On the integrability of two- and three-dimensional dynamical systems with a quadratic right-hand side in cases of resonances in linear parts and in cases of general position

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Аннотация. We use a heuristic method that allows one to a prior determine the cases of integrability of an autonomous dynamical systems with a polynomial right-hand side. We demonstrate the capabilities of the method using examples of two and three-dimensional dynamic systems with quadratic nonlinearity. A significant advance relative to our previous works is the possibility of studying systems of a general form, without resonances in the linear parts.

Introduction

In previous works [1, 2] a technique was described for constructing systems of algebraic equations for the parameters of an ODE system with resonance in its linear part. It was experimentally shown that with relations on the parameters obtained as a result of solving such a system, it is usually possible to find explicit expressions for the first integrals of the ODE.

This report discusses the use of this technique to search for first integrals of two and three-dimensional systems with quadratic nonlinearity on the right-hand sides.

1. Two-dimensional case

For the center case

$$\begin{aligned} \dot{x} &= \quad y + a_1 x^2 + a_2 x\, y + a_3 y^2, \\ \dot{y} &= -x + b_1 x^2 + b_2 x\, y + b_3 y^2, \end{aligned} \tag{1}$$

the first integrals can be calculated for the systems:

$$
\begin{array}{lll}
1 & \dot{x} = y + a_1x^2 - 2\,b_3xy + a_3y^2, & \dot{y} = -x + b_1x^2 - 2\,a_1xy + b_3y^2; \\
2 & \dot{x} = y + a_1x^2 + a_2xy - a_1y^2, & \dot{y} = -x - b_3x^2 + b_2xy + b_3y^2. \\
3 & \dot{x} = y + a_2xy, & \dot{y} = -x + b_1x^2 + b_3y^2; \\
4 & \dot{x} = y + a_1x^2 - 2\,b_3xy, & \dot{y} = -x + b_1x^2 - 2\,a_1xy + b_3y^2; \\
5 & \dot{x} = y + a_1x^2 - 2b_3xy - a_1y^2, & \dot{y} = -x + b_1x^2 - 2a_1xy + b_3y^2; \\
6 & \dot{x} = y + a_1x^2 + a_2xy - a_1y^2, & \dot{y} = -x - b_3x^2 - 2a_1xy + b_3y^2. \\
7 & \dot{x} = y + a_2xy - b_2y^2, & \dot{y} = -x - a_2x^2 + b_2xy; \\
8 & \dot{x} = y + a_1x^2 + \frac{1}{5}\,(-3a_1 + b_2)y^2, & \dot{y} = -x + b_2xy; \\
9 & \dot{x} = y + \frac{1}{5}\,b_2y^2, & \dot{y} = -x + b_2xy; \\
10 & \dot{x} = y - b_2xy + \frac{1}{5}\,b_2y^2, & \dot{y} = -x - \frac{1}{5}\,b_2x^2 + b_2xy; \\
11 & \dot{x} = y + b_2xy + \frac{1}{5}\,b_2y^2, & \dot{y} = -x + \frac{1}{5}\,b_2x^2 + b_2xy; \\
12 & \dot{x} = y + a_1x^2 - \frac{5}{6}\,a_1y^2, & \dot{y} = -x - \frac{7}{6}\,a_1xy; \\
13 & \dot{x} = y + a_1x^2 - \frac{2}{3}\,a_1y^2, & \dot{y} = -x - \frac{4}{3}\,a_1xy.
\end{array}
$$

In case 2 above we can prove the existence of an integrating factor, but the first integral is too cumbersome to handle. We are grateful to Prof. M.V. Demina for the calculation of the first integral in case 6.

For the saddle case

$$
\begin{aligned}
\dot{x} &= \alpha x + a_1x^2 + a_2x\,y + a_3y^2, \\
\dot{y} &= -y + b_1x^2 + b_2x\,y + b_3y^2,
\end{aligned} \tag{2}
$$

at the resonance 1:1, i.e. at $\alpha = 1$ we got the first integrals for the systems:

