On multidimensional analogs of Euler (Tait-Bryan) angles and Grassmanians.

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joint work with
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Super-task.

The initial problem is a construction of "good-comfortable" coordinates on the space of Hermitian matrices $\mathcal{H}$.

This space has a natural stratification in accordance with the conjugated classes

$$\mathcal{H} = \bigsqcup_{J} \mathcal{O}_J \text{ , where } \mathcal{O}_J := \bigcup_{g \in SU(N)} g^{-1} J g.$$

Here J runs all different diagonal matrices with the ordered diagonal values $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_N$.

Subject of the talk.

I introduce the parametrization of a fixed orbit. It is a parametrization of the all splittings of $\mathbb{C}^N$ on the sum of mutually orthogonal subspaces of the fixed dimensions:

$$\mathbb{C}^N = (e_0) \overset{\perp}{\oplus} (e_1) \overset{\perp}{\oplus} (e_2) \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} (e_{m-1}) \overset{\perp}{\oplus} (e_m),$$

where $\mathsf{dim}\ (e_k) = n_k, \sum_k n_k = N$.

In the simplest case when $N = 3, n_k = 1$ and everything is real, it is the parameterization of $SO(3) \subset SU(3)$ (locally). The problem solved by L. Euler.

Euler angles.

A problem solved by Euler is the parametrization of the mutual positions of two orthonormal bases in $\mathbb{R}^3$. He introduced three subsequent rotations around the axes and three angles of the rotations, see Fig. 1.

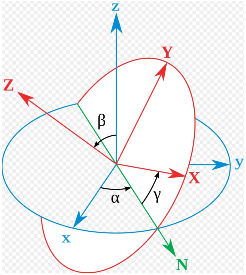

Figure: Euler angles α, β, γ.

It was L. Euler who introduced a principal concept that is "the line of nodes" $\mathcal{N}$.

Notations.

Let $\{\mathrm{x},\mathrm{y},\mathrm{z}\}$ and $\{\mathrm{X},\mathrm{Y},\mathrm{Z}\}$ be the initial and the finite positions of the orthogonal semi-axes. The intermediate positions we denote by the primes, like $\{\mathrm{x}'',\mathrm{y}'',\mathrm{z}''\}$. letters in brackets like (x,y) are enveloping subspaces (planes). Let $\rho_\kappa(\phi)$ be a rotation around the axis κ, $\kappa \in \{\mathrm{x},\mathrm{y},\mathrm{z},\mathrm{x}',\mathrm{y}',\dots\}$ on the angle ϕ. The subsequent rotations on Euler angles that coincide $\{\mathrm{x},\mathrm{y},\mathrm{z}\}$ and $\{\mathrm{X},\mathrm{Y},\mathrm{Z}\}$ are:

$$\{\mathrm{x},\mathrm{y},\mathrm{z}\} \overset{\rho_{\mathrm{z}}(\alpha)}{\to} \{\mathrm{x}',\mathrm{y}',\mathrm{z}\} \overset{\rho_{\mathrm{x}'}(\beta)}{\to} \{\mathrm{x}''=\mathrm{x}',\mathrm{y}'',\mathrm{Z}\} \overset{\rho_{\mathrm{Z}}(\gamma)}{\to} \{\mathrm{X},\mathrm{Y},\mathrm{Z}\}.$$

The axes of the rotations are z,Z and the line of nodes $\mathcal{N} = \mathrm{x}' = \mathrm{x}''$:

$$\mathcal{N} := (\mathrm{x},\mathrm{y}) \cap (\mathrm{X},\mathrm{Y})$$

that is an intersection of the initial and the finite positions of XY-coordinate planes.

Generalization. First attempt (unsuccessful).

Let us change the one-dimensional subspaces (axes) x, y, z by the multidimensional subspaces $(e_1), (e_2), (e_3)$, and the rotation

$$\rho_z(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

around the axis by some unitary transformation that is identical on E_3 and "rotates" (turns, maps, sends) subspaces (e_1) to (E_1) and (e_2) to (E_2).

$$(e_1) \oplus (e_2) = (E_1) \oplus (E_2) \sim \mathbb{C}^{n_1+n_2}$$

The disadvantage:
There is no subspace among e_k which dimension is equal to the dimension of $(e_1, e_2) \cap (E_1, E_2)$ that is the analog of the line of nodes $\mathcal{N}$.

Tait–Bryan angles.

