# Finite groups and quantum mechanics: evolution and decomposition of quantum systems

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Hilbert

# Quantum evolution

- $\mathbf{i}\hbar \frac{\partial}{\partial t} |\psi_t\rangle = H |\psi_t\rangle \rightsquigarrow |\psi_t\rangle = U_t |\psi_0\rangle$
  cyclic group $U_t = \mathrm{e}^{-\mathbf{i}\frac{H}{\hbar}t} = \left(\mathrm{e}^{-\mathbf{i}\frac{H}{\hbar}}\right)^t = \mathtt{E}^t$
- Without empirical losses Banks, the evolution generator $\mathtt{E}$ can be represented by an element of a finite group
  - specifically, $\mathtt{E}$ is an element of a representation of the cyclic group $\mathbb{Z}_n$

Mathematical reasons

- any linear representation of a finite group is unitary
- any linear representation of a finite group is a subrepresentation of some permutation representation

Advantages in describing reality

- finite groups have more expressive power than Lie groups: any Lie group can be approximated by finite groups, but not vice versa

# Finite groups vs Lie groups: $\mathbb{Z}_n$ vs $\mathrm{U}(1)$

- $\mathbb{Z}_n \approx \mathrm{U}(1)$ for large $n$

- $\mathbb{Z}_n \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$, if $n = n_1 n_2$ and $\gcd(n_1, n_2) = 1$

$$\Downarrow$$

$$\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{\ell_1}} \times \cdots \times \mathbb{Z}_{p_m^{\ell_m}}$$

  - $n = p_1^{\ell_1} \cdots p_m^{\ell_m}$ is prime factorization of $n$
  - The fact that $\mathbb{Z}_{p^\ell} \cong \mathrm{GF}(p^\ell)$, a Galois field, plays a crucial role in quantum mechanics.
  - Topologically, $\mathbb{Z}_n$ is a discrete multidimensional torus, resembles the circle $\mathrm{U}(1)$ topology only if $n$ is a prime number.

# Regular permutation representation of $\mathbb{Z}_N$

- Generator

$$X = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \qquad X|_{N=2} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ a Pauli matrix}$$

- Position or ontic ('t Hooft) or computational (quantum informatics) basis

$$\mathrm{B}_X = \{|0\rangle, \ldots, |N-1\rangle\}$$

- Position operator in ontic basis

$$\widehat{x} = \sum_{x=0}^{N-1} x\,|x\rangle\langle x| = \mathrm{diag}\,(0, 1, \ldots, N-1)$$

- Generator of evolution with velocity $v$: $X_v = X^v$

$$\widehat{x}_t = X_v^t \widehat{x}_0 X_v^{-t}$$

in components $\quad x_t = x_0 + vt \mod N$

# Pontryagin dual group $\widetilde{\mathbb{Z}}_N$

- Generator

$$Z = \widetilde{X} = FXF^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix} \qquad Z|_{N=2} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$F$ is the Fourier transform and $\omega = e^{2\pi \mathbf{i}/N}$ is the $N$th base root of unity

- Momentum basis

$$B_Z = \left\{ \left|\widetilde{0}\right\rangle, \left|\widetilde{1}\right\rangle, \ldots, \left|\widetilde{N-1}\right\rangle \right\}$$

- Momentum operator in momentum basis

$$\widehat{p} = \sum_{p=0}^{N-1} p \left|\widetilde{p}\right\rangle\left\langle\widetilde{p}\right| = \operatorname{diag}(0, 1, \ldots, N-1)$$

# Cyclic permutation paired with its Pontryagin dual gives rise to quantum behavior

- Bases $\mathrm{B}_X$ and $\mathrm{B}_Z$ are mutually unbiased

$$\left|\left\langle \widetilde{\ell} \,\middle|\, k \right\rangle\right|^2 = \frac{1}{N}$$

- $X, Z$ generate a projective representation of $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N \cong \mathbb{Z}_N \times \mathbb{Z}_N$ on $N$-dimensional Hilbert space $\mathcal{H}_N$

