The Euler top and the Lagrange top as two special cases of the Galois top

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Rigid body motion about a fixed point

Consider the (rotational) motion of a rigid body about a fixed point O. Denote by $\boldsymbol{\omega}$ and $\boldsymbol{m}$ the pseudovectors of its angular velocity and the angular momentum, respectively.

The pseudovectors $\boldsymbol{\omega}$ and $\boldsymbol{m}$ might be expressed in a (rotating) coordinate system which directing unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are aligned along the principal axes of inertia, passing through the point O,[1] that is,

$$\boldsymbol{\omega} = p\,\boldsymbol{i} + q\,\boldsymbol{j} + r\,\boldsymbol{k}\ ,\ \ \boldsymbol{m} = A\,p\,\boldsymbol{i} + B\,q\,\boldsymbol{j} + C\,r\,\boldsymbol{k}, \tag{1}$$

where A, B and C are the principal moments of inertia, corresponding to the principal directions $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$, respectively.[2]

The expression for the angular momentum, given in (1), is based on the identities

$$\boldsymbol{m} = \int \boldsymbol{r} \times \boldsymbol{\omega} \times \boldsymbol{r}\, d\rho =^{3} \int r^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{r})\, \boldsymbol{r}\, d\rho = J\, \boldsymbol{\omega},\ \ J = (J^n_m),\ \ J^n_m := \int \delta^n_m r^2 - r_m r_n\, d\rho,^{4}$$

where $\boldsymbol{r}$ is the radius vector of an "infinitesimal" mass element $d\rho$,[5] the (raw and column) indices m and n run over the values $1, 2, 3$ and δ^n_m is the Kronecker symbol, that is, δ^n_m vanishes, unless it acquires the unit value when $m = n$ (on the diagonal of J). Thus, the matrix J, expressing the tensor of inertia of a rigid body, with respect to an orthogonal basis, fixed in it, being real and symmetric,[6] is unitarily diagonalizable.

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[1] The principal axes of inertia need not pass through the centre of mass, contrary to a false, yet common belief notably shared, among others, by Richard Feynman.

[2] We note that a reversal in the ordering of moments of inertia (ascendingly or descendingly) corresponds to swapping the chirality between a "right-handed" and a "left-handed" coordinate system.

[3] Although the cross product is non associative we need not place brackets in this special (associative) case.

[4] We have tacitly assumed an orthogonal coordinate system.

[5] We use the same letter (bolded and not) to (respectively) denote both a vector and its modulus.

[6] Note, moreover, that J is positive definite.

We point out that once the matrix I of the tensor of inertia of a rigid body, about its centre of mass G, has been calculated, the Huygens-Steiner theorem enables a swift calculation of the matrix J of the tensor of inertia about any other point O, as

$$J = I + \begin{pmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_3 r_1 \\ -r_1 r_2 & r_3^2 + r_1^2 & -r_2 r_3 \\ -r_3 r_1 & -r_2 r_3 & r_1^2 + r_2^2 \end{pmatrix} \int d\rho, \;\; \boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} := O - G.^{7}$$

Note that the principal axes of inertia about the point O do not (in general) coincide with the principal axes of inertia about the centre of mass G. However, if O lies on a principal axis through G then the directions of the principal axes through O would coincide with the directions of the principal axes through G.[8]

[7] The coordinates r_1, r_2 and r_3 of the radius vector $\boldsymbol{r}$ are presummed to be calculated in the same coordinate system, centred at G, in which I was caculated. Thus, the calculation of J merely requires the coordinates of O (in the said coordinate system) and body's (total) mass $\int d\rho$.

[8] That is, the (unordered) set of directions of the principal axes is preserved in this special case.

The notion of the Galois top

We consider a heavy top, in a uniform gravitational field, which fixed point O lies at the Galois axis [2, 4, 12, 16, 17]. We shall call such a top the "Galois top".

If we impose a coordinate system which origin is the body's centre of mass G and its axes are aligned along the principal axes of inertia (which share G as their common point) then the coordinates of the point O would be expressed as

$$O = O(d) := d\left(g_1 := \sqrt{\frac{C(A-B)}{B(A-C)}},\, 0,\, g_3 := \sqrt{\frac{A(B-C)}{B(A-C)}}\right), \qquad (2)$$

where d is the distance between G and O, whereas A, B and C are the principal moments of inertia about the centre of mass G.

