On coefficients of the Berenstein-Kazhdan decoration functions for classical groups

Gleb Koshevoy, Denis Mironov

Institute for Information Transmission Problems Russian Academy of Sciences
Moscow Center for Continuous Math Education

Polynomial Computer Algebra 2024
April 19, 2024

Notation

G simple, simply connected (simply-laced) algebraic group over $\mathbb{C}$
$B,\ B^{-} \subset G$ its Borel subgroups, $T := B \cap B^{-}$ the maximal torus,
$W = \mathrm{Norm}_G(T)/T$ Weyl group,
$U,\ U^{-}$ be unipotent radicals,
$A = (a_{i,j})$ the Cartan matrix of G with an index set $I = \{1, 2, \cdots, n\}$.

$\mathfrak{g} = \mathrm{Lie}(G)$ with generators $e_i,\ f_i,\ h_i\ (i \in I)$,
Cartan subalgebra $\mathfrak{h}$,
canonical pairing $\langle , \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$.
Λ_i denote the i-th fundamental weight, that is, $\langle h_j, \Lambda_i \rangle = \delta_{j,i}$.

$P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ the weight lattice,
$P_+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$ the positive weight lattice,
$P^* = \oplus_{i \in I} \mathbb{Z} h_i$ the dual weight lattice,
$\{\alpha_i\}\ (i \in I)$ the set of simple roots.

Notation

For each $\lambda \in P_+$, let $V(\lambda)$ denote the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ,

$U_q(\mathfrak{g})$ quantized universal enveloping algebra with generators E_i, F_i $(i \in I)$ and K_λ $(\lambda \in P)$,
$U_q(\mathfrak{g})^- \subset U_q(\mathfrak{g})$ be the subalgebra generated by $\{F_i\}_{i \in I}$,

It is well-known that $U_q(\mathfrak{g})^-$ has the crystal base $(L(\infty), B(\infty))$.

For two integers l, $m \in \mathbb{Z}$ such that $l \leq m$, one sets
$[l, m] := \{l, l+1, \cdots, m-1, m\}$.

A birational map

$B^-_{w_0}$, where w_0 is the longest element in W, and an open embedding $(\mathbb{C}^\times)^N \hookrightarrow B^-_{w_0}$ associated with a reduced word $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ of w_0 for $i \in I$ and $t \in \mathbb{C}$, we put

$$x_i(t) := \exp(te_i),\ y_i(t) := \exp(tf_i) \in G.$$

There exists the canonical embedding $\phi_i : SL_2(\mathbb{C}) \to G$ satisfying

$$x_i(t) = \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right), \quad y_i(t) = \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right).$$

$$t^{h_i} := \phi_i\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \in T$$

$$x_{-i}(t) := y_i(t)t^{-h_i} = \phi_i\left(\begin{pmatrix} t^{-1} & 0 \\ 1 & t \end{pmatrix}\right) \in G$$

for $i \in I$ and $t \in \mathbb{C}^\times$.

A birational map

For each $i \in I$ one can construct a representative of a simple reflection $s_i \in W = \mathrm{Norm}_G(T)/T$ by

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1) \in \mathrm{Norm}_G(T)$$

For $w \in W$, one can define a representative $\overline{w} \in \mathrm{Norm}_G(T)$ by the rule

$$\overline{uv} = \overline{u} \cdot \overline{v} \quad \text{if } l(uv) = l(u) + l(v),$$

where l is the length function on W.
We define a variety $B^-_{w_0} := B^- \cap U\overline{w_0}U$.

A birational map

One defines a map $\theta_{\mathbf{i}}^{-} : (\mathbb{C}^{\times})^{N} \to G$ associated with a reduced word $\mathbf{i} = (i_1, \cdots, i_N)$ of $w_0 \in W$ by

$$\theta_{\mathbf{i}}^{-}(t_1, \cdots, t_N) := x_{-i_1}(t_1) \cdots x_{-i_N}(t_N). \tag{1}$$

Proposition ([2])

The map $\theta_{\mathbf{i}}^{-}$ is an open embedding from $(\mathbb{C}^{\times})^{N}$ to $B_{w_0}^{-}$.