- Direct calculation $\longrightarrow$ $ZX = \omega XZ$, the Weyl commutation relation

# Canonical commutation relations

- Heisenberg commutation relation

$$\boxed{[\widehat{x},\, \widehat{p}] = \mathbf{i}\hbar\,\mathbb{1}} \quad \Longrightarrow \quad \dim \mathcal{H} = \infty$$

observables $\widehat{x}$, $\widehat{p}$ are Hermitian

- Weyl commutation relation

$$\dim \mathcal{H} = N < \infty \quad \Longrightarrow \quad \boxed{ZX = \omega XZ}$$

observables $X$, $Z$ are unitary

Weyl

# Basic constructions with $X$ and $Z$

- Weyl–Heisenberg group

$$\mathrm{H}(N) = \left\{\tau^k X^v Z^m\right\}, \quad \tau = -\omega^{1/2} = -\,\mathrm{e}^{\pi\mathbf{i}/N},$$

$$v, m \in \mathbb{Z}_N, \quad k \in \mathbb{Z}_{\overline{N}}, \quad \overline{N} = \begin{cases} N, & N \text{ is odd}, \\ 2N, & N \text{ is even}. \end{cases}$$

- Finite phase space $T^2$

  $2D$ discrete torus of size $N \times N$

  - $X$ shifts "positions" and gives phases to "momenta"
    Position-space Schrödinder evolution: $|\psi_t\rangle = \left(\theta^k X^v\right)^t |\psi_0\rangle$
  - $Z$ shifts "momenta" and gives phases to "positions"
    Momentum-space Schrödinder evolution: $|\psi_t\rangle = \left(\theta^k Z^m\right)^t |\psi_0\rangle$

- Symplectic group $\mathrm{Sp}(2, \mathbb{Z}_N)$

  symplectic transformations of $T^2$, quantum prototype of canonical transformations in Hamiltonian mechanics

- Clifford group $\mathrm{C}\ell(N) \cong \mathrm{H}(N) \rtimes \mathrm{Sp}(2, \mathbb{Z}_N)$

  normalizer of $\mathrm{H}(N)$ in $\mathrm{U}(N)$, $\mathrm{Aut}(\mathrm{H}(N))$

# Minimal example $N = 2$

$$\overbrace{X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{\text{Hermitian, period} = 2}, \quad \overbrace{XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbf{i}\sigma_y}^{\text{anti-Hermitian, period} = 4}$$

- Weyl–Heisenberg group

$$\mathrm{H}(2) = \{\pm 1, \pm \mathbf{i}\} \times \{\mathbb{1}, X, Z, XZ\} \cong (\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \qquad |\mathrm{H}(2)| = 16$$

- Phase space

$$T^2 = \mathbb{Z}_2 \times \widetilde{\mathbb{Z}}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad |T^2| = 4$$

- Symplectic transformations

$$\mathrm{Sp}(2, \mathbb{Z}_2) \cong \mathsf{S}_3 \qquad |\mathrm{Sp}(2, \mathbb{Z}_2)| = 6$$

- Clifford group

$$\mathrm{C}\ell(2) \cong \mathrm{H}(2) \rtimes \mathrm{Sp}(2, \mathbb{Z}_2) \qquad |\mathrm{C}\ell(2)| = 96$$

# Dimension $N = p^\ell$: $\mathcal{H}_{p^\ell} \cong \mathcal{H}_p \otimes \cdots \otimes \mathcal{H}_p$

- Galois field $\mathrm{GF}\left(p^\ell\right) \equiv$ finite field $\mathbb{F}_{p^\ell}$

  Galois number $\alpha = \alpha_0 + \alpha_1\varepsilon + \cdots + \alpha_{\ell-1}\varepsilon^{\ell-1} \in \mathbb{F}_{p^\ell}, \quad \alpha_k \in \mathbb{Z}_p$

  Addition — same as for $\ell$-dimensional vector space

  Multiplication — modulo an irreducible polynomial

$$\Phi(\varepsilon) = \phi_0 + \phi_1\varepsilon + \cdots + \phi_{\ell-1}\varepsilon^{\ell-1} + \varepsilon^\ell \in \mathbb{Z}_p[x]$$

