Finite groups and quantum mechanics: evolution and decomposition of quantum systems

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Abstract. Quantum mechanics is based on two main points: (1) the assumption that the evolution of a closed system is described by unitary transformations in Hilbert space, and (2) the idea of observation, formalized in the concept of an observable and canonical commutation relations between pairs of conjugate observables. We call such conjugate pairs complementary, since they form the basis of Borh's complementarity principle.

The combined use of complementary observables allows us to obtain the maximum available information about the state of a quantum system. Complementary observables are related to such issues as the uncertainty principle, the principle of least action, the path integral formulation of quantum mechanics, mutually unbiased bases etc.

Replacing a continuous unitary group with a finite permutation group in the quantum formalism [1–4] allows us to reduce the description of evolution to the group of cyclic permutations $\mathbb{Z}_N$. The product of $\mathbb{Z}_N$ and its Pontryagin dual, $\widetilde{\mathbb{Z}}_N$, has a nontrivial projective representation, which allows to describe quantum interferences taking into account phase differences.

Thus, by starting with just a cyclic permutation, we obtain a complete finite version of quantum mechanics, including unitary evolution and the complementarity principle. Finite structures that stem from cyclic permutations are naturally found in various fields, including quantum computer science and signal processing. These finite structures were first discovered, within the framework of continuous quantum mechanics, by Weyl when he constructed an analogue of the Heisenberg canonical commutation relations suitable for finite-dimensional Hilbert spaces.

The generator of the *regular representation* of $\mathbb{Z}_N$ on the N-dimensional Hilbert space $\mathcal{H}_N$ is the cyclic permutation matrix

$$X = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The matrix X is related to the basis $B_X = \{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$, called *the position basis.* This basis has other names, such as *ontic* (G. 't Hooft) or *initial*, or *computational* (quantum computer science).

In this basis, the position operator has the diagonal form $\widehat{x} = \sum_{x=0}^{N-1} x\,|x\rangle\langle x|$.
If $\gcd(v, N) = 1$, then the matrix $X_v = X^v$ defines a cyclic evolution on the eigenvalues of the position operator $\widehat{x}_t = X_v^t \widehat{x}_0 X_v^{-1}$. In the components we have

$$x_t = x_0 + vt \mod N.$$

Therefore, the parameter v can be interpreted as "velocity".

The generator of the Pontryagin dual group $\widetilde{\mathbb{Z}}_N$ is

$$Z = \widetilde{X} = FXF^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix},$$

where F is the Fourier transform and $\omega = \mathrm{e}^{2\pi\mathbf{i}/N}$ is the Nth base root of unity. The basis $B_Z = \left\{\left|\widetilde{0}\right\rangle, \left|\widetilde{1}\right\rangle, \ldots, \left|\widetilde{N-1}\right\rangle\right\}$ formed by the eigenvectors of Z is called *the momentum basis.*

The bases B_X and B_Z are interconnected by the Fourier transform, and they are *mutually unbiased*.

A direct calculation reveals that $XZ = \omega ZX$. This is precisely *the Weyl canonical commutation relation.* The operators X and Z generate a non-trivial *projective representation* of the group $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N \cong \mathbb{Z}_N \times \mathbb{Z}_N$ on the space $\mathcal{H}_N$.

The main constructs derived from the matrices X and Z are:

- *Weyl–Heisenberg group*

  $$\mathrm{H}(N) = \left\{\tau^k X^v Z^m\right\},$$

  where $\tau = -\omega^{1/2} = -\,\mathrm{e}^{\pi\mathbf{i}/N}$, $v, m \in \mathbb{Z}_N$, $k \in \mathbb{Z}_{\overline{N}}$, $\overline{N} = \begin{cases} N, & N \text{ is odd}, \\ 2N, & N \text{ is even}. \end{cases}$
- *Finite position-momentum phase space* T^2 is a 2D discrete torus of size $N \times N$.
- *Symplectic group* $\mathrm{Sp}(2, \mathbb{Z}_N)$ is the group of symplectic transformations of the phase space T^2.
- *Clifford group* $\mathrm{C}\ell(N) \cong \mathrm{H}(N) \rtimes \mathrm{Sp}(2, \mathbb{Z}_N)$ is the normalizer of $\mathrm{H}(N)$ in $\mathrm{U}(N)$. It is the group of all symmetries of the group $\mathrm{H}(N)$: $\mathrm{C}\ell(N) \cong \mathrm{Aut}(\mathrm{H}(N))$.

The properties of the operators X and Z and the constructions derived from them, as well as the possibility of decomposing the corresponding quantum system into subsystems, are determined by the structure of the group $\mathbb{Z}_N$. A decomposition of a cyclic group into smaller groups has the form

$$\mathbb{Z}_N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m}, \tag{1}$$

where $N = n_1 \cdot n_2 \cdot \ldots \cdot n_m,\ \gcd(n_i, n_j) = 1$. The canonical decomposition takes the form $\mathbb{Z}_N \cong \mathbb{Z}_{p_1^{\ell_1}} \times \cdots \times \mathbb{Z}_{p_m^{\ell_m}}$, where $N = p_1^{\ell_1} \cdots p_m^{\ell_m}$ is the prime factorization.

Mappings that provide isomorphism (1), which can be considered as an isomorphism of rings, can be calculated using the Chinese remainder theorem. Namely, for $k \in \mathbb{Z}_N$ we have the mappings

$$k \mapsto (r_1, \ldots, r_m),$$
$$(r_1, \ldots, r_m) \mapsto k = \sum_{i=1}^{m} r_i N_i^{-1} N_i \mod N,$$

where $r_i = k \mod n_i \in \mathbb{Z}_{n_i}$, $N_i = N/n_i \in \mathbb{Z}_N$, $N_i^{-1} \in \mathbb{Z}_{n_i}$ is the *multiplicative inverse* of N_i within $\mathbb{Z}_{n_i}$.

Dual mappings, which are more useful for many problems, have the form

$$k \mapsto (k_1, \ldots, k_m),$$
$$(k_1, \ldots, k_m) \mapsto k = \sum_{i=1}^{m} k_i N_i \mod N, \qquad (2)$$

where $k_i = r_i N_i^{-1} \in \mathbb{Z}_{n_i}$. For example, the equation

$$\frac{k}{N} = \sum_i \frac{k_i}{n_i} \mod 1, \qquad (3)$$

which follows from (2), helps us to understand the additivity of the energy in a composite quantum system: in representing the frequency of a system as a sum of frequencies of subsystems, frequencies can be interpreted as corresponding energy levels in accordance with the Planck relation, which states the equivalence of energy and frequency.

References

[1] Kornyak V.V. *Quantum models based on finite groups.* IOP Conf. Series: Journal of Physics: Conf. Series **965**, 012023, 2018. arXiv:1803.00408 [physics.gen-ph]

[2] Kornyak V.V. *Modeling Quantum Behavior in the Framework of Permutation Groups.* EPJ Web of Conferences **173**, 01007, 2018. arXiv:1709.01831 [quant-ph]

[3] Kornyak V.V. *Mathematical Modeling of Finite Quantum Systems.* In: Adam G. *et al* (eds) MMCP 2011. LNCS, **7125**. Springer, 2012. arXiv:1107.5675 [quant-ph]

[4] *Banks T.* Finite Deformations of Quantum Mechanics. arXiv:2001.07662 [hep-th], 20 p., 2020.

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