Generalized minors and $\mathbf{i}$-trails

Let $G_0 := U^- TU \subset G$ denote the open subset whose elements $x \in G_0$ are uniquely decomposed as $x = [x]_-[x]_0[x]_+$ with some $[x]_- \in U^-$, $[x]_0 \in T$ and $[x]_+ \in U$.

Definition ([4])

For $u, v \in W$ and $i \in I$, the generalized minor $\Delta_{u\Lambda_i, v\Lambda_i}$ is defined as the regular function on G such that

$$\Delta_{u\Lambda_i, v\Lambda_i}(x) = ([\overline{u}^{-1} x \overline{v}]_0)^{\Lambda_i}$$

for any $x \in \overline{u} G_0 \overline{v}^{-1}$. Here, for $t \in \mathbb{C}^\times$ and $j \in I$, we define $(t^{h_j})^{\Lambda_i} = (t^{\Lambda_i(h_j)})$ and extend it to the group homomorphism $T \to \mathbb{C}^\times$.

For calculations of generalized minors, one can use $\mathbf{i}$-*trails* [2].

Generalized minors and $\mathbf{i}$-trails

Definition

For a finite dimensional representation V of $\mathfrak{g}$, two weights γ, δ of V and a sequence $\mathbf{i} = (i_1, \cdots, i_l)$ of indices from I, a sequence $\pi = (\gamma = \gamma_0, \gamma_1, \cdots, \gamma_l = \delta)$ is said to be a *pre-$\mathbf{i}$-trail* from γ to δ if $\gamma_1, \cdots, \gamma_{l-1} \in P$ and for $k \in [1, l]$, it holds $\gamma_{k-1} - \gamma_k = c_k \alpha_{i_k}$ with some nonnegative integer c_k.

For $k \in [1, l]$

$$c_k = \frac{\gamma_{k-1} - \gamma_k}{2}(h_{i_k}). \tag{2}$$

For a pre-$\mathbf{i}$-trail $\pi = (\gamma_0, \gamma_1, \cdots, \gamma_l)$ and $k \in [1, l]$, we put

$$d_k(\pi) := \frac{\gamma_{k-1} + \gamma_k}{2}(h_{i_k}). \tag{3}$$

One obtains $d_k(\pi) = c_k + \gamma_k(h_{i_k}) \in \mathbb{Z}$ by (2). If $\gamma_{k-1} = s_{i_k}\gamma_k$ then $d_k(\pi) = 0$.

Generalized minors and $\mathbf{i}$-trails

Definition ([2])

We consider the setting of Definition 8. If a pre-$\mathbf{i}$-trail π from γ to δ satisfies the condition

- $e_{i_1}^{c_1} e_{i_2}^{c_2} \cdots e_{i_l}^{c_l}$ is a non-zero linear map from V_δ to V_γ,

then π is said to be an $\mathbf{i}$-*trail* from γ to δ, where $V = \oplus_\mu V_\mu$ is the weight decomposition of V.

Lemma ([8])

Let γ, δ be weights of a finite dimensional representation V of $\mathfrak{g}$. Let $\mathbf{i} = (i_1, \cdots, i_l)$ be a sequence of indices from I and $\pi = (\gamma_0, \gamma_1, \cdots, \gamma_l)$, $\pi' = (\gamma'_0, \gamma'_1, \cdots, \gamma'_l)$ be two pre-$\mathbf{i}$-trails from γ to δ. If $d_k(\pi) = d_k(\pi')$ for all $k \in [1, l]$ then $\pi = \pi'$.

Generalized minors and $\mathbf{i}$-trails

For a sequence $\mathbf{i} = (i_1, \cdots, i_l)$ of indices from I and $t_1, \cdots, t_l \in \mathbb{C}^\times$, just as in (1), we set

$$\theta_{\mathbf{i}}^-(t_1, \cdots, t_l) := x_{-i_1}(t_1) \cdots x_{-i_l}(t_l) \in G.$$

Then the following theorem holds:

Theorem ([2])

For u, $v \in W$ and $i \in I$, it holds

$$\Delta_{u\Lambda_i, v\Lambda_i}(\theta_{\mathbf{i}}^-(t_1, \cdots, t_l)) = \sum_\pi C_\pi t_1^{d_1(\pi)} \cdots t_l^{d_l(\pi)},$$

where C_π is a positive integer and π runs over all $\mathbf{i}$-trails from $-u\Lambda_i$ to $-v\Lambda_i$ in $V(-w_0\Lambda_i)$.