Replacing these principal moments (about the centre of mass G), appearing on the right-hand side of (2) with the principal moments of inertia about the point O would yield the coordinates of the centre of mass G with respect to a coordinate system which origin is the (fixed) point O and its axes are aligned along the principal axes of inertia about the point O.[9]

[9] Although only the principal axis, corresponding to the intermediate moment of inertia, about the centre of mass G, would transform to a colinear principal axis about the point O, as further explained in [8].

The Euler equations for the Galois top

Denoting with $\boldsymbol{n}$ the unit vertical upwards (in the direction opposing the direction of gravity), the exerted torque $\boldsymbol{\tau}$ (that is, the moment of external forces) is calculated as the cross product of the radius vector $\boldsymbol{d} = d(g_1, 0, g_3)$, eminating from the fixed point O towards the centre of mass G, and the weight of the top $\boldsymbol{w}$ (that is, the gravitational force):[10]

$$\boldsymbol{\tau} = w\,\boldsymbol{n} \times \boldsymbol{d}.$$

Expressing $\boldsymbol{n}$ via the Euler angles as

$$\boldsymbol{n} = \sin\phi \sin\theta\, \boldsymbol{i} + \cos\phi \sin\theta\, \boldsymbol{j} + \cos\theta\, \boldsymbol{k},$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the principal axes, corresponding to the principal moments of inertia A, B and C (about the point O), respectively, we might, accordingly,[11] write down the Euler equations as

$$\begin{aligned} A\,\dot{p} + (C - B)\,q\,r &= g_3 \cos\phi \sin\theta, \\ B\,\dot{q} + (A - C)\,r\,p &= g_1 \cos\theta - g_3 \sin\phi \sin\theta, \\ C\,\dot{r} + (B - A)\,p\,q &= -g_1 \cos\phi \sin\theta, \end{aligned}$$

where p, q and r are the projections of the angular velocity $\boldsymbol{\omega}$ upon the principal axes $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, respectively.

[10] That gravitational force w is applied to the centre of mass. Its magnitude coincides with the magnitude of the weight, as defined by the third General Conference on Weights and Measures, in 1901 [18, p. 46].

[11] Upon assuming that $O \neq G$ (so that $d \neq 0$) and that the units of distance and force are so chosen so that the product wd coincides with unity.

The Lagrange top as a special case of the Galois top I

Permitting both ascending and descending ordering of the principal moments of inertia, the Lagrange top arises if $A = B$. The corresponding Euler equations become

$$\begin{aligned} B\,\dot p + (C - B)\,r\,q &= \cos\phi\sin\theta,\\ B\,\dot q + (B - C)\,r\,p &= -\sin\phi\sin\theta,\\ C\,\dot r &= 0. \end{aligned}$$

Thus, r is constant along with the two constants $c = B\,\dot\psi(\sin\theta)^2 + C\,r\cos\theta$ and $h = B(\dot\psi\,\sin\theta)^2 + {} + B(\dot\theta)^2 + C\,r^2 + 2\,w\,d\cos\theta$,[12] and so $\cos\theta$ is an elliptic function (of time) which satisfies the differential equation:

$$B\,(\dot x)^2 = 2\,w\,d\,x^3 + \left(\left(1 - \frac{C}{B}\right)\,C\,r^2 - h\right)x^2 + 2\left(\frac{C\,r\,c}{B} - w\,d\right)x + h - C\,r^2 - \frac{c^2}{B}.$$