$$
\begin{array}{lll}
1 & \dot{x} = x + b_2b_3/a_2\,x^2 + a_2xy + a_3y^2, & \dot{y} = -y + a_3b_2^3/a_2^3\,x^2 + b_2xy + b_3y^2; \\
2 & \dot{x} = x + a_1x^2 + a_3y^2, & \dot{y} = -y + b_1x^2 + b_3y^2; \\
3 & \dot{x} = x - \frac{1}{2}\,b_2x^2 + a_2xy + a_3y^2, & \dot{y} = -y + b_1x^2 + b_2xy - \frac{1}{2}\,a_2y^2; \\
4 & \dot{x} = x + a_1x^2, & \dot{y} = -y + b_1x^2 + b_2xy; \\
5 & \dot{x} = x + 2b_2x^2 + a_2xy + a_3y^2, & \dot{y} = -y + \frac{a_2b_2}{a_3}x^2 + b_2xy + 2a_2y^2; \\
6 & \dot{x} = x - b_2/2x^2 + a_2xy, & \dot{y} = -y + b_2xy - a_2/2y^2; \\
7 & \dot{x} = x + 2b_2x^2 + a_2xy, & \dot{y} = -y + b_2xy + 2a_2y^2.
\end{array}
$$

For the resonance $2:1$, i.e. at $\alpha = 2$ we got the first integrals for the systems:

$$
\begin{array}{lll}
1 & \dot{x} = 2x + \frac{2b_2b_3x^2}{a_2+b_3} + a_2xy, & \dot{y} = -y + b_2xy + b_3y^2; \\
2 & \dot{x} = 2x - \frac{2}{3}\,b_2x^2 - 4b_3xy, & \dot{y} = -y + b_1x^2 + b_2xy + b_3y^2; \\
3 & \dot{x} = 2x + a_1x^2 - b_3xy, & \dot{y} = -y + b_1x^2 + b_3y^2; \\
4 & \dot{x} = 2x + a_1x^2, & \dot{y} = -y + b_1x^2 + b_2xy; \\
5 & \dot{x} = 2x + \frac{4}{3}\,b_2x^2 + \frac{1}{2}\,b_3xy, & \dot{y} = -y + b_2xy + b_3y^2; \\
6 & \dot{x} = 2x + \frac{1}{2}\,b_3xy, & \dot{y} = -y + b_1x^2 + b_3y^2; \\
7 & \dot{x} = 2x + a_3y^2, & \dot{y} = -y.
\end{array}
$$

The results above were got by solving the algebraic systems on the system parameters. Each of these algebraic system was calculated for a certain resonance, i.e. for the fixed natural parameter α. But the form of all these equations, their variables, are the same. The idea arises to look for a general solution of systems for several resonances. We solved a combined system for 1:1, 2:1 and 3:1 resonances.

For each set of parameters obtained as a result of solving this unified system, it was possible to calculate the first integrals of system (2) for an arbitrary (symbolic) α. These systems are:

$$
\begin{array}{rll}
1 & \dot{x} = \alpha x + a_1 x^2, & \dot{y} = -y + b_1 x^2 + b_3 y^2; \\
2 & \dot{x} = \alpha x + a_1 x^2, & \dot{y} = -y + b_1 x^2 + b_2 xy; \\
3 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y + b_3 y^2; \\
4 & \dot{x} = \alpha x + 2b_2 x^2 + a_3 y^2, & \dot{y} = -y + b_2 xy; \\
5 & \dot{x} = \alpha x + a_2 xy, & \dot{y} = -y + b_2 xy; \\
6 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y; \\
7 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y - \frac{1}{2} a_2 y^2; \\
8 & \dot{x} = \alpha x + b_2 x^2 + a_2 xy, & \dot{y} = -y + b_2 xy + a_2 y^2; \\
9 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y + a_2 y^2; \\
10 & \dot{x} = \alpha x + a_2 xy, & \dot{y} = -y + b_1 x^2 + 2a_2 y^2; \\
11 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y + 2a_2 y^2.
\end{array}
$$

The first integrals for the systems above were calculated by the procedure DSolve of the MATHEMATICA-11 system or by hand using the Darboux method.

Three dimension case

First we considered resonant cases of the system

$$
\begin{aligned}
\dot{x} &= M_x\, x + a_2\, x\, y + a_4\, x\, z + a_5\, y\, z, \\
\dot{y} &= -M_y\, y + b_2\, x\, y + b_4\, x\, z + b_5\, y\, z, \\
\dot{z} &= -z + c_2\, x\, y + c_4\, x\, z + c_5\, y\, z
\end{aligned}
\tag{3}
$$

with natural M_x, M_y on the square table $\{1,2,3\} \times \{1,2,3\}$

In the two-dimensional case, we struggled to evaluate each integral. But here we limited ourselves to calculations only using the DSolve procedure of the MATHEMATICA 13.3.1.0 system. The result is in the table

N	M_x	M_y	Alg.solutions	Integrals	% Success
8	1	1	23	19	83
8	1	2	16	12	75
8	1	3	25	19	76
8	2	1	57	49	86
8	2	2	34	29	85
8	2	3	43	35	81
9	3	1	60	51	85
9	3	2	63	58	92
10	3	3	43	38	88

"N" here is the normal form order, "Alg.solutions" is a number of rational solutions of the corresponding algebraic system, the "Integrals" is a number of success solutions by the MATHEMATICA and "Success" is the percentage of successfully integrated cases to the total number of solutions of the corresponding algebraic system.