  - Frobenius transformation $\sigma : \alpha \longrightarrow \alpha^p$
  - Galois group $\mathrm{Gal}(\ell) = \left\{1, \sigma, \ldots, \sigma^{\ell-1}\right\} \cong \mathbb{Z}_\ell$
  - Galois conjugates $\alpha \xrightarrow{\sigma} \alpha^p \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \alpha^{p^{\ell-1}} \xrightarrow{\sigma} \alpha$
  - Trace $\mathrm{tr}(\alpha) = \alpha + \alpha^p + \cdots + \alpha^{p^{\ell-1}} \in \mathbb{Z}_p$

- Generalized Pauli matrices

$$X_\alpha = \sum_{\gamma \in \mathbb{F}_{p^\ell}} |\gamma + \alpha\rangle\langle\gamma|, \quad Z_\beta = \sum_{\gamma \in \mathbb{F}_{p^\ell}} \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}\mathrm{tr}(\beta\gamma)} |\gamma\rangle\langle\gamma|$$

$$Z_\beta X_\alpha = \mathrm{e}^{\frac{2\pi \mathrm{i}}{p}\mathrm{tr}(\alpha\beta)} X_\alpha Z_\beta$$

# Chinese remainder theorem

$$0 \le k < N = n_1 \cdot n_2 \cdots n_m, \quad \gcd(n_i, n_j) = 1$$
$$\Downarrow$$
$$k \leftrightarrow (r_1, r_2, \ldots, r_m)\,, \quad r_i = k \mod n_i$$

- ring isomorphism $\mathbb{Z}_N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m}$
  isomorphic map $(r_1, r_2, \ldots, r_m) \mapsto k \in \mathbb{Z}_N$
  $$k = \sum_i \underbrace{r_i N_i^{-1}}_{k_i \in \mathbb{Z}_{n_i}} N_i \mod N$$
  $N_i = N/n_i \in \mathbb{Z}_N$
  $N_i^{-1} \in \mathbb{Z}_{n_i}$ is the multiplicative inverse of $N_i$ within $\mathbb{Z}_{n_i}$
  usually by the extended Euclidean algorithm
- dual map $k \leftrightarrow (k_1, k_2, \ldots, k_m)\,,\ k_i \in \mathbb{Z}_{n_i}$
  $$k = \sum_i k_i N_i \mod N$$
  $$\Downarrow$$
  $$\frac{k}{N} = \sum_i \frac{k_i}{n_i} \mod 1 \ \rightsquigarrow \ \text{additivity of energy}$$

# Additivity of energy of a composite quantum system

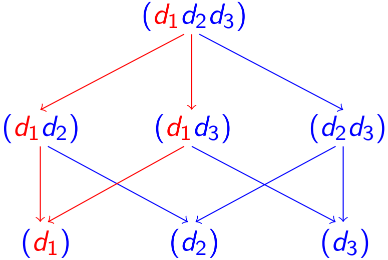
- Planck relation $E = h\nu$, energy = frequency
- Energy additivity $E(A \cup B) = E(A) + E(B) + \Delta E(A, B)$
- Hamiltonian
$$U = \mathrm{e}^{2\pi \mathbf{i} H} \sim \mathrm{diag}\left(\mathrm{e}^{2\pi \mathbf{i} E_0}, \mathrm{e}^{2\pi \mathbf{i} E_1}, \ldots\right)$$
all eigenvalues of $U \in \mathrm{H}(n)$ are $n$th roots of unity:
$$H_n \sim \mathrm{diag}\left(E_{n,k_0}, E_{n,k_1}, \ldots, E_{n,k_{n-1}}\right), \qquad E_{n,k_i} = \frac{k_i}{n}$$
- Composite system
$$U_N = U_{n_1} \otimes U_{n_2} \otimes \cdots \otimes U_{n_m}$$
$$\big\downarrow \log$$
$$H_N = H_{n_1} \otimes \mathbb{1}_{n_2} \otimes \cdots \otimes \mathbb{1}_{n_m} + \mathbb{1}_{n_1} \otimes H_{n_2} \otimes \cdots \otimes \mathbb{1}_{n_m} + \cdots + \mathbb{1}_{n_1} \otimes \mathbb{1}_{n_2} \otimes \cdots \otimes H_{n_m}$$
Additivity of energy as dual map in Chinese remainder theorem
$$E_{N,k} = \sum_i E_{n_i,k_i} \Longleftrightarrow \frac{k}{N} = \sum_i \frac{k_i}{n_i} \mod 1$$