With x viewed as a variable of the cubic polynomial on the right-hand side of the latter equation we might express its (three) roots as $x_s = \alpha_1/6 + \gamma(4\,s)$, $s \in \{0, 1, 2\}$, where

$$\gamma(s) := \frac{(\alpha_1^2 - 6\,\alpha_2)/\beta_s + \beta_s}{6},\;\; \beta_s := e^{s\,\pi\,i/6}\,\sqrt[3]{\alpha_1^3 - 9\,\alpha_1\,\alpha_2 + 54\,\alpha_3 + \sqrt{-27\,d}},\;\; i := \sqrt{-1},$$

$$\alpha_1 := \frac{h + (C/B - 1)\,C\,r^2}{w\,d},\;\; \alpha_2 := 2\left(\frac{C\,r\,c}{w\,d\,B} - 1\right),\;\; \alpha_3 := \frac{c^2/B + C\,r^2 - h}{w\,d},$$

and $d := \alpha_2^2(\alpha_1^2 - 8\,\alpha_2) - 4\,\alpha_3(\alpha_1^3 - 9\,\alpha_1\alpha_2 + 27\,\alpha_3)$ is the discriminant of the cubic polynomial $2\,x^3 - \alpha_1\,x^2 + \alpha_2\,x - \alpha_3$.

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[12] Although we have stipulated that the product $w\,d$ is equal to one we still explicitly placed it on the right-hand side of the latter equation in order to emphasize it as a unit of energy.

The Lagrange top as a special case of the Galois top II

So if we put

$$x_s(t) := k\,\mathcal{R}\left(\sqrt{\frac{kwd}{2B}}\,(t - T_s)\,,\sqrt{\frac{\gamma(1)}{\gamma(-1)}}\,\right) + x_0,$$

$$T_s := \frac{i^s\pi\sqrt{2B/(\sqrt{3}\,wd)}}{2M(\sqrt{\gamma(-2s+1)},\sqrt{\gamma(-2s-1)}\,)},$$

$$k := \sqrt{(x_1 - x_0)(x_2 - x_0)} = \sqrt{3\gamma(1)\gamma(-1)} = \sqrt{\frac{(\alpha_1^2 - 6\,\alpha_2)^2/\beta_0^2 + \alpha_1^2 - 6\,\alpha_2 + \beta_0^2}{12}},$$

where $\mathcal{R}(\cdot,\cdot)$ is the Galois essential elliptic function, as defined in [1, 3, 5, 6, 13], and $M(\cdot,\cdot)$ is the arithmetic-geometric mean (of its two variables), then $x_s(0) = x_s$ and each (elliptic) function $x_s(\cdot)$ does, indeed, satisfy the differential equation for $\cos\theta$. The index s might, in fact, be regarded as an integer modulo 3, so we have three functions $x_s(\cdot)$ which might be matched one with other via a corresponding argument-shift. The argument-shift is real-valued only between the functions $x_1(\cdot)$ and $x_2(\cdot)$. The real half-period T_0 of these two functions separates their consecutive extrema: a minimum for either function is a maximum for the other. Thus, for real-valued time the range of either function $x_1(\cdot)$ or $x_2(\cdot)$ is contained in the (closed) interval $[x_1, x_2]$, whereas the function $x_0(\cdot)$ is unbounded near odd multiples of T_0 and has x_0 as its (local) extremum at each even multiple of T_0.

The real half-period T_0 is also shared with the (elliptic) functions:

$$p^2 + q^2 = \frac{h - C\,r^2 - 2\,wdx_s}{B},\ \ \dot\phi \pm \dot\psi = \frac{(B-C)\,r}{B} \pm \frac{c \pm C\,r}{B\,(1 \pm x_s)}.$$

A "relevant digression" on the "current state" of affairs

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Analytical Solution to the Lagrange Top

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Abstract

In this article, the exact periodic and bounded solutions for the motions of Lagrange top with initial conditions are obtained. These solutions are expressed in terms of the Weierstrass elliptic function.

Let the effective potential be

$$U_{\text{eff}} = \frac{1}{2I'}\frac{(M_z - M_3\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta \tag{1.7}$$

and

$$E' = \frac{I'\dot{\theta}^2}{2} + U_{eff}. \tag{1.8}$$

Consequently,

$$E = E' + \frac{M_3^2}{2I_3} \tag{1.9}$$

In view of (1.8) and (1.7), we have

$$\dot{\theta}^2\sin^2\theta = \frac{2}{I'}\{E'\sin^2\theta - \frac{1}{2I'}((M_z - M_3\cos\theta)^2 - Mgl\sin^2\theta\}. \tag{1.10}$$

