Divergent Fourier Series and summation in Finite Terms using the Krylov Method in CAS

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Ksaverii Malyshev
(SINP MSU, Patrice Lumumba RUDN)

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Problem formulation

We investigate the application of computer algebra systems to the summation of trigonometric Fourier series. Fourier series associated with problems of mathematical physics are not analytic functions of a complex argument. Therefore, an attempt to find the sum of the Fourier series in closed form using CAS leads to transcendental functions. At the same time, often these sums are elementary functions of a real variable, piecewise given elementary functions. This class of functions is not included in the class of elementary Liouville functions.
In this talk, we present the first functions of the «Kryloff for Sage» software package, which make it possible to determine at least some of the cases in which the Fourier series represents an elementary function of a real argument.

Start point: simplest Green's function

$$\begin{cases} \dfrac{\partial^2 g}{\partial t^2} = \dfrac{\partial^2 g}{\partial x^2}, 0 < x < \pi, t > \tau, 0 \leq \tau < +\infty; \\ g|_{t=\tau} = 0,\ \dfrac{\partial g}{\partial t}\Big|_{t=\tau} = \delta(x - s), 0 < x < \pi, 0 < s < \pi; \\ g|_{x=0} = 0,\ g|_{x=l} = 0,\ \tau < t < +\infty \end{cases} \quad (1)$$

Here $\delta(x - s)$ is Dirac delta function. The Green's function g can be found as a Fourier series [Tikhonov, Samarskii]:

$$g = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n(t - \tau) \sin ns \sin nx. \quad (2)$$

This series has been studied by many authors, but is not presented in the available literature in finite terms.

Start point: simplest Green's function

Theorem

The Green's function g can be presented in finite terms as the expression:

$$g = \frac{1}{2\pi}\left(\operatorname{atan}\left(\cot\frac{t-\tau-x+s}{2}\right)+\operatorname{atan}\left(\cot\frac{t-\tau+x-s}{2}\right)-\right.$$
$$\left.-\operatorname{atan}\left(\cot\frac{t-\tau-x-s}{2}\right)-\operatorname{atan}\left(\cot\frac{x+s+t-\tau}{2}\right)\right)$$

Alternative expression has the form:

$$g = \frac{1}{2}\left(\left[\frac{x-s+T}{2\pi}\right]+\left[\frac{-x+s+T}{2\pi}\right]\right.$$
$$\left.-\left[\frac{-x-s+T}{2\pi}\right]-\left[\frac{x+s+T}{2\pi}\right]\right).$$

Here $[\cdot]$ means «floor», $T = t - \tau$.

Our Research

Leaving aside further mathematical studies of Green's functions, we fix the following questions:

- Can we find such kind of expressions in finite terms of Fourier series in moderm CAS?
- What kind or expressions for series can we find for Green's functions and other Fourier series using computer instruments?
- Is it possible to systematize the cases in which the sum of the series presents in finite terms? Does such a systematization allow implementation in CAS?

Fourier series summation, g

$$sum\left(\frac{1}{n}\cdot\sin(n\cdot x)\cdot\sin(n\cdot s)\cdot\sin(n\cdot t),\ n=1..\text{infinity}\right) \text{ assuming } x>0,\ s>0,\ t>0;$$

$$\frac{I}{8}\Big(\ln\left(1-e^{-I(t+x+s)}\right)-\ln\left(1-e^{-I(-t+x+s)}\right)-\ln\left(1-e^{I(-t-x+s)}\right)+\ln\left(1-e^{I(t-x+s)}\right)-\ln\left(1-e^{-I(t-x+s)}\right)+\ln\left(1-e^{-I(-t-x+s)}\right)$$
$$+\ln\left(1-e^{I(-t+x+s)}\right)-\ln\left(1-e^{I(t+x+s)}\right)\Big)$$

The simplest Green's function g in finite terms in CAS Maple'2019. We can see satisfactory result, but too difficult for users. We cannot find this result without assuming, because this representation does not hold for arbitrary complex values of x, s, t.