By this theorem and lemma, for each monomial M in $\Delta_{u\Lambda_i, v\Lambda_i}(\theta_{\mathbf{i}}^-(t_1, \cdots, t_l))$, there uniquely exists a corresponding $\mathbf{i}$-trail π from $-u\Lambda_i$ to $-v\Lambda_i$ satisfying $M = t_1^{d_1(\pi)} \cdots t_l^{d_l(\pi)}$

Geometric crystal structure on $B^-_{w_0}$

$$\gamma_i : B^-_{w_0} \to \mathbb{C}^\times, \quad \varepsilon_i : B^-_{w_0} \to \mathbb{C}^\times, \quad \overline{e}_i : \mathbb{C}^\times \times B^-_{w_0} \to B^-_{w_0}$$

on $B^-_{w_0} = B^- \cap U\overline{w_0}U$ define $\mathfrak{g}$-geometric crystal $(B^-_{w_0}, \{\overline{e}_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ ([1], Section 3 [8]).
The variety $T \cdot B^-_{w_0}$ has a positive structure $\theta_{\mathbf{i}} : T \times (\mathbb{C}^\times)^{l(w_0)} \to T \cdot B^-_{w_0}$ associated with each reduced word $\mathbf{i}$ of w_0 so that we obtain a crystal $X_*(T \times (\mathbb{C}^\times)^{l(w_0)})$ by the tropicalization functor.

Geometric crystal structure on $B^-_{w_0}$

The *Berenstein-Kazhdan decoration function* Φ_{BK} on $T \cdot B^-_{w_0}$ is defined as

$$\Phi_{BK} = \sum_{i \in I} \frac{\Delta_{w_0 \Lambda_i, s_i \Lambda_i}}{\Delta_{w_0 \Lambda_i, \Lambda_i}} + \sum_{i \in I} \frac{\Delta_{w_0 s_i \Lambda_i, \Lambda_i}}{\Delta_{w_0 \Lambda_i, \Lambda_i}}. \tag{4}$$

Here, Λ_i is the i-th fundamental weight, for u, $v \in W$, the function $\Delta_{u\Lambda_i, v\Lambda_i}$ is a generalized minor.

Geometric crystal structure on $B^-_{w_0}$

Let us define a regular function Φ^h_{BK} on $B^-_{w_0}$ as follows:

$$\Phi^h_{\mathrm{BK}} := \sum_{i \in I} \Delta_{w_0 \Lambda_i, s_i \Lambda_i}.$$

In [12], Kanakubo and Nakashima proved that the function Φ^h_{BK} is an upper half-decoration on the geometric crystal $B^-_{w_0}$.

Geometric crystal structure on $B^-_{w_0}$

An open embedding $\theta^-_{\mathbf{i}} : (\mathbb{C}^\times)^N \hookrightarrow B^-_{w_0}$ in Proposition 6, which gives a positive structure on $(B^-_{w_0}, \Phi^h_{\mathrm{BK}})$. Thus, one obtains a crystal $\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}$

$$\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}} := \{z \in X_*((\mathbb{C}^\times)^N) | \mathrm{Trop}(\Phi^h_{\mathrm{BK}} \circ \theta^-_{\mathbf{i}})(z) \geq 0\},$$

$$\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}} = (\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}, \{\tilde{e}_i\}_{i \in I}, \{\tilde{f}_i\}_{i \in I}, \{\tilde{\varepsilon}_i\}_{i \in I}, \{\tilde{\varphi}_i\}_{i \in I}, \{\tilde{\gamma}_i\}_{i \in I}). \tag{5}$$

Here, we omitted the notation of restrictions $|_{\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}}$ for $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$, $\tilde{\varphi}_i$ and $\tilde{\gamma}_i$.

Geometric crystal structure on $B_{w_0}^-$

Theorem ([12])

For each reduced word $\mathbf{i}$ of the longest element w_0, the set $\mathbb{B}_{\theta_{\mathbf{i}}^-,\Phi_{\mathrm{BK}}^h}$ is a ${}^L\mathfrak{g}$-crystal isomorphic to the crystal $B(\infty)$.