Let $u = \cos\theta$, $du = -\sin\theta d\theta$. Then, from (1.10),

$$\dot{u}^2 = \left(\frac{2}{I'}E' - \frac{2Mgl}{I'}u\right)(1-u^2) - \left(\frac{M_z}{I'} - \frac{M_3}{I'}u\right)^2 \tag{1.11}$$

Introducing the notations

$$a = \frac{M_3}{I'}, b = \frac{M_z}{I'}, \alpha = \frac{2}{I'}E', \beta = \frac{2Mgl}{I'} \tag{1.12}$$

expression (1.11) takes the form

$$\dot{u}^2 = (1-u^2)(\alpha - \beta u) - (b - au)^2 \tag{1.13}$$

In view of (1.13), $\dot{u}^2$ is a cubic polynomial in u as follows:

$$\dot{u}^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \tag{1.14}$$

Taking the derivative w.r.t. t and taking into account $\dot{u} \neq 0$, we get the Helmholtz oscillator equation

2 Analytical Solution

In this section we will solve the initial value problem

$$\dot{u}^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2),\ u(0) = u_0 \text{ and } u'(0) = \dot{u}_0 \quad (2.18)$$

Let

$$u(t) = A + B\wp(t - t_0; g_2, g_3), \quad (2.19)$$

where $\wp(t) = \wp(t - t_0; g_2, g_3)$ is the Weierstrass elliptic function. This function satisfies the nonlinear ode $\wp'(t)^2 = 4\wp^3(t) - g_2\wp(t) - g_3$. The numbers g_2 and g_3 are called the Weierstrass elliptic invariants.

Inserting equation (2.19) into (2.18), gives

$$\begin{aligned} &-aA^2 - 2aAb + b^2 - B^2 g_3 - \alpha - A^2\alpha + A\beta - A^3\beta - \\ &B\left(2aA + 2ab + Bg_2 + 2A\alpha - \beta + 3A^2\beta\right)\wp(t) - \\ &B^2(a + \alpha + 3A\beta)\wp(t)^2 - B^2(-4 + B\beta)\wp(t)^3 = 0. \end{aligned}$$

Equating the coefficients of $\wp^j(t)$ to zero, gives an algebraic system. Solving it using the initial conditions, we obtain the following solutio :

$$\begin{aligned} &A = \tfrac{-a-\alpha}{3\beta},\ B = \tfrac{4}{\beta},\ g_2 = \tfrac{1}{12}\left(a^2 + \alpha^2 + 2\alpha a - 6ab\beta + 3\beta^2\right). \\ &g_3 = \tfrac{1}{432}\left(-2a^3 - 2\alpha^3 - 6\alpha a^2 + 18a^2 b\beta - 6\alpha^2 a - 9a\beta^2 - 36\alpha\beta^2 + 18\alpha ab\beta + 27b^2\beta^2\right). \\ &t_0 = \pm\wp^{-1}\left(\begin{array}{c} \tfrac{1}{12}\left(a + \alpha + 3u_0\beta\right); \tfrac{1}{12}\left((a+\alpha)^2 + 3\beta^2 - 6ab\beta\right), \\ \tfrac{1}{432}\left(-2(a+\alpha)^3 + 18ab\beta(a+\alpha) - 9\left(-3b^2 + a + 4\alpha\right)\beta^2\right) \end{array}\right). \end{aligned}$$

Finally, the solution to the top problem reads

$$\theta(t) = \cos^{-1}\left(-\frac{1}{3\beta}\left(a + \alpha - 12\wp\left(\begin{array}{c} t - t_0; \tfrac{1}{12}\left((a+\alpha)^2 + 3\beta^2 - 6ab\beta\right), \\ \tfrac{1}{432}\left(-2(a+\alpha)^3 + 18ab\beta(a+\alpha) - 9\left(-3b^2 + a + 4\alpha\right)\beta^2\right) \end{array}\right)\right)\right).$$

Hi! Semjon,

I am using Excel spreadsheets to calculate the numerical integrals of my "Gompertz-like" integrals. I guess that if I add the specifics of my excel

Spreadsheets, and I flesh out the VBscript code for skewness that I refer to in appendix A, then it will be pretty obvious that this is about computer tools that can analyze the noise distribution of threshold comparators. The stitching-up together the data acquired with N (twelve) sine wave periods can also be shown as an automated process (in fact, I did it manually but it would not be difficult at all to make it automatic in Excel… or whatever other programing language is the favorite… e.g. Python…) Maybe we can also "append" the spreadsheets to the article in electronic format?

