A set of tools developed based on the Calogero system for testing numerical methods for solving many-body problems

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Abstract. A set of testing tools has been developed to examine the behavior of numerical solutions in the vicinity of the collision point and to verify the conservation of integrals of motion when solving many-body problems using numerical methods. The proposed set of tests is based on the Calogero system and implemented in the fdm package integrated into the Sage computer algebra system. The first application of the developed set of tests revealed that the effective accuracy order found using Alshina's method has sharp jumps that have no theoretical explanation.

It is well known that in the general case the many-body problem is not an example of an integrable dynamic system. For this reason, the number of examples on which numerical methods for solving it can be tested is quite limited. Most of the existing tests do not allow to investigate the phenomenon of collision of bodies, and among the integrals of motion, in fact, it is only possible to test the conservation of energy [1, 2]. In the present paper, a tool based on the Calogero system [3] that can be used for testing difference schemes regarding these questions is proposed. The Calogero system is a one-dimensional system of many particles of the same mass interacting with each other with potential $\varphi \sim \frac{1}{(q_i - q_j)^2}$, where q_i is the coordinate of the i-th particle. This is the only system where the many-body problem has an analytical solution for any number of particles. Due to the purely algebraic properties of solutions and integrals of motion, this system is a very convenient example for creating tests for numerical methods. The set of tests proposed in this article was developed in the Sage computer algebra system called *fdm for sage* [4]. To apply this tool, the 5–body problem was chosen, in which the initial momenta are close to zero.

The initial problem for the Calogero system in the *fdm* package is specified separately, using the `Initial_problem` class [4], and the specially added `calogero_problem` function. The optional arguments of this function are: a set

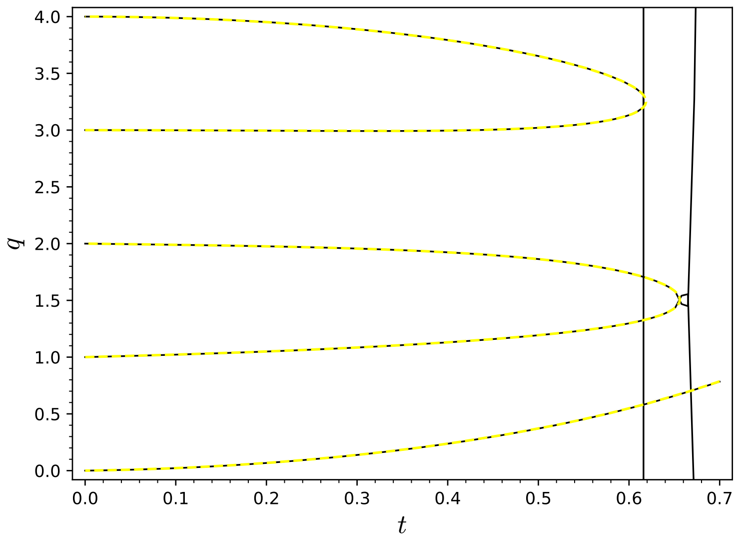

FIGURE 1. Analytical (dashed yellow line) and numerical (solid line) solutions for the 5-body problem.

of initial coordinates and momenta, the number of particles, the final time (the initial time we assume to be $t = 0$), and the interaction coefficient.

In order to obtain particle trajectories numerically, the 4th order Runge–Kutta method was used. Analytically, solutions of the equations of motion for the Calogero system can be found as eigenvalues of the matrix [3]

$$\hat{M} = \hat{Q}|_{t=0} + t\hat{L}|_{t=0},$$

where

$$\hat{Q} = q_i\delta_{ij},$$
$$\hat{L} = p_i\delta_{ij} + \sqrt{-b}\Big(\frac{1-\delta_{ij}}{q_i - q_j}\Big),$$

p_i is the i-th particle momenta and b is the interaction coefficient. A comparison of the numerical solutions with the analytical ones on the example of the 5-body problem is presented on the Fig.1. As it is clearly seen, a quite well degree of coincidence takes place.

The moment of collision of bodies can be found as the value of t for which the characteristic equation for the matrix $\hat{M}$ has several equal roots. For simplicity, we restricted ourselves to considering the collision of two bodies. The ways for numerically determining the position of moving singular points and diagnostics of the type of singularity in them are described in [5]–[9]. We used Alshina's method [5] to plot the effective accuracy order p_{eff} versus time. This method is based

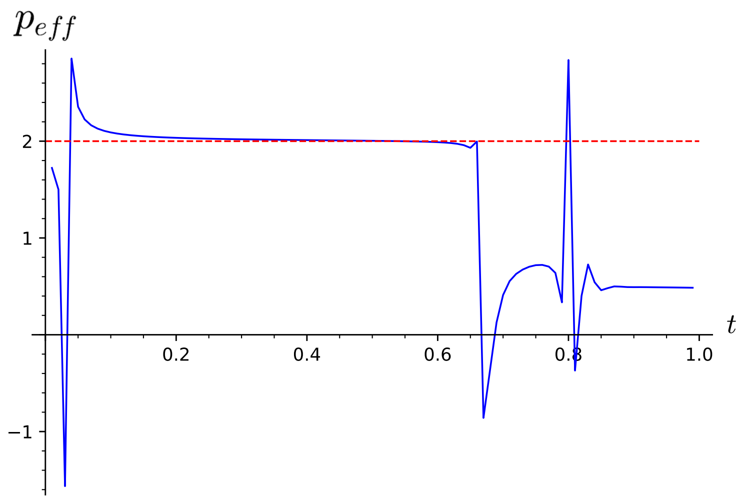

FIGURE 2. Problem of 5 bodies, effective order according to Alshina.

on the CROS scheme, which has a number of useful properties and is well suited for determining p_{eff} both before and after the singular point. In that part of the domain of the solution function where it's smoothness is not violated, p_{eff} should coincide with its theoretical value, that is, with the scheme order. At the singular point, according to [5], p_{eff} sharply changes its value, becoming equal to the order of the pole, which is a special feature of the CROS scheme. The obtained graph of $p_{eff}(t)$ is presented on the Fig.2 and shows that p_{eff} indeed behaves as expected, with the exception of a few sharp fluctuations (at the beginning of the plot and after the collision point) that have no theoretical explanation. Determining the cause of these jumps may be the subject of further research.

The integrals of motion in the Calogero system are rational functions and their number coincides with the number of particles, which makes this dynamic system completely integrable. Any symplectic Runge–Kutta scheme preserves linear and quadratic integrals of motion. In the Calogero system, the integrals of motion can be found as traces of the matrix $\hat{L}$ powers:

$$F_k = tr(\hat{L}^k).$$

For $k = 1$ the corresponding integral is linear and is preserved by any Runge–Kutta scheme. For $k = 2$ and $k = 3$, the integrals F_k are neither linear nor quadratic, and none of the Runge–Kutta methods provides their conservation. Having plotted the time dependence for F_2 and F_3, we saw that as time approaches the moment of collision, their absolute values increase monotonically. It should be noted that when numerically integrating the classical, three–dimensional many–body problem, sharp jumps in the integrals of motion are often observed. As it turned out, the test based on the Calogero system cannot display these features due to its simplicity.

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