# Quantum mereology

Mereology: part-to-whole and part-to-part relations within a whole

$$\mathcal{H}_{d_1 d_2 d_3} = \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3}$$

Hasse diagram



1. Appendix
    - David Hilbert
    - Gerard ‘t Hooft
    - Hermann Weyl
    - Tom Banks
    - Ordinary view of finite QM
    - MUBs
    - SIC POVMs

# David Hilbert

To 1

David Hilbert. *On the infinite*

"Our principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought — a remarkable harmony between being and thought."

# Gerard 't Hooft

*We postulate the existence of an ontological basis. It is an orthonormal basis of Hilbert space that is truly superior to the basis choices that we are familiar with. In terms of an ontological basis, the evolution operator for a sufficiently fine mesh of time variables, does nothing more than permute the states.*

The Cellular Automaton Interpretation of Quantum Mechanics. Springer, 2016, p. 66

To 3

# Hermann Weyl

*Our general principle allows for the possibility that the Abelian rotation group is entirely discontinuous, or that it may even be a finite group. . . .*
*Because of these results I feel certain that the general scheme of quantum kinematics formulated above is correct. But the field of discrete groups offers many possibilities which we have not as yet been able to realize in Nature; perhaps these holes will be filled by applications to nuclear physics.*

The Theory of Groups and Quantum Mechanics.
1928, transl. 1950 Dover, p. 276

To 6

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# Finite Deformations of Quantum Mechanics

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**Abstract**

We investigate modifications of quantum mechanics (QM) that replace the unitary group in a finite dimensional Hilbert space with a finite group and determine the minimal sequence of subgroups necessary to approximate QM arbitrarily closely for general choices of Hamiltonian. This mathematical study reveals novel insights about 't Hooft's Ontological Quantum Mechanics, and the derivation of statistical mechanics from quantum mechanics. We show that Kornyak's proposal to understand QM as classical dynamics on a Hilbert space of one dimension higher than that describing the universe, supplemented by a choice of the value of a naturally conserved quantum operator in that classical evolution, can probably be a model of the world we observe.

To 1

# Ordinary view of finite QM

*Quantum state spaces are continuous, but they have some intriguing realisations of discrete structures hidden inside. . . . The structures we are aiming at are known under strange acronyms such as 'MUB' and 'SIC'.*

# MUBs

## Mutually unbiased bases

$$\left|\left\langle \psi_j^m \,|\, \psi_k^n \right\rangle\right|^2 = \frac{1}{N}; \quad m \neq n; \quad \overbrace{m, n = 1, \ldots, K}^{\text{bases}} \leq N + 1, \quad \overbrace{j, k = 1, \ldots, N}^{\text{vectors}}$$

Maximal number, $N + 1$, of MUBs is achieved when $N = p^\ell$

Total number of vectors in complete set is $N^2 + N$

Associated with finite affine planes with $N^2$ points and $N^2 + N$ lines

# SIC POVMs

## Symmetric informationally complete positive operator-valued measures

$$N^2 \text{ vectors } |\psi_j\rangle \in \mathcal{H}_n$$

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{N\delta_{jk}+1}{N+1}, \quad j,k = 1,\ldots,N^2$$

Zauner's conjecture:

1. In every dimension there exists a SIC which is an orbit of the Weyl–Heisenberg group
2. There exists a SIC of this kind, where the individual vectors are invariant under a Clifford group element of order 3

Associated with finite projective planes