Then we solved the unified algebraic system of 329 equations, found its 10 solutions, and found that system MATHEMATICA-11 solves all corresponding systems of ODEs of the form (3) except one. These systems with arbitary M_x and M_y are:

1	$\dot{x} = M_x x + a_2 x\cdot y + a_4 x\cdot z + a_5 y\cdot z,$	$\dot{y} = -M_y y + b_5 y\cdot z,$	$\dot{z} = -z + c_5 y\cdot z;$
2	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_2 x\cdot y + b_4 x\cdot z,$	$\dot{z} = -z + c_4 x\cdot z;$
3	$\dot{x} = M_x x + a_2 x\cdot y + a_4 x\cdot z + a_5 y\cdot z,$	$\dot{y} = -M_y y + a_4 y\cdot z,$	$\dot{z} = -z - a_2 y\cdot z;$
4	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_2 x\cdot y,$	$\dot{z} = -z + c_4 x\cdot z;$
5	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_4 x\cdot z,$	$\dot{z} = -z + c_4 x\cdot z;$
6	$\dot{x} = M_x x,$	$\dot{y} = -M_y y,$	$\dot{z} = -z + c_4 x\cdot z + c_5 y\cdot z;$
7	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_2 x\cdot y + b_5 y\cdot z,$	$\dot{z} = -z;$
8	$\dot{x} = M_x x + a_4 x\cdot z,$	$\dot{y} = -M_y y + b_4 x\cdot z + a_4 y\cdot z,$	$\dot{z} = -z;$
9	$\dot{x} = M_x x + a_5 y\cdot z,$	$\dot{y} = -M_y y + b_2 x\cdot y,$	$\dot{z} = -z - b_2 x\cdot z;$ Nonintegrable?
10	$\dot{x} = M_x x,$	$\dot{y} = -M_y y,$	$\dot{z} = -z + c_4 x\cdot z.$

2. The general three-dimension system

Finally, we considered the general case of a three-dimensional system with 20 parameters

$$\begin{aligned}
\dot{x} &= \quad M_x x + a_1 x^2 + a_2 x\cdot y + a_3 y^2 + a_4 x\cdot z + a_5 y\cdot z + a_6 z^2,\\
\dot{y} &= -M_y y + b_1 x^2 + b_2 x\cdot y + b_3 y^2 + b_4 x\cdot z + b_5 y\cdot z + b_6 z^2,\\
\dot{z} &= \quad - z + c_1 x^2 + c_2 x\cdot y + c_3 y^2 + c_4 x\cdot z + c_5 y\cdot z + c_6 z^2.
\end{aligned}$$

Calculating the normal form up to 6th order for 4 pairs $\{M_x, M_y\}=\{1,1\}$, $\{1,2\}$, $\{2,1\}$ and $\{2,2\}$, we got a system of 121 equations with 18 parameters. We calculated 174 of its solutions. For 109 of them the MATHEMATICA-13 system calculated solutions of the corresponding ODEs.

3. Conclusion

There are many cases of integrability of dynamical systems, and the corresponding exact solutions can be very useful in applications, for example, in problems of chemical kinetics [3].

Список литературы

[1] Edneral V.F., *Integrable Cases of the Polynomial Liénard-type Equation with Resonance in the Linear Part.* Mathematics in Computer Science **17** 19 (2023). https://doi.org/10.1007/s11786-023-00567-6

[2] Aranson A.B., Edneral V.F., *On integrability of the resonant cases of the generalized Lotka–Volterra system.* International Conference on Polynomial Computer Algebra PCA-2023, VVM publishing, St. Petrsburg, ISBN: 978-5-9651-1473-3 (2023) 38–42.

[3] Корзухин М. Д., Жаботинский А. М. *Математическое моделирование химических и экологических автоколебательных систем* // М.: Наука (1965).

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