$\mathbf{i}$-trails and BK decoration functions

The main result of ([9], Theorem 4.4) allows us, for all reduced words $\mathbf{i}$, to get all monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$ explicitly in the following cases (the numbering of Dynkin diagram is same as in [6]), which covers a significantly wide range of indices $i \in I$ comparing with type A due to Postnikov, and minuscule weights due to [8].

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
i	all $i \in I$	all $i \in I$	all $i \in I$	all $i \in I$	$1, 2, 4, 5, 6$	$1, 5, 6, 7$	$1, 7$	$1, 4$	all $i \in I$

Motivation to compute coefficients in BK and GHKK

The Kanakubo-Koshevoy-Nakashima [9] algorithm allows us, for any reduced decomposition of the longest element w_0 of the Weyl group for the classical ABCD Dynkin types, compute all monomials in Berenstein-Kazdan decoration function (as well as in the Gross-Hacking-Keel-Kontsevic potential).
The upshot of this computation is that upon going to tropicalization, we get the inequalities defining string cone for upper canonical basis due to Kashiwara and the cone for the theta basis due to Gross-Hacking-Keel-Kontsevich.

For type A, the coefficients with all monomials are 1 (all weights are minuscule).

For types BCD, this is not typically the case. Note for all types, there exist special reduced decompositions of w_0 such that coefficients with all monomials in BK-decoration are equal to 1, so-called *nice decompositions* due to Littelmann.

Motivation to compute coefficients in BK and GHKK

Knowing coefficients allows us

1. to detect redundant inequalities in defining the string cone (theta-cone)
2. to define the geometric crystal
3. to get a crossing formula for defining crystal actions on the Lusztig canonical basis
4. to compute Euler characteristics of quiver Grassmannians for quivers corresponding reduced decompositions and injective modules (Caldero-Chapoton cluster character formula)
5. to characterize faces with interior integer points of Newton polytopes of Beretstein-Kazhdan decoration function

Motivation to compute coefficients in BK and GHKK

Caldero-Chapoton cluster character formula

$$F_M(t_1, \dots, t_N) = \sum_{\mathbf{e}} \chi(Gr_{\mathbf{e}}(M)) \prod_{j \in I} t_j^{e_j}$$

where M is a representation of a quiver Q of dimension $\mathbf{d} = (d_i)_{i \in V(Q)}$, $Gr_{\mathbf{e}}(M)$ is the quiver Grassmannian of subrepresentations $N \subset M$ of dimension $\mathbf{e}$, χ the Euler characteristic.

Coefficients of BK decoration functions for classical types *ABCD*

In case of $\mathfrak{g}$ is of type A_n, all coefficients with $\mathbf{i}$-trails in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$ are equal to 1.

Theorem

In case of $\mathfrak{g}$ is of classical type B_n, C_n or D_n, coefficients with $\mathbf{i}$-trails in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$ are of the form 2^k, $k \geq 0$.

For any reduced decomposition $\mathbf{i}$, the portion of $\mathbf{i}$-trails π with coefficients bigger than 1 is rather small, and we have an algorithm for computing such cases.

Sketch of proof

By using cluster algebra tools, we relate the BK decoration function and sum of F-polynomials of injective modules for fundamental weights Λ_i (a quiver corresponding to reduced decomposition of w_0 following Berenstein-Fomin-Zelevinsky). Then, using tools from Kanakubo-Koshevoy-Nakashima, we show that in formula (5.8) from Derksen-Weyman-Zelevinsky

$$F_M(t_1, \ldots, t_N) = \sum_{\mathbf{e}', r, s} \chi(Z_{\mathbf{e}', r, s}(M)) t_k^r (1 + t_k)^{s-r} \prod_{j \neq k} t_j^{e'_j}$$

where $Z_{\mathbf{e}',r,s}(M)$ denotes some sunset of the quiver Grassmannian $Gr_{\mathbf{e}}(M)$, χ is the Euler characteristic, we have $s \leq 2$ and $r = 0$.