Fourier series summation, g

Терминал - IPython: Документы/article-3

Файл Правка Вид Терминал Вкладки Справка

```
sage: + y)) + cos(t + 2*x + y)*e^(cos(t - x + y)) - I*e^(cos(t - x + y))*sin(2*t + x + 2*y) - I*e^(cos(t - x + y))*sin(t + 2*x
....:  + y))*sin(3*t + 3*y))*cos(sin(t - x + y)) + 1/4*((I*cos(3*t) - sin(3*t))*cos(2*t + x + y)*e^(cos(-t + x + y)) + (I*cos(
....: 3*t) - sin(3*t))*cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) + (cos(3*t) + I*sin(3*t))*e^(cos(-t + x + y))*sin(2*t + x + y) +
....:  (cos(3*t) + I*sin(3*t))*e^(cos(-t + x + y))*sin(t + 2*x + 2*y) + (-I*cos(2*t + x + y)*e^(cos(-t + x + y)) - I*cos(t + 2
....: *x + 2*y)*e^(cos(-t + x + y)) - e^(cos(-t + x + y))*sin(2*t + x + y) - e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*cos(3*x +
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....: n(3*t + 3*x))*e^(cos(-t - x + y)) + I*cos(3*y)*e^(cos(-t - x + y)) - e^(cos(-t - x + y))*sin(3*y))*cos(2*t + 2*x + y) +
....: ((-I*cos(3*t + 3*x) + sin(3*t + 3*x))*e^(cos(-t - x + y)) + I*cos(3*y)*e^(cos(-t - x + y)) - e^(cos(-t - x + y))*sin(3*y
....: ))*cos(t + x + 2*y) - ((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(-t - x + y)) - I*e^(cos
....: (-t - x + y))*sin(3*y))*sin(2*t + 2*x + y) - ((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(
....: -t - x + y)) - I*e^(cos(-t - x + y))*sin(3*y))*sin(t + x + 2*y))*cos(sin(-t - x + y)) + 1/4*(cos(2*t + 2*x + 2*y) - I*si
....: n(2*t + 2*x + 2*y))*sin(3*t + 3*x + 3*y) - 1/4*(cos(2*t + x + 2*y) - I*sin(2*t + x + 2*y))*sin(3*t + 3*y) + 1/4*(cos(3*t
....:  + 3*x) + I*sin(3*t + 3*x))*sin(2*t + 2*x + y) - 1/4*(cos(3*t) + I*sin(3*t))*sin(2*t + x + y) - 1/4*(cos(3*x) + I*sin(3*
....: x))*sin(t + 2*x + y) - 1/4*(cos(3*y) + I*sin(3*y))*sin(t + x + 2*y) - 1/4*(cos(t + 2*x + 2*y) - I*sin(t + 2*x + 2*y))*si
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....: sin(2*t + 2*x + 2*y) + I*e^(cos(t + x + y))*sin(t + x + y))*cos(3*t + 3*x + 3*y) - cos(2*t + 2*x + 2*y)*e^(cos(t + x + y
....: )) + cos(t + x + y)*e^(cos(t + x + y)) + (I*cos(2*t + 2*x + 2*y)*e^(cos(t + x + y)) - I*cos(t + x + y)*e^(cos(t + x + y)
....: ) + e^(cos(t + x + y))*sin(2*t + 2*x + 2*y) - e^(cos(t + x + y))*sin(t + x + y))*sin(3*t + 3*x + 3*y) + I*e^(cos(t + x +
....:  y))*sin(2*t + 2*x + 2*y) - I*e^(cos(t + x + y))*sin(t + x + y) - 2*e^(cos(t + x + y)))*sin(sin(t + x + y)) + 1/4*((cos(
....: 3*x) + I*sin(3*x))*cos(2*t + x + 2*y)*e^(cos(t - x + y)) - (cos(3*x) + I*sin(3*x))*cos(t + 2*x + y)*e^(cos(t - x + y)) +
....:  (-I*cos(3*x) + sin(3*x))*e^(cos(t - x + y))*sin(2*t + x + 2*y) + (I*cos(3*x) - sin(3*x))*e^(cos(t - x + y))*sin(t + 2*x
....:  + y) - (cos(2*t + x + 2*y)*e^(cos(t - x + y)) - cos(t + 2*x + y)*e^(cos(t - x + y)) - I*e^(cos(t - x + y))*sin(2*t + x
....: + 2*y) + I*e^(cos(t - x + y))*sin(t + 2*x + y))*cos(3*t + 3*y) + (-I*cos(2*t + x + 2*y)*e^(cos(t - x + y)) + I*cos(t + 2
....: *x + y)*e^(cos(t - x + y)) - e^(cos(t - x + y))*sin(2*t + x + 2*y) + e^(cos(t - x + y))*sin(t + 2*x + y))*sin(3*t + 3*y)
....:  + 2*e^(cos(t - x + y)))*sin(sin(t - x + y)) - 1/4*((cos(3*t) + I*sin(3*t))*cos(2*t + x + y)*e^(cos(-t + x + y)) - (cos(
....: 3*t) + I*sin(3*t))*cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) - (I*cos(3*t) - sin(3*t))*e^(cos(-t + x + y))*sin(2*t + x + y)
....:  - (-I*cos(3*t) + sin(3*t))*e^(cos(-t + x + y))*sin(t + 2*x + 2*y) - (cos(2*t + x + y)*e^(cos(-t + x + y)) - cos(t + 2*x
....:  + 2*y)*e^(cos(-t + x + y)) - I*e^(cos(-t + x + y))*sin(2*t + x + y) + I*e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*cos(3*x
....:  + 3*y) - (I*cos(2*t + x + y)*e^(cos(-t + x + y)) - I*cos(t + 2*x + 2*y)*e^(cos(-t + x + y)) + e^(cos(-t + x + y))*sin(2
....: *t + x + y) - e^(cos(-t + x + y))*sin(t + 2*x + 2*y))*sin(3*x + 3*y) - 2*e^(cos(-t + x + y)))*sin(sin(-t + x + y)) + 1/4
....: *(((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(-t - x + y)) - I*e^(cos(-t - x + y))*sin(3*
....: y))*cos(2*t + 2*x + y) - ((cos(3*t + 3*x) + I*sin(3*t + 3*x))*e^(cos(-t - x + y)) - cos(3*y)*e^(cos(-t - x + y)) - I*e^(
....: cos(-t - x + y))*sin(3*y))*cos(t + x + 2*y) + ((-I*cos(3*t + 3*x) + sin(3*t + 3*x))*e^(cos(-t - x + y)) + I*cos(3*y)*e^(
....: cos(-t - x + y)) - e^(cos(-t - x + y))*sin(3*y))*sin(2*t + 2*x + y) + ((I*cos(3*t + 3*x) - sin(3*t + 3*x))*e^(cos(-t - x
....:  + y)) - I*cos(3*y)*e^(cos(-t - x + y)) + e^(cos(-t - x + y))*sin(3*y))*sin(t + x + 2*y) - 2*e^(cos(-t - x + y)))*sin(si
....: n(-t - x + y)) + 1/4*I*cos(t + x + y) + 1/4*sin(t + x + y))
```

