Divergent Fourier series and summation in finite terms using the A.N. Krylov method in CAS

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Abstract. New procedures of the «Kryloff for Sage» software package, created to accelerate the convergence of Fourier series, are presented. We implement divergent Fourier series into our package, considered as generalized functions from the space of distributions $D'(-\pi,\pi)$. This not only expands the class of series that can be studied using symbolic calculations, but also creates the possibility of symbolically summing the Fourier series for which in the previous version of the «Kryloff for Sage» package it was only possible to speed up the convergence, without finding an expression in a closed form.

Introduction

This work is a continuation of previous ones aimed at creating programs for symbolic analysis of Fourier series. We consider the formulation of the problem in which the function represented by the Fourier series is not known, but must be determined from its Fourier coefficients. Fourier coefficients are given in symbolic form. This problem can be solved by standard series summation functions built into modern computer algebra systems for a wide class of symbolic expressions of Fourier coefficients. However, some difficulties arise along this path: when applied to Fourier series, CAS usually produce results in terms of complex-valued higher transcendental functions [1]. At the same time, often the indicated Fourier series represent piecewise polynomial and other piecewise elementary expressions. Therefore, it is relevant to develop symbolic algorithms capable of finding finite expressions for the sums of Fourier series precisely among real-valued piecewise elementary functions.

In our previous report [2], we talked about A.N. Krylov accelerating the convergence of some special Fourier series, the coefficients of which are rational functions of the harmonic number. For them, a simple version of the A.N. Krylov method was implemented in the CAS `Sage` [3] in the form of several functions of the «Kryloff for Sage» software package. In that version of the program the acceleration of convergence led to a closed-form expression for a very narrow, but not empty class of Fourier series. In the proposed work, we complement these considerations with new functions that can solve some of the summation problems that were previously beyond our capabilities.

We consider Fourier series as elements of the space of distributions $D'(-\pi,\pi)$ [4]. This is correct provided there is no more than a power-law increase in Fourier coefficients [5]. Then the series under consideration can be differentiated term by term any number of times: if a series diverging in the classical sense arises, then we treat it as an element of the space $D'(-\pi,\pi)$ in which this series is convergent. By differentiating the Fourier series term by term, one can try to look for a differential equation of which it may be a solution, cf. [6, p.224-226, 235], cf. [7] and references therein. On this path, it is possible to formulate several simbolic procedures that allow one to reconstruct, using the Fourier series, an inhomogeneous boundary value problem that it satisfies as an element of the space $D'(-\pi,\pi)$. The inhomogeneity of the differential equation will be an element of the space $D'(-\pi,\pi)$. To obtain an expression for the sum, it is necessary to solve this boundary value problem. The Fourier series is represented as a convolution of distributions. In some cases, this approach makes it possible to obtain the desired expressions for the sums of Fourier series in the form of elementary real-valued functions.

The report will present the implementation of the outlined strategy using CAS `Sage`, and experiments on it in computer algebra systems `WolframAlpha` [8] and `Maple` [9].

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