There is (classical) modification of the Euler's method named after P. G. Tait and G. H. Bryan. The main difference is another line of nodes $\mathcal{N}_{\mathrm{TB}} := (\mathrm{x}, \mathrm{y}) \cap (\mathrm{Y}, \mathrm{Z})$:

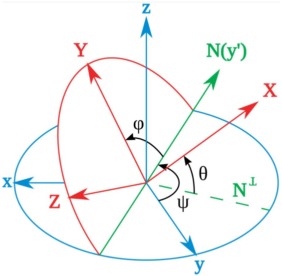

Figure: Tait-Bryan angles ϕ, θ, ψ.

$$\{\mathrm{x}, \mathrm{y}, \mathrm{z}\} \xrightarrow{\rho_{\mathrm{z}}(\psi)} \{\mathrm{x}', \mathrm{y}', \mathrm{z}\} \xrightarrow{\rho_{\mathrm{y}'}(\theta)} \{\mathrm{X}, \mathrm{y}', \mathrm{z}'\} \xrightarrow{\rho_{\mathrm{X}}(\phi)} \{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}.$$

Dimension of $\mathcal{N}_{\rm TB}$.

The subspaces $\mathcal{N}_{\rm TB}$ and $({\rm Y})$ have the same dimension:
$\textsf{dim}\ \mathcal{N}_{\rm TB} =$

$\textsf{dim}\ ({\rm x}, {\rm y}) + \textsf{dim}\ ({\rm Y}, {\rm Z}) - {\rm N} = ({\rm n_x} + {\rm n_y}) + ({\rm n_y} + {\rm n_z}) - ({\rm n_x} + {\rm n_y} + {\rm n_z})$

$= {\rm n_y}$, and can be coincided by an element of the orthogonal group of the space $({\rm z})^{\perp}$.

Generalization of the set of rotations of plane.

The position of the ordered pair of the orthogonal lines on a plane can be parametrized by the angle between two corresponding lines that is a point of $P\mathbb{R}^2$.

All the splittings of $\mathbb{C}^{n_1+n_2}$ on two orthogonal subspaces $(e_1) \overset{\perp}{\oplus} (e_2)$, $\mathsf{dim}\ (e_1) = n_1$, $\mathsf{dim}\ (e_2) = n_2$ can be naturally identified with the points of the Grassmanian $G(n_1, n_1 + n_2) \simeq G(n_2, n_1 + n_2)$.

Multidimensional version of Tait–Bryan rotations.

Let the space $\mathbb{C}^N$ be split on $m+1$ mutually orthogonal subspaces:

$$\mathbb{C}^N = (e_0) \overset{\perp}{\oplus} (e_1) \overset{\perp}{\oplus} (e_2) \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} (e_{m-1}) \overset{\perp}{\oplus} (e_m),$$

that we will transform, step-by-step, to "the fixed" splitting

$$\mathbb{C}^N = (E_0) \overset{\perp}{\oplus} (E_1) \overset{\perp}{\oplus} (E_2) \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} (E_{m-1}) \overset{\perp}{\oplus} (E_m)$$

with the same dimensions of the corresponding subspaces. The first set of m steps $\Phi_1, \ldots, \Phi_m$ (unitary) moves all subspaces $(e_1), \ldots, (e_m)$ to the fixed subspace $(E_1, \ldots, E_m)$ and, consequently, (e_0) to (E_0).

Step 1.

Let us denote $\mathcal{N}_1 := (e_0, e_1) \cap (E_0)^{\perp} \sim (e_1)$ and transform $\mathbb{C}^N$ by a unitary transformation Φ_1 that is non-trivial on (e_1, e_0) and is identical on its orthogonal complement that is $(e_2, \ldots, e_m)$.

Φ_1 moves (e_1) to $\mathcal{N}_1 =: (e_1')$ and e_0 moves to some (e_0^1) that is orthogonal to (e_1').
So $\Phi_1 \in SU(N)$ moves (e_1) "inside" (e_1, e_0) into the given subspace of $(E_0)^{\perp}$. This subspace is $\mathcal{N}_1$.

Step 2.

Let us repeat the same action with (e_2) and (e_0^1).
Φ_2 is identical on $(\mathrm{e}_1', \mathrm{e}_3, \ldots, \mathrm{e}_\mathrm{m})$ and transforms whole $(\mathrm{e}_2, \mathrm{e}_0^1)$ to itself.
We denote $(\mathrm{e}_2, \mathrm{e}_0^1) \cap (\mathrm{E}_0)^\perp =: \mathcal{N}_2$, and set Φ_2 a unitary transformation that moves (e_2) to $\mathcal{N}_2 \subset (\mathrm{E}_0)^\perp$

$$\Phi_2 : (\mathrm{e}_2) \to \mathcal{N}_2 =: (\mathrm{e}_2'), (\mathrm{e}_0^1) \to (\mathrm{e}_0^2).$$

Iterations.

We continue in the same way.
Φ_k is identical on $(e'_1, \dots, e'_{k-1}, e_{k+1}, \dots, e_m)$ and transforms whole (e_k, e_0^{k-1}) to itself.
We denote $(e_k, e_0^{k-1}) \cap (E_0)^{\perp} =: \mathcal{N}_k$, and set Φ_k a unitary transformation that moves (e_k) to $\mathcal{N}_k \subset (E_0)^{\perp}$

$$\Phi_k : (e_k) \to \mathcal{N}_k =: (e'_k), (e_0^{k-1}) \to (e_0^k).$$

Last step of the set.