I cannot believe how nobody so far has realized that the addition of a constant (which is what differentiates your R) makes the norm constant midway between the poles… it is so much more symmetric than all the other elliptic functions! Keep your head up, it is a just a matter of time until skeptical realize it!...

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23 Weierstrass Elliptic and Modular Functions

Weierstrass Elliptic Functions

23.6 Relations to Other Functions | 23.8 Trigonometric Series and Products

§23.7 Quarter Periods

23.7.1 $$\wp\left(\tfrac{1}{2}\omega_1\right) = e_1 + \sqrt{(e_1 - e_3)\,(e_1 - e_2)} = e_1 + \omega_1^{-2}(K(k))^2 k',$$

23.7.2 $$\wp\left(\tfrac{1}{2}\omega_2\right) = e_2 - \mathrm{i}\sqrt{(e_1 - e_2)\,(e_2 - e_3)} = e_2 - \mathrm{i}\omega_1^{-2}(K(k))^2 k k',$$

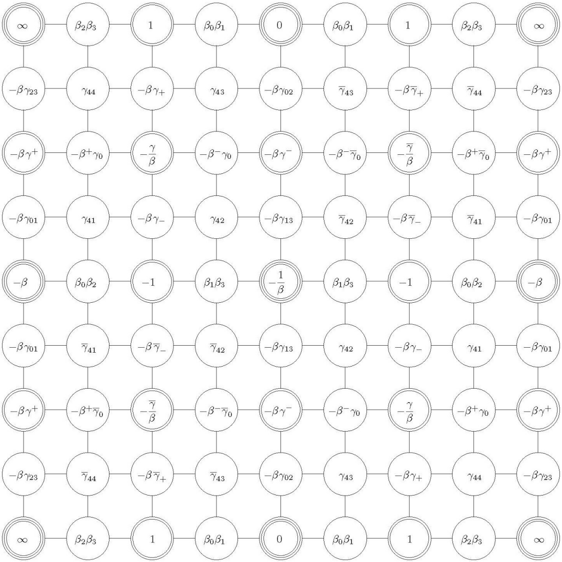
23.7.3 $$\wp\left(\tfrac{1}{2}\omega_3\right) = e_3 - \sqrt{(e_1 - e_3)\,(e_2 - e_3)} = e_3 - \omega_1^{-2}(K(k))^2 k,$$

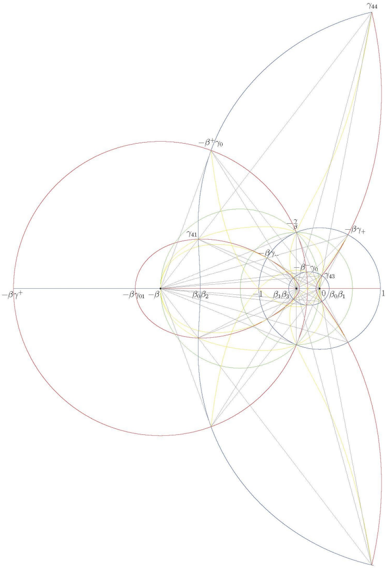
where k, k' and the square roots are real and positive when the lattice is rectangular; otherwise they are determined by continuity from the rectangular case.

A printed companion is available.