The simplest Green's function g in finite terms in CAS Sage.

Another Green's function, $\tilde{g}$

Consider the series:

$$\tilde{g} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(k+\frac{1}{2}\right)x\sin\left(k+\frac{1}{2}\right)s\sin\left(k+\frac{1}{2}\right)(t-\tau).$$

In finite terms:

$$\tilde{g} = \frac{1}{4}\left(\operatorname{sign}\sin\frac{x-s+t-\tau}{2}+\operatorname{sign}\sin\frac{-x+s+t-\tau}{2}+\right.$$
$$\left.\operatorname{sign}\sin\frac{x+s-t+\tau}{2}+\operatorname{sign}\sin\frac{-x-s-t+\tau}{2}\right).$$

This test turned out to be much more difficult.

Another Green's function, $\tilde{g}$

$$> sum\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)\cdot\sin\left(\left(n+\frac{1}{2}\right)\cdot s\right)\cdot\sin\left(\left(n+\frac{1}{2}\right)\cdot t\right)}{\left(n+\frac{1}{2}\right)}, n=0..infinity\right) \text{ assuming } x>0, s>0, t>0$$

$$\Big(\tag{65}$$

$$-\frac{I}{4}\Big(\arctan\left(\sqrt{-e^{-I(s-x-t)}}\right)\sqrt{-e^{I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{-I(s-x+t)}}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{I(s-x-t)}}\sqrt{-e^{-I(s+x-t)}}\sqrt{-e^{-I(s+x+t)}}$$

$$e^{-\frac{I}{2}(s-x-t)}$$

$$-\sqrt{-e^{-I(s-x-t)}}\Big(\sqrt{-e^{I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{-I(s-x+t)}}\arctan\left(\sqrt{-e^{I(s-x-t)}}\right)e^{\frac{I}{2}(s-x-t)}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{-I(s+x-t)}}$$

$$\sqrt{-e^{-I(s+x+t)}}+\Big(e^{-\frac{I}{2}(s-x+t)}\sqrt{-e^{-I(s+x-t)}}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{-I(s-x+t)}}\right)$$

$$+\sqrt{-e^{-I(s-x+t)}}\Big(e^{-\frac{I}{2}(s+x-t)}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{-I(s+x-t)}}\right)$$

$$-\sqrt{-e^{-I(s+x-t)}}\Big(e^{\frac{I}{2}(s-x+t)}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{I(s-x+t)}}\right)$$

$$+\sqrt{-e^{I(s-x+t)}}\Big(e^{\frac{I}{2}(s+x-t)}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{I(s+x-t)}}\right)$$

$$+\sqrt{-e^{I(s+x-t)}}\Big(e^{-\frac{I}{2}(s+x+t)}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{-I(s+x+t)}}\right)-e^{\frac{I}{2}(s+x+t)}\sqrt{-e^{-I(s+x+t)}}\arctan\left(\sqrt{-e^{I(s+x+t)}}\right)\Big)\Big)\Big)\Big)\Big)$$

$$\sqrt{-e^{I(s-x-t)}}\Big)\Big)\Big)\Big/$$

$$\left(\sqrt{-e^{-I(s-x-t)}}\sqrt{-e^{I(s+x+t)}}\sqrt{-e^{-I(s+x-t)}}\sqrt{-e^{-I(s-x+t)}}\sqrt{-e^{I(s-x-t)}}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s-x+t)}}\right)$$

Fourier testing results: tipycal series from textbook

This is the series for motion of finite string. It satisfies the initial condition $u(0,x) = \varphi(x) = x^2(1-x)$.

$$u = \sum_{n=1}^{\infty} \frac{8 \cdot (-1)^{n+1} - 4}{\pi^3 n^3} \sin(\pi n x) \cos(\pi n c t)$$

Maple'2019 is able to convert the infinite series in symbolic expression, namely

$$u = \frac{2i}{\pi^3} \Big(\mathrm{Li}_3(-e^{i\cdot\pi(x+ct)}) - \mathrm{Li}_3(-e^{-i\cdot\pi(x+ct)}) + \\ + \mathrm{Li}_3(-e^{i\cdot\pi(x-ct)}) - \mathrm{Li}_3(-e^{-i\cdot\pi(x-ct)}) \Big) - \\ - \frac{i}{\pi^3} \Big(\mathrm{Li}_3(e^{-i\cdot\pi(x+ct)}) - \mathrm{Li}_3(e^{i\cdot\pi(x+ct)}) + \\ + \mathrm{Li}_3(e^{-i\cdot\pi(x-ct)}) - \mathrm{Li}_3(e^{i\cdot\pi(x-ct)}) \Big).$$

Here $\mathrm{Li}_3(z)$ is Euler's polylogarithm.