After these m steps we get the same problem with the smaller dimension $N \to N - n_0$, $m \to m - 1$.
On the step number m we define Φ_m that is identical on $(e'_1, \ldots, e'_{m-1}) \supset (E_0)^\perp$ and transforms whole (e_m, e_0^{m-1}) to itself.
We denote $(e_m, e_0^{m-1}) \cap (E_0)^\perp =: \mathcal{N}_m$, and set Φ_m a unitary transformation that moves (e_m) to $\mathcal{N}_m \subset (E_0)^\perp$

$$\Phi_m : (e_m) \to \mathcal{N}_m =: (e'_m), (e_0^{m-1}) \to (e_0^m).$$

We see that all m mutually orthogonal subspaces split $(E_0)^\perp$ on the direct sum, consequently $(e_0^m) = (E_0)$.

Reduction of the dimension $N \to N - n_0$, $m \to m - 1$.

$$\mathbb{C}^{N-n_0} \sim (e'_1) \overset{\perp}{\oplus} (e'_2) \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} (e'_{m-1}) \overset{\perp}{\oplus} (e'_m),$$

that we will transform, step-by-step, to "the fixed" splitting of the same space

$$(E_1) \overset{\perp}{\oplus} (E_2) \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} (E_{m-1}) \overset{\perp}{\oplus} (E_m).$$

The next set of $m - 1$ steps unitary moves all subspaces $(e'_2), \ldots, (e'_m)$ to the fixed subspace $(E_1)^\perp = (E_2, \ldots, E_m)$ and, consequently, (e'_1) to (E_1).
By acting in this way we will get the space $(E_{m-1}E_m)$ split on two orthogonal subspaces $(e^{(m)}_{m-1})$ and $(e^{(m)}_m)$ isomorphic to (E_{m-1}) and (E_m) correspondingly. We transform them to (E_{m-1}) and (E_m) by unitaty transformation.

Summary.

The space of Hermitian matrices $\mathcal{H}$ is parametrized by the eigen-numbers of the corresponding eigen-spaces and points of Grassmanians:

$$\begin{array}{rc}
G(n_1, n_1+n_0), G(n_2, n_2+n_0), \dots, G(n_m, n_m+n_0) & \lambda_0 \\
G(n_2, n_2+n_1), \dots, G(n_m, n_m+n_1) & \lambda_1 \\
\vdots & \vdots \\
G(n_m, n_m+n_{m-1}) & \lambda_{m-1} \\
0 & \lambda_m
\end{array}$$

Example: $n_k = 1\ \forall k$.

The manifold of subspaces on $\mathbb{C}^2$ is the Riemann sphere $P\mathbb{C}^2 = \mathbb{C}P^1 \ni (z:1)$, or Bloch sphere:
$\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta,\ \theta \in [0,\pi], \phi \in [0, 2\pi]$ in the spherical coordinates.
The connection is $(z:1) = (e^{i\phi}\sin\theta/2 : \cos\theta/2)$. The Cartesian coordinates of the point on the sphere P_1, P_2, P_3 corresponding to $z = x + iy$ are given by the stereographic projection

$$P_1 = \frac{2x}{1+x^2+y^2}, P_2 = \frac{2y}{1+x^2+y^2}, P_3 = \frac{1-x^2-y^2}{1+x^2+y^2}.$$

We get $N(N-1)/2$ complex numbers and N real numbers that are the eigenvalues, that is N^2 real parameters. It is the dimension of $U(N)$.

Torus action.

There is a natural action of the torus T^N on the Grassmanian $G(n, N)$:

$$(x_1, x_2, \dots, x_N) \to (t_1 x_1, t_2 x_2, \dots, t_n x_N),$$

where $(x_1, x_2, \dots, x_N) \in (e_1, \dots, e_N) \in G(n, N)$, $(t_1 x_1, t_2 x_2, \dots, t_n x_N) \in T^N$.
There are two versions of the torus action, namely $|t_k| = 1$ – we call it "a real torus", and $t_k \in \mathbb{C}^*$, that is "a complex torus".
Complex torus action, evidently, has thee orbits on $\mathbb{C}P^1$. The factorization moves $\mathbb{C}P^1$ to the discreet set of three points, say $(0 : 1), (1 : 0), (1 : 1)$.
Real torus action gives some realization of the presented complex theory. In the case of one-dimensional subspaces (e_k) the Bloch spheres become the unit circles parametrized by the real angle ϕ and $U(N)$ becomes $O(N)$.

The End.

Thank You!:)