661, 164–178. Springer, Cham (2019)

[6] Gutnik, S.A., Sarychev, V.A.: Symbolic methods for studying the equilibrium orientations of a system of two connected bodies in a circular orbit. Programming and Computer Software **48**(2), 73–79 (2022)

[7] Gutnik, S.A., Sarychev, V.A.: Computer algebra methods for searching the stationary motions of the connected bodies system moving in gravitational field. Math. Comput. Sci. **16**, 1–15 (2022)

[8] Sarychev, V.A, Mirer, S.A., Sazonov, V.V.: Plane oscillations of a gravitational system satellite-stabilizer with maximal speed of response. Acta Astronaut. **3**(9-10), 651–669 (1976)

[9] Zlatoustov, V.A., Markeev, A.P.: Stability of planar oscillations of a satellite in an elliptic orbit. Celest. Mech. **7**(1), 31–45 (1973)

[10] Penkov, V.I.: Compensation for eccentric oscillations of a satellite with a gravitational stabilization system. Cosmic Research **15**(3), 376–383 (1977)

[11] Clark, J.P.C.: Response of a two-body gravity gradient system in a slightly eccentric orbit. J. Spacecraft and Rockets. **7**(3), 294–298 (1970)

[12] Wolfram, S.: An elementary introduction to the Wolfram Language. 2nd ed. Champaign, IL, USA, Wolfram media (2017)

[13] Wolfram Mathematica website. http://www.wolfram.com/mathematica

[14] Nayfeh, A.H.: Introduction to Perturbation Techniques. John Wiley and Sons, New York (1993)

[15] Prokopenya, A.N.: Symbolic computation in studying stability of solutions of linear differential equations with periodic coefficients. Programming and Computer Software **33**(2), 60–66 (2007)

Sergey A. Gutnik
76, Prospekt Vernadskogo, 119454
Department of Mathematics, Econometrics and Informatics
MGIMO University
Moscow, Russia
e-mail: s.gutnik@inno.mgimo.ru