Fourier testing results: tipycal series from textbook

But *obviously*, at every moment of time it is a piecewise polynomial function! It is obviously, because we can convert the product $sin \cdot cos$ to the sum of sines. The symbolic expression of the function u in finite terms again requires the piecewise constructions sign or $\arctan(\cot)$. There is an alternative: work in the field of complex numbers $\mathbb{C}$ and use special functions, or work in the field of real numbers $\mathbb{R}$ and use piecewise elementary functions. The bridge between the two representations in finite terms is the Fourier series.

A.N. Krylov's technique Fourier series

The direct application of CAS to the summation of Fourier series can lead to difficulties.

One can try to overcome them by changing the formulation of the problem: instead of the summation problem in the finite terms, consider the problem of accelerating the convergence of the Fourier series. As Krylov wrote, this technique «often leads to the representation of the sum of the proposed series in closed form under the guise piecewice function».

Krylov's technique in moderm mathemathical physics

- А. Б. Нерсесян "Ускорение сходимости разложений по собственным функциям", Докл. НАН Армении. 2007.
- А. П. Хромов, М. Ш. Бурлуцкая, "Классическое решение методом Фурье смешанных задач при минимальных требованиях на исходные данные", Изв. Сарат. ун-та. Нов. сер. Сер. Математика. Механика. Информатика, 14:2 (2014), 171–198
- Adcock B. Modified Fourier expansions: theory, construction and application / Trinity Hall University of Cambridge, 2010.

A.N. Krylov's method [L.V.Kantorovich, V.I. Krylov]

$$\sum_{n=1}^{\infty}\left(U\left(\frac{1}{n}\right)\sin nx + V\left(\frac{1}{n}\right)\cos nx\right)$$

$$U\left(\frac{1}{n}\right) = u_1\frac{1}{n} + u_2\frac{1}{n^2} + O\left(\frac{1}{n^3}\right), u_1, u_2 \in \mathbb{R};$$

$$V\left(\frac{1}{n}\right) = v_1\frac{1}{n} + v_2\frac{1}{n^2} + O\left(\frac{1}{n^3}\right), v_1, v_2 \in \mathbb{R};$$

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi - x}{2}, x \in (0, 2\pi]$$

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = -\log 2\left|\sin\frac{x}{2}\right|, .$$

We can see, that the simplest Green's function g is the subject, when this scheme has only ONE nontrivial step.

Implementation in Sage

We will implement the above convergence acceleration scheme, assuming that the given functions U and V are good enough. In practice, the Fourier coefficient is a function of n, and it may not be possible to expand it into a series in terms of $\frac{1}{n}$. Let's take a class of functions where such a possibility exists: rational functions of n. The main direction of our work is the symbolic study of a series, and this class of functions has additional advantages:

- The built-in functions of Sage allow to determine the membership of a polynomial ring and its field of quotients. This allows, sometimes, to determine: is a given Fourier series a (piecewise) elementary function?
- The fact that the series diverges can be easily established.
- For this case, standard symbolic tools give the answer in the form of transcendental functions, and comparison of the results may be of independent interest.

Implementation in Sage

For simplicity, today we will leave aside the question of trigonometric transformations and the question of the periodic extension of a piecewise function to a straight line.

Main functions of the program

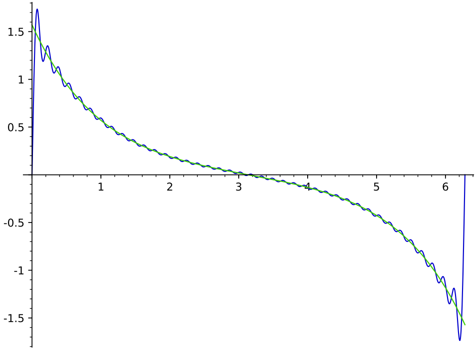
Here U is the function $U(n)$ given by symbolic expression. The prefixes c and s are used to indicate which series is being considered: by cosines (c) or by sines (s).

- `is_elementary_c(V)|`
- `c_series(V,M)|, summation_c(V)|. Finding a partial sum of order M and a sum in closed form using sage tools.`
- `kryloff_c_slow(V,k)| Returns the expression in closed form for the slowly converging part of the Fourier series. The Taylor polynomial of order` k `is used. Only those terms are singled out that lead to expressions in elementary functions (logarithm and Bernoulli polynoms).`
- `c_rapid(V,k,M)| Returns the order M partial sum of the accelerated convergence series.`

Examples

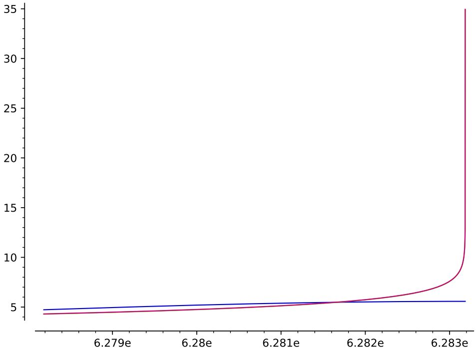
```
sage: load('kryloff-1.sage')\\
sage: U=x/(x^2+1)\\
sage: summation_s(U)\\
1/4*(imag_part(hypergeometric((2, -I + 1, I + 1), (-I + 2, I
sage: f=s_series(U,40)\\
sage: g=kryloff_s_slow(U,5)+s_rapid(U,5,3)\\
sage: m=plot([f,g],z,0,2*pi)
```

Examples



The series $\sum_{n=1}^{\infty} \frac{n}{n^2+1} \sin nx$. Blue line: partial sum of 40 terms. Green line: the sum of elementary part of 5 terms and 3 terms accelerated series.

