Combinatorial Ky Fan theorem for sphere bundles

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(By a joint work with Rade Zivaljevic)

Combinatorial statements, such as theorems of Carathéodory, Radon, Helly, Sperner, Tucker, Ky Fan, etc., are fundamental results of combinatorial (algebraic) topology, accessible to non-specialists, which are immediately applicable to mathematical economics, data science, game theory, graph theory, mathematical optimization, computational geometry, and other fields.

The Ky Fan theorem is also a disguised combinatorial counterpart of the Borsuk-Ulam theorem. Recall its usual set-up: the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$ is triangulated and the triangulation is assumed to be centrally symmetric. There is a labeling (coloring) of vertices of this triangulation

$$\lambda : Vert(S^n) \to \{\pm 1, ..., \pm N\}$$

which is

- *antipodal*, $\lambda(-v) = -\lambda(v) \ \ \forall v \in Vert(S^n)$, and
- $\lambda(v) \neq -\lambda(w)$ for each pair $\{v, w\}$ of adjacent vertices of the triangulation.

The *alternating number* $Alt(\sigma)$ of a simplex is the number of sign changes in the labels of its vertices (which are ordered by the absolute values). For example $Alt(-1, 2, 3, -4) = 2; \ \ Alt(-1, 2, -3, 4) = 3$, etc.

Clearly, the alternating numbers of a simplex and its antipodal one are equal. The maximal possible alternating number is n, and these simplices come in pairs. The Ky Fan theorem states that $n < N$, and the number of (pairs of) simplices with alternating number n is odd.

We shall address the following questions:

When is it possible to replace the triangulated sphere by some other triangulated manifold with a free $\mathbb{Z}_2$-action? What happens if one replaces a unique sphere S^n by a parameterized continuous family of spheres, that is, by the total space of some spherical bundle over a smooth manifold?

Our main results are:

- For a spherical bundle, there are "many" simplices with alternating number n; taken together, they form a closed pseudomanifold which is topologically as complicated as the base of the bundle.
- For non-trivial bundles one expects simplices with alternating numbers bigger than n. How much bigger depends on the Stiefel-Whitney classes of the bundle.
- Some explicit examples will be provided. They include spherical bundles associated to the tangent bundles of selected real and complex projective spaces.

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