23.6 Relations to Other Functions | 23.8 Trigonometric Series and Products

The essential elliptic function as a conformal map from an interior of a rectangle to a half plane $\mathbb{H}$





The Euler top as a special case of the Galois top I

The Euler top is the special case of the Galois top, corresponding to $O = G$, so that the torque $\boldsymbol{\tau}$ vanishes. Guided by [2], we put

$$f(t, A, B, C) := \sqrt{\frac{BC}{(A-B)(A-C)}}\, i\, \mathcal{S}(t, A, B, C),$$

$$\mathcal{S}(t, A, B, C) := \mathcal{S}(t, I(A, B, C), I(A, C, B)), \ \ \mathcal{S}(t, \mu, \nu) := \frac{\sqrt{\mu\,\nu}}{i}\, \mathcal{S}\left(i\sqrt{\mu\,\nu}\left(t + \frac{\pi}{2M(\mu, \nu)}\right), \frac{\mu}{\nu}\right),$$

where $\mathcal{S}(\cdot, \cdot)$ is the Galois alternative elliptic function, as defined in [1, 3, 5, 6, 13], and $I(A, B, C) := \sqrt{\frac{(A-B)(m^2 - C\,h)}{ABC}}$.[13]

Note that permuting the principal moments A, B and C, as the (three) variables upon which the value of $-I(A, B, C)$ depends, would yield the six values $\mathcal{S}(T(B, C, A), A, B, C)$, $\mathcal{S}(T(C, A, B), A, B, C)$, $\mathcal{S}(T(C, A, B), B, C, A)$, $\mathcal{S}(T(A, B, C), B, C, A)$, $\mathcal{S}(T(A, B, C), C, A, B)$ and $\mathcal{S}(T(B, C, A), C, A, B)$, where

$$T(A, B, C) := \frac{\pi}{2M(I(A, B, C),\ I(A, C, B))}.$$

For fixed principal moments A, B and C, a root of $\mathcal{S}(\cdot, A, B, C)$, and hence of $f(\cdot, A, B, C)$ (when viewed as a function of its first variable), would coincide with (the half-period) $T(A, B, C)$, that is,

$$\mathcal{S}(T(A, B, C), A, B, C) = 0.$$

[13] Observe that $I(A, B, C)^2 + I(B, C, A)^2 + I(C, A, B)^2 = 0$.

Observing that

$$\dot{\mathcal{S}}(t,\mu,\nu)=\mathcal{S}(t,\mu,\nu)\,\mathcal{S}(t,i(\mu+\nu),i(\mu-\nu))=\mathcal{S}(t,\sqrt{\mu^2-\nu^2},i\,\nu)\,\mathcal{S}(t,\sqrt{\nu^2-\mu^2},i\,\mu),$$

we might verify that the equalities

$$A\dot{f}(t,A,B,C)=A\sqrt{\frac{BC}{(A-B)(A-C)}}\,i\,\dot{\mathcal{S}}(t,A,B,C)=A\sqrt{\frac{BC}{(A-B)(A-C)}}\,i\,\mathcal{S}(t,B,C,A)\,\mathcal{S}(t,C,A,B)=$$

$$=(C-B)\sqrt{\frac{CA}{(B-C)(B-A)}}\,\sqrt{\frac{AB}{(C-A)(C-B)}}\,\mathcal{S}(t,B,C,A)\,\mathcal{S}(t,C,A,B)=$$

$$=(B-C)\,f(t,B,C,A)\,f(t,C,A,B)$$

are preserved under cyclic permutations of the principal moments of inertia A, B and C. Thereby, these (three) cyclic permutations would generate, via acting on $f(t,A,B,C)$, the (three) coordinates $p=\boldsymbol{\omega}\cdot\boldsymbol{i}$, $q=\boldsymbol{\omega}\cdot\boldsymbol{j}$ and $r=\boldsymbol{\omega}\cdot\boldsymbol{k}$ of the angular velocity (in body's frame) of Euler top.

We now put

$$g(t,B,C,A):=\frac{2\,i\,CA}{C-A}\,\mathcal{T}\left(t,\;I(B,C,A),I(B,A,C)\right),\;\;\mathcal{T}(t,\mu,\nu):=\mathcal{S}\left(t,\;\frac{\mu-\nu}{2\,i},\;\frac{\mu+\nu}{2\,i}\right),$$

and observe that the four values $\pm(g_3\,C\,f(t,C,A,B)\pm g_1\,A\,f(t,A,B,C))$ would coincide with the four values $g(t,B,C,A)$, $g(t+2\,T(C,A,B),B,C,A)$, $g(t+2\,T(A,B,C),B,C,A)$, $g(t+2\,T(B,C,A),B,C,A)$.