Examples: logarithm singularity



The series $\sum_{n=1}^{\infty} \frac{1}{n+1} \cos nx$.

Blue line: partial sum of 400 terms. Green line: the sum of elementary part of 5 terms and 5 terms accelerated series. Red line: elementary answer (but this is not basic series).

Summary: results, presented in the talk

1. We present a simple emplementation of A.N. Krylov's metod of Fourier series convergence acceleration
2. This emlementation can give closed-form representation for several cases of Fourier series
3. Functions of the package «Kryloff for Sage» can be adapted to another eigenfunctions: $\sin(n+\frac{1}{2})$ and other cases of series, typical for mathematical physics

Unsolved problems

1. The presented implementation of convergence acceleration can lead to finite expressions for series in a very narrow class of cases. Even the series $\sum_{n=1}^{\infty} \frac{1}{n+1} \cos nx$ does not fall into it.
2. Acceleration of convergence based on the Taylor expansion of the Fourier coefficient in a Taylor series cannot, in the general case, lead to acceleration of convergence to an arbitrary order of decrease in the Fourier coefficient.

Therefore, further research into the issue is necessary.

The main idea

Let us remember why it was necessary to accelerate the convergence of the Fourier series in the first place? This procedure was absolutely necessary for working with solutions presented in the form of series, since each calculation of the derivative of such a series usually slows down convergence. At the same time, it is not difficult to encounter problems in which the corresponding differentiation of the required order leads to a divergent series. In this case, the desired function itself can be arbitrarily smooth within the domain of consideration. Accelerating convergence singled out the slowly converging parts of the Fourier series; they can be replaced with finite expressions, and remove the problem of finding the derivative in the form of a divergent series.

The main idea

Note that the trigonometric Fourier series we are studying can be considered not as ordinary functions, but as elements of the distribution space $D'(-\pi, \pi)$ [Laurent-Moïse Schwartz, Teorie des distributions, 1950]. That is, as generalized functions. Every generalized function has derivatives of any order. It can be proven that the Fourier series of a generalized function, if it is locally integrable, can be obtained using the usual Fourier formulas. It can be proven that this Fourier series will converge in the distribution space $D'(-\pi, \pi)$. It can be proven that in the sense of the space $D'(-\pi, \pi)$ it can be differentiated term-by-term any number of times! In this case, in the classical sense, divergent Fourier series, with which Krylov struggled, will inevitably arise.

The main idea. Differetiation of the basic series.

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\sum_{n=1}^{\infty}\cos nx=-\frac{1}{2}+\pi\delta(x), x\in[-\pi,\pi]$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\cos nx}{n}=-\frac{d}{dx}\left(\log 2\left|\sin\frac{x}{2}\right|\right)=-\sum_{n=1}^{\infty}\sin nx=-\frac{1}{2}\cot\frac{x}{2},$$

x $\in[-\pi,0)\cup(0,\pi]$

These equalities should be understood as equalities of generalized functions from the distribution space $D'(-\pi,\pi)$. Justifications for them as equalities for elements of distribution spaces can be found in [Gelfand, Shilov], [Antosik, Mikusinsky, Sikorski].

The main idea. Basic divergent series.

$$\sum_{n=1}^{\infty} \cos nx = -\frac{1}{2} + \pi\delta(x), x \in [-\pi, \pi]$$

$$\sum_{n=1}^{\infty} \sin nx = \frac{1}{2} \cot \frac{x}{2},$$

$$\sum_{n=1}^{\infty} n \cos nx = -\frac{1}{4} \left(\sin \frac{x}{2}\right)^{-2},$$

$$\sum_{n=1}^{\infty} n \sin nx = -\pi\delta(1, x), x \in [-\pi, \pi].$$

These equalities should be understood as equalities of generalized functions from the distribution space $D'(-\pi, \pi)$. Justifications for them as equalities for elements of distribution spaces can be found in [Gelfand, Shilov], [Antosik, Mikusinsky, Sikorski].

The main idea. How we will use divergent series.

We will take advantage of the opportunity to express the indicated series in closed form, and the opportunity to differentiate the Fourier series term by term any number of times. If the Fourier series has the form

$$\sum_{n=1}^{\infty} \frac{P}{Q} \sin nx = y, x \in [-\pi, \pi], P \in \mathbb{Q}[n], Q \in \mathbb{Q}[n],$$

then it is not difficult to find an expression for the differential polynomial L_Q that "annihilates" the denominator in the fraction $\frac{P}{Q}$, so that for y we get the equation

$$L_Q y = f_{PQ}, x \in [-\pi, \pi].$$

On the right side is the distribution f, which is expressed in final form. Note that if the polynomial Q in the denominator of the Fourier coefficient depends only on n^2, then the expression for f is entirely determined by the numerator - the polynomial P.

