# Paradoxes of Game Semantics

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Traditionally, game semantics was developed to confirm or complement some existing semantics of logical systems. Usually it was established that the Verifier has a winning strategy in the game associated with the formula A when A is derivable in some corresponding deductive system or true in a model-theoretical sense.

It is well known that the restrictions on the classes of strategies used by Verifier and Falsifier may break this relationship, so the question of “admissible” restrictions has been studied: for example whether it is possible to consider only computable strategies and still obtain the same “adequate” semantics.

The games with backward moves (i.e., where the Verifier is permitted to “replay” if it comes to a “bad” position) were proposed to improve correspondence between classical provability and existence of recursive winning strategies [1, 4, 5].

**Proposition 9** ( [4]). For every first order language $L$, every decidable $L$-structure $M$ and every set $\Sigma$ of axiom sequents validated by semantic games on $M$, any proof $\phi$ in $\vdash \Sigma$ is validated by games on $M$ (i.e., one may construct a winning strategy for Verifier in these games).

We study the effect of restrictions from a different point of view: how much the semantics may be deformed due to some natural asymmetry between players. We consider various known kinds of semantic games, for example, games with backward moves. It turns out that in many semantic games, in particular, the games with backward moves, and under some conditions the Verifier may win even if the formula is not true in ordinary logical semantics.

The following general result may be used to describe paradoxal situations of this kind.

- Assume that Verifier can compute any general recursive function and knows (and can compute) a universal function $U(x, y)$ for the strategies $f$ of Falsifier, i.e., every $f = U(k, -)$ for some $k$.
- Assume that if Verfier knows the strategy of Falsifier it can win. That is, Verifier can compute another function $W(x, y)$ such that $v_k = W(k, -)$ wins against $f_k = U(k, -)$.

- Here we may assume that $y \in N$ are the codes of partial plays (they may include backward moves).
- **Theorem.** In the conditions listed above the Verifier has a recursive strategy that wins against any strategy of the Falsifier.

  Two significant examples (not based on this theorem) are considered as well. **Example 1.**Let us consider the following (false) formula:

$$\phi = \exists x \forall y.(y \leq \mathcal{A}(x)).$$

Let here $\mathcal{A}$ be the Ackermann's function.

- In games without backward moves, since for Verifier the history is empty, the strategies of Verifier are just natural numbers (values of x). The strategies of Falsifier are functions $f : N \to N$. If Verifier chooses $x$ then the answer of Falsifier is $f(x)$. Let the strategies of Falsifier be limited to the class of PR functions.

  The formula is *false* on $N$, but there is no winning strategy for Falsifier because $\mathcal{A}$ grows faster than any PR function. There is no winning strategy for Verifier as well, because for any $x$ there exists some PR function $f$ such that $f(x) > \mathcal{A}(x)$.
- Let us consider the games with backward moves. Now the strategies of Verifier are functions on histories (not empty after replay). And the strategy that takes the values $0, ..., n...$ (during $n$-th replay) is winning for Verifier against all PR-strategies of Falsifier because $\mathcal{A}(n)$ will outgrow any PR-function. (If Falsifier has memory, its strategies are PR functions on histories, but histories may be coded by numbers of the lists of values. Enumeration is PR, and with $0, ..., n...$ as values of $y$ the resulting function is also PR.)

  **Example 2.** (A more extreme example.)
- **Generalized Ramsey theorem.**
- Recall [2]: A set of integers, $S$, is large if $S$ is non-empty, and (if $s$ is its least element) $S$ has at least $s$ elements.
- $A$ being a set, $b \in N$, $A^{[b]}$ denotes the set of all subsets of $A$ of cardinality $b$. If $F : A^{[b]} \to X$, a subset $B$ of $A$ is homogeneous for $F$ if $F$ is constant on $B^{[b]}$. Each integer $n$ is, as usual, identified with the set of integers less than $n$. For $a, b, c \in N, a, b, c > 0$, $a \to (large)^b_c$ means that for every map $F : a^{[b]} \to c$ there is a large homogeneous set for $F$ of cardinality greater than $b$ (this relation is PR).
- $(\forall b, c \geq 1)(\exists a \geq 1)(a \to (large)^b_c)$ is the generalized Ramsey's theorem. It is not provable in Peano Arithmetic, but provable in second order arithmetic [2].
- We may consider the game associated with its classical negation

$$(\exists b, c \geq 1)(\forall a \geq 1)\neg(a \to (large)^b_c)$$

  This formula is *false* but the value of $a$ (that gives a counterexample) grows faster that any function that is provably general recursive in Peano Arithmetic.

- Thus, like in Example 1, in games without backward moves there is no winning strategy for Falsifier in the class of provably general recursive functions.
- In games with backward moves the simple strategy for Verifier (take the values $0, 1, ...$ for subsequent replays) wins against any provably general recursive strategy for Falsifier.

**Conclusion.** It may be argued that completely automated verification (not checked by humans) is related to semantic games in Game Semantics. This underlines the relevance of "perversions" considered above.

## References

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