The Galois top invariant of motion

Aside from the well-known "classical" invariants of motion,[14] the Galois top possesses its "own" invariant. Too often the trivial "geometrical" constraint, that is, the condition that the modulus of $\boldsymbol{n}$ is equal to one, is added (as a third invariant) to the "vertical" projection of the angular momentum $c := \boldsymbol{m} \cdot \boldsymbol{n}$ and (twice) the energy $h := \boldsymbol{\omega} \cdot \boldsymbol{m} + 2\, w\, \boldsymbol{d} \cdot \boldsymbol{n}$ (which are constants).

The "fourth" invariant of the Galois top is

$$g := e^{\beta \int_0^t q(t)\,dt} g(t, B, C, A), \;\; \beta := \frac{g_3 g_1 B(C-A)}{CA} = \sqrt{\frac{(A-B)(B-C)}{CA}},$$

where $g(t, B, C, A) := g_3 C\, r(t) + g_1 A p(t)$ is the projection of the angular momentum upon the Galois axis.[15]

We might, in particular, recall the case of the Euler top for which we do, indeed, have

$$\beta\, q(t) g(t, B, C, A) + \dot{g}(t, B, C, A) =$$

$$= \left(\beta \sqrt{\frac{CA}{(B-C)(B-A)}}\; i + 1 \right) \mathcal{S}(t, B, C, A) g(t, B, C, A) \equiv 0.$$

---

[14] An archaic term "first integral" is still much in use instead of the term "invariant".

[15] Thus, $g(t, B, C, A)$, along with the invariant $g = g(0, B, C, A)$, is either identically zero or never zero!

Instead of a conclusion: on citing, reciting and more reciting

V. V. Kozlov

NON-EXISTENCE OF AN ADDITIONAL ANALYTIC INTEGRAL IN THE PROBLEM OF THE MOTION OF AN UNSYMMETRICAL HEAVY SOLID ABOUT A FIXED POINT

If the ellipsoid of inertia is not an ellipsoid of revolution, the equations of motion of a solid are not integrable by Liouville quadratures. This result considerably strengthens the Poincaré—Husson theorem concerning the absence of an algebraic integral.

В. В. любит цитировать А. Пуанкаре: «Нет задач решенных и не решенных. А есть задачи, более решенные и менее решенные». Вот на этой оптимистической ноте мы и закончим наш очерк и предоставим возможность читателю познакомиться с оригинальными работами самого Валерия Васильевича.

А. В. Борисов, С. В. Болотин, А. А. Килин,
И. С. Мамаев, Д. В. Трещев
Январь, 2010

НЕЛИНЕЙНАЯ ДИНАМИКА, 2010, Т. 6, № 3, с. 461–473

Козлов, Валерий Васильевич (род. 1.01.1950) — русский математик и механик, академик РАН (с 2000 г.). В цикле работ, объединенных в монографии «Методы качественного анализа в динамике твердого тела» (МГУ, 1980), доказал несуществование аналитических интегралов уравнений Эйлера–Пуассона, а также указал динамические эффекты, препятствующие интегрируемости этих уравнений, — расщепление сепаратрис, рождение большого числа невырожденных периодических решений. Эти исследования «закрыли» проблему Пуанкаре, поставленную им в «Новых методах небесной механики» (т. 1), открыв тем самым новую эпоху в динамике твердого тела: на первый план вышли методы качественного исследования, а не поиск частных решений заданной алгебраической структуры.

В. В. Козлов

В. В. Козловым предложены также новые методы анализа интегрируемых систем, основанные на использовании геометрической теоремы Лиувилля–Арнольда и теоремы Вейля о равномерном распределении. Обосновывая метод Ковалевской, В. В. Козлов доказал ряд утверждений, связывающих ветвление общего решения на комплексной плоскости времени с несуществованием однозначных первых интегралов (гипотеза Пенлеве–Голубева). Для нахождения периодических решений в динамике твердого тела им впервые были применены вариационные методы. В. В. Козловым был создан целый цикл работ, связанных с движением тела в жидкости, в которых были решены многие классические вопросы, а также предложены новые модели и методы, опирающиеся на качественные исследования. Сейчас занимается исключительно вопросами статистической механики.

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