Finding of the sum

1. Reconstruct the operator L_Q and the right-hand side f_P.
2. If the right-hand side is composed only of the Dirac delta function and its derivatives, the given Fourier series is found in elementary functions.
3. If the right-hand side contains a cotangent and its derivatives, the ability to express the series in elementary functions depends on the spectrum of the operator L_Q. If there are irrational eigenvalues, or multiple eigenvalues, expressions in elementary functions cannot be composed.

Fundamental Fourier Series

$$\sum_{n=1}^{\infty} \frac{1}{n+\lambda} \sin nx = y, \qquad \sum_{n=1}^{\infty} \frac{1}{n-\lambda} \sin nx = z.$$

"Annihilation":

$$\frac{d^2 y}{dx^2} + \lambda^2 y = \pi\delta(1,x) + \frac{\lambda}{2}\cot\frac{x}{2}, \quad y(-\pi) = 0, y(\pi) = 0;$$

$$\frac{d^2 z}{dx^2} + \lambda^2 z = \pi\delta(1,x) - \frac{\lambda}{2}\cot\frac{x}{2}, \quad y(-\pi) = 0, y(\pi) = 0; .$$

The ability to find an expression for the sums in elementary functions is reduced to the ability to find an integral of the form

$$\int \sin \lambda x \cot \frac{x}{2} dx.$$

This is possible if lambda is a rational number [Liouvill, Hardy].

Fundamental Fourier Series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\lambda^2} \sin nx = y, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2-\lambda^2} \sin nx = z.$$

"Annihilation":

$$\frac{d^2 y}{dx^2} - \lambda^2 y = -\frac{1}{2} \cot \frac{x}{2}, \quad y(-\pi) = 0, y(\pi) = 0;$$

$$\frac{d^2 z}{dx^2} + \lambda^2 z = -\frac{1}{2} \cot \frac{x}{2}, \quad y(-\pi) = 0, y(\pi) = 0; .$$

The ability to find an expression for the sums in elementary functions is reduced to the ability to find an integrals of the form

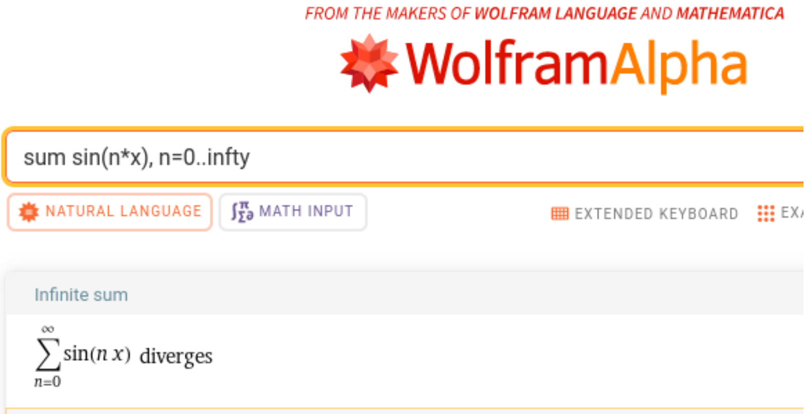
$$\int \sin \lambda x \cot \frac{x}{2} dx, \quad \int \exp \lambda x \cot \frac{x}{2} dx$$

Fundamental Fourier Series

More complex simple fractions correspond to transcendental functions associated with multiple integration of the form indicated above, and quadratures of the form

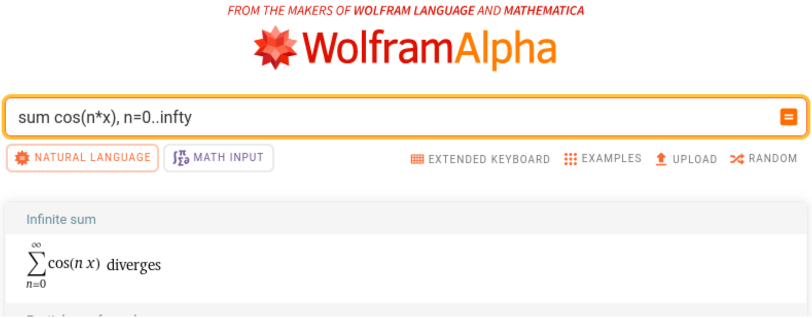
$$\int x^d \sin \lambda x \cot \frac{x}{2} dx, \lambda \in \mathbb{C}, d \in \mathbb{N}.$$

Divergent series, WolframAlpha



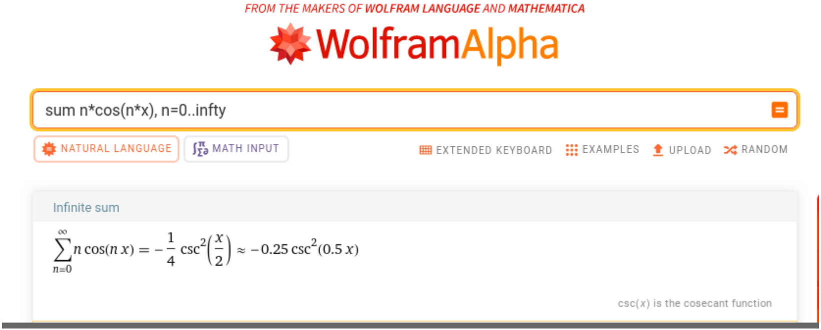
This is consistent with the results of mathematical analysis.