Algorithm for computing coefficients of BK decoration functions for classical types *ABCD*

From the proof of Main Theorem we provide an algorithm to compute all coefficients in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$. Firstly, we use the algorithm of [9] (see also [11]) to get monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$, and edge-colored directed graph $\overline{DG}$. Then we apply the following procedure:

```
set S=all monomials
set k=1
while S is not empty
     S1=get all pairs (a,b) of S,
        such that a*b is perfect square Laurent monomial
     for each pair (a,b)∈S1 set coefficient of √(a∗b) to 2^k
     set S=S1
```

Algorithm for computing coefficients of BK decoration functions for classical types $ABCD$

To elaborate why this algorithm always stops we use correspondence between monomials with coefficients 2^k and k-dimensional faces of Newton polytope of $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^{-}(t_1, \cdots, t_N)$ ([3]), so it runs no more than length of w_0 cycles.

Algorithm for computing coefficients of BK decoration functions for classical types $ABCD$

This procedure can also be used to compute Gross-Hacking-Keel-Kontsevich potential with proper coefficients [11] (set same coefficients for corresponding monomials) and prove that coefficients of Gross-Hacking-Keel-Kontsevich potential take the form 2^k, $k \geq 0$.

Algorithm complexity

From [11] we know that complexity of computing $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$ consisting of K monomials is

$$O(r^4 K) \sim O(r^2 * \textbf{length of string representation})$$

where **length of string representation** $\sim O(r^2 K)$. Overall complexity of computing coefficients is bounded by product of number of cycles (length $w_0 \sim r^2$) and square of number of monomials

$$O(r^2 * K^2) \leq O(\textbf{length of string representation}^2).$$

This means that whole Berenstein-Kazhdan decoration function and Gross-Hacking-Keel-Kontsevich potential computation algorithm is polynomial (square) in length of string representation of answer.

[1] A.Berenstein, D.Kazhdan, Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases, Quantum groups, 13–88, Contemp. Math., 433, Amer. Math. Soc., Providence, RI, (2007).

[2] A. Berenstein, A. Zelevinsky, Tensor product multiplicities, canonical bases and totally positive varieties, Invent. Math. 143(1), 77–128 (2001) arXiv:math/9912012.

[3] Fei, J., Combinatorics of F-polynomials, Preprint. arXiv:1909.10151

[4] S.Fomin, A.Zelevinsky, Double Bruhat cells and total positivity, J. Amer. Math. Soc. 12, no. 2, 335–380 (1999).

[5] Gross, M., Hacking, P., Keel, S., Kontsevich, M., Canonical bases for cluster algebras. J. Am. Math. Soc. 31, 497–608 (2018)

[6] V.G.Kac, *Infinite-dimensional Lie algebras*, third edition. Cambridge University Press, Cambridge, xxii+400 pp, (1990).

[7] Y.Kanakubo, Polyhedral realizations for $B(\infty)$ and extended Young diagrams, Young walls of type $\mathrm{A}^{(1)}_{n-1}$, $\mathrm{C}^{(1)}_{n-1}$, $\mathrm{A}^{(2)}_{2n-2}$, $\mathrm{D}^{(2)}_{n}$, Algebr. Represent. Theory, https://doi.org/10.1007/s10468-022-10172-z, 1–48 (2022).

[8] Y.Kanakubo, G.Koshevoy, T.Nakashima, An algorithm for Berenstein-Kazhdan decoration functions and trails for minuscule representations, J. Algebra, 608, 106–142 (2022).

[9] Y.Kanakubo, G.Koshevoy, T.Nakashima, An Algorithm for Berenstein–Kazhdan Decoration Functions and Trails for Classical Lie Algebras, International Mathematics Research Notices, Vol. 2024, No. 4, pp. 3223–3277

[10] G.Koshevoy and B. Schumann, Redundancy in string cone inequalities and multiplicities in potential functions on cluster varieties, *J. of Algebraic Combinatorics* (2022)

[11] G.Koshevoy and D.Mironov, F-polynomials and Newton polytopes, 2022 24th International Symposium on Symbolic and Numeric Algorithms for Scientific Computing (SYNASC), Hagenberg / Linz, Austria, 2022, pp. 51-54

[12] Y. Kanakubo, T. Nakashima, Half Potential on Geometric Crystals and Connectedness of Cell Crystals, Transform. Groups 28, 327–373 (2023).

[13] T.Nakashima, A.Zelevinsky, Polyhedral realizations of crystal bases for quantized Kac-Moody algebras, Adv. Math. **131**, no. 1, 253–278, (1997).

[14] https://github.com/mironovd/crystal