Fourier series summation, WolframAlpha



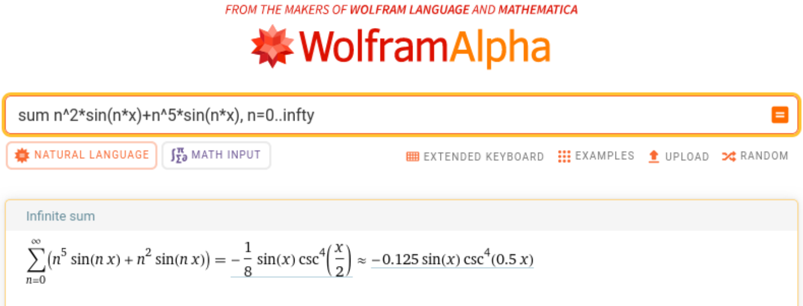
This is consistent with the results of mathematical analysis.

Fourier series summation, WolframAlpha



This is something unexpected!

Fourier series summation, WolframAlpha



This is something unexpected!

L.A. Dikii, 1954

в) Для собственных функций можно также получить серию тождеств, аналогичных тождествам Гельфанда—Левитана для собственных значений. Заметим, что

$$\sum_{1}^{\infty} n^{-s} \sin 2n x = \frac{1}{2i\Gamma(s)\sin(s-1)\pi} \int_L (-z)^{s-1} \frac{e^z \sin 2x}{e^{2z} - 2e^z \cos 2x + 1}\, dz,$$

где контур L приходит из $+\infty$, огибает начало координат в положительном направлении и уходит в $+\infty$. При $s = -2k+1$ полюс $\Gamma(s)$ уничтожается нулем $\sin(s-1)\pi$, а интеграл равен нулю, так как он приводится к

$$\oint (-z)^{-2k} \frac{e^z \sin 2x}{e^{2z} - 2e^z \cos 2x + 1}\, dz.$$

Дробь, стоящая под интегралом,— четная функция. Мы получим:

$$\sum n^{2k-1} \sin 2n x = 0$$

(в смысле аналитического продолжения). Точно так же получим, что при $k > 0$

$$\sum_{1} n^{2k} \cos 2n x = 0.$$

При $k = 0$ эта сумма равна $-\frac{1}{2}$. Эти соображения приводят к теоремам 7 и 8.

Поступило
10. IV. 1954

L. A. Dikii, "The zeta function of an ordinary differential equation on a finite interval", Izv. Akad. Nauk SSSR Ser. Mat., 19:4 (1955), 187–200.

Fundamental Differential Equation, Kamke

$$\frac{d^2y}{dx^2} + \lambda^2 y = -\frac{1}{2}\cot\frac{x}{2}, y(-\pi) = 0, y(\pi) = 0;$$

2.9] $ay'' + \ldots$ 365

2.7. $y'' - 2y = 4x^2 \exp x^2.$

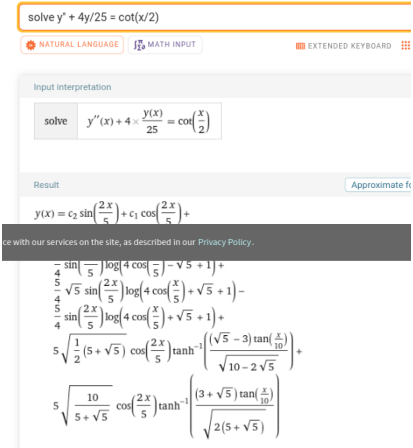
$$y = \exp x^2 + C_1 \exp x\sqrt{2} + C_2 \exp\left(-x\sqrt{2}\right).$$

2.8. $y'' + a^2 y = \operatorname{ctg} ax.$

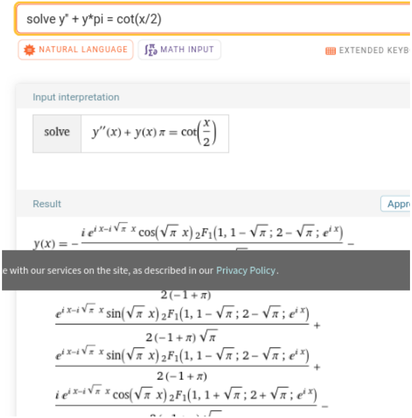
$$y = C_1 \cos ax + C_2 \sin ax + \frac{\sin ax}{a^2} \ln\left|\frac{1 - \cos ax}{\sin ax}\right|.$$

For our purposes, only one value of a is suitable, a=1/2.

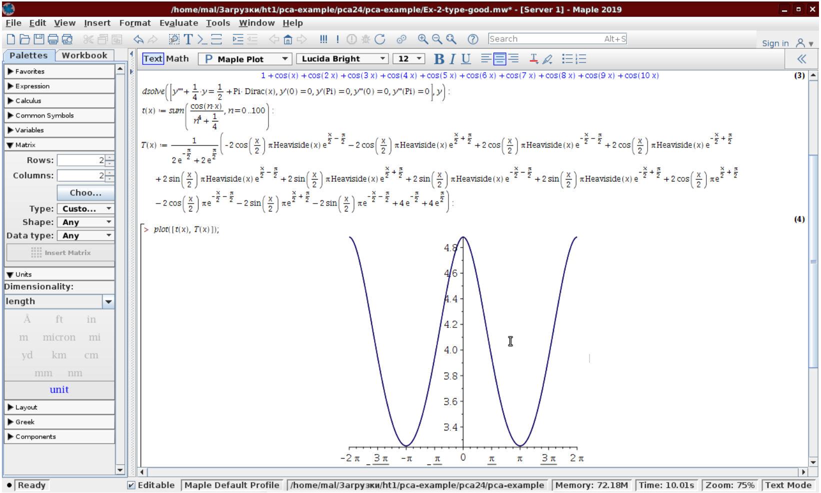
Fundamental Differential Equation, WolframAlpha



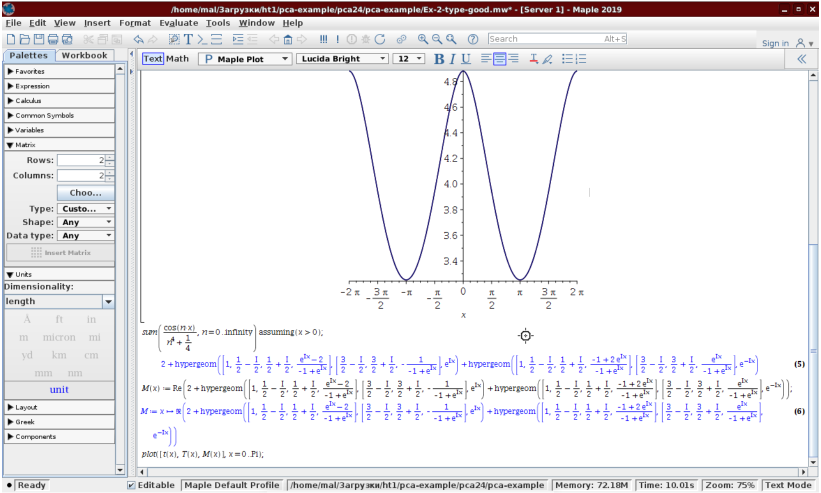
Fundamental Differential Equation, WolframAlpha



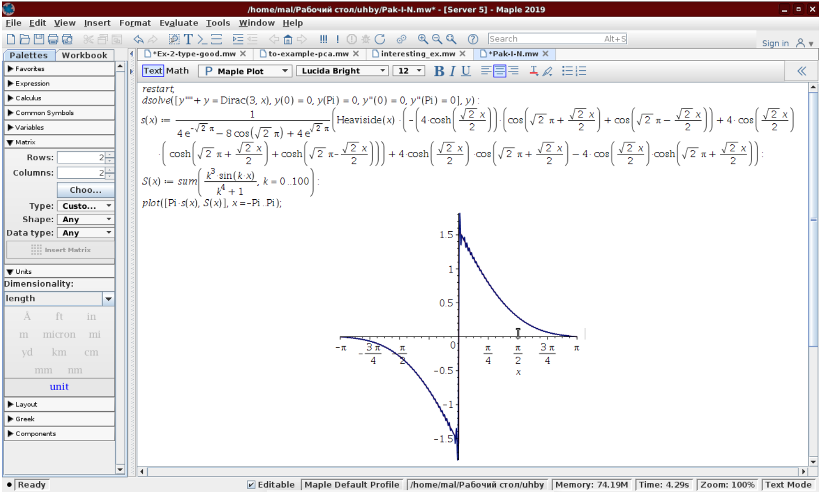
Example-1, cos-series



Example-1, cos-series



Example-2, I.N. Pak Series



The series from the article I. N. Pak, "On the sums of trigonometric series", Uspekhi Mat. Nauk, 35:2(212) (1980), 91–144. Kernel has

New functions in «Kryloff for Sage»

1. Function «SeriesToODE» to reconstruct the differential equation for the Fourier series
2. Function «TheElementaryConvolution» for finding a solution to a boundary value problem with a delta function and its derivatives on the right sides. Based on the explicit form of the Cauchy function and solving a system of linear algebraic equations.
3. Function «KryloffDecomposition» for finding the sums of the fundamental components of a Fourier series. Built on the basis of Ostrogradsky decomposition and integration ODE by CAS Sage.

Summary: results, presented in the talk

1. The difficulties of symbolic summation of series can in some cases be circumvented by using finite expressions for the sums of divergent Fourier series.
2. The basis of this approach is the theory of generalized functions based on the space D', within the framework of which work with a certain class of divergent series becomes updated and acquires a purely algebraic character
3. Some functions for summing series have been implemented. Certain problems that could not be solved in final form using the conventional version of the convergence acceleration method are now available for solution.

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Thank you for your time and attention!