On coefficients of the Berenstein-Kazhdan decoration functions for classical groups

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Abstract. For type A_n, $SL_{n+1}(\mathbb{C})$, all coefficients of the BK decoration function are equal one, because all weights are minuscule. This is not the case for other types. For types BCD, we prove that the coefficients are powers of two (zero power is one). We propose an algorithm which computes these coefficients. The complexity of the algorithm is comparable with the complexity of writing the BK decoration function.

1. i-trails and generalized minors

1.1. Notation

Let G be a simply connected connected simple algebraic group or rank r, B, $B^- \subset G$ its Borel subgroups, $T := B \cap B^-$ the maximal torus, $W = \mathrm{Norm}_G(T)/T$ Weyl group, U, U^- be unipotent radicals of B, B^-, $A = (a_{i,j})$ the Cartan matrix of G with an index set $I = \{1, 2, \cdots, n\}$. We define $\mathfrak{g} = \mathrm{Lie}(G)$ with Chevalley generators e_i, f_i, h_i $(i \in I)$, a Cartan subalgebra $\mathfrak{h}$ and the canonical pairing $\langle , \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$. Let Λ_i denote the i-th fundamental weight, that is, $\langle h_j, \Lambda_i \rangle = \delta_{j,i}$ and $P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ be the weight lattice, $P_+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$ the positive weight lattice, $P^* = \oplus_{i \in I} \mathbb{Z} h_i$ the dual weight lattice, $\{\alpha_i\}$ $(i \in I)$ the set of simple roots. For each $\lambda \in P_+$, let $V(\lambda)$ denote the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra with generators E_i, F_i $(i \in I)$ and K_λ $(\lambda \in P)$ and $U_q(\mathfrak{g})^- \subset U_q(\mathfrak{g})$ be the subalgebra generated by $\{F_i\}_{i \in I}$. It is well-known that $U_q(\mathfrak{g})^-$ has the crystal base $(L(\infty), B(\infty))$. For two integers l, $m \in \mathbb{Z}$ such that $l \leq m$, one sets $[l, m] := \{l, l+1, \cdots, m-1, m\}$.

1.2. A birational map

Let us recall a definition of $B^-_{w_0}$, where w_0 is the longest element in W, and an open embedding $(\mathbb{C}^\times)^N \hookrightarrow B^-_{w_0}$ associated with a reduced word $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ of w_0.

First, for $i \in I$ and $t \in \mathbb{C}$, we put

$$x_i(t) := \exp(te_i),\ y_i(t) := \exp(tf_i) \in G.$$

There exists the canonical embedding $\phi_i : SL_2(\mathbb{C}) \to G$ satisfying

$$x_i(t) = \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right), \quad y_i(t) = \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right).$$

Using the embedding, one puts

$$t^{h_i} := \phi_i\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \in T$$

and

$$x_{-i}(t) := y_i(t)t^{-h_i} = \phi_i\left(\begin{pmatrix} t^{-1} & 0 \\ 1 & t \end{pmatrix}\right) \in G$$

for $i \in I$ and $t \in \mathbb{C}^\times$. One can construct a representative of a simple reflection $s_i \in W = \mathrm{Norm}_G(T)/T$ by

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1) \in \mathrm{Norm}_G(T)$$

for each $i \in I$. For $w \in W$, one can define a representative $\overline{w} \in \mathrm{Norm}_G(T)$ by the rule

$$\overline{uv} = \overline{u} \cdot \overline{v} \quad \text{ if } l(uv) = l(u) + l(v),$$

where l is the length function on W. We define a variety $B^-_{w_0} := B^- \cap U\overline{w_0}U$. One defines a map $\theta^-_{\mathbf{i}} : (\mathbb{C}^\times)^N \to G$ associated with a reduced word $\mathbf{i} = (i_1, \cdots, i_N)$ of $w_0 \in W$ by

$$\theta^-_{\mathbf{i}}(t_1, \cdots, t_N) := x_{-i_1}(t_1) \cdots x_{-i_N}(t_N). \tag{1.1}$$

Proposition 1.1 ([2]). *The map $\theta^-_{\mathbf{i}}$ is an open embedding from $(\mathbb{C}^\times)^N$ to $B^-_{w_0}$.*

1.3. Generalized minors and i-trails

Let $G_0 := U^-TU \subset G$ denote the open subset whose elements $x \in G_0$ are uniquely decomposed as $x = [x]_-[x]_0[x]_+$ with some $[x]_- \in U^-$, $[x]_0 \in T$ and $[x]_+ \in U$.

Definition 1.2 ([4]). For $u, v \in W$ and $i \in I$, the generalized minor $\Delta_{u\Lambda_i, v\Lambda_i}$ is defined as the regular function on G such that

$$\Delta_{u\Lambda_i, v\Lambda_i}(x) = ([\overline{u}^{-1}x\overline{v}]_0)^{\Lambda_i}$$

for any $x \in \overline{u}G_0\overline{v}^{-1}$. Here, for $t \in \mathbb{C}^\times$ and $j \in I$, we define $(t^{h_j})^{\Lambda_i} = (t^{\Lambda_i(h_j)})$ and extend it to the group homomorphism $T \to \mathbb{C}^\times$.

For calculations of generalized minors, one can use **i**-*trails* [2]. Here in this subsection, we take $\mathbf{i} = (i_1, \cdots, i_l)$ as a sequence of indices from I. Let us review pre-**i**-trails and **i**-trails.

Definition 1.3. For a finite dimensional representation V of $\mathfrak{g}$, two weights γ, δ of V and a sequence $\mathbf{i} = (i_1, \cdots, i_l)$ of indices from I, a sequence $\pi = (\gamma = \gamma_0, \gamma_1, \cdots, \gamma_l = \delta)$ is said to be a *pre-*$\mathbf{i}$*-trail* from γ to δ if $\gamma_1, \cdots, \gamma_{l-1} \in P$ and for $k \in [1, l]$, it holds $\gamma_{k-1} - \gamma_k = c_k\alpha_{i_k}$ with some nonnegative integer c_k.

We can easily check that for $k \in [1, l]$, it holds

$$c_k = \frac{\gamma_{k-1} - \gamma_k}{2}(h_{i_k}). \tag{1.2}$$

Definition 1.4 ([2]). We consider the setting of Definition 1.3. If a pre-$\mathbf{i}$-trail π from γ to δ satisfies the condition

- $e_{i_1}^{c_1} e_{i_2}^{c_2} \cdots e_{i_l}^{c_l}$ is a non-zero linear map from V_δ to V_γ,

then π is said to be an $\mathbf{i}$-*trail* from γ to δ, where $V = \oplus_\mu V_\mu$ is the weight decomposition of V.

For a pre-$\mathbf{i}$-trail $\pi = (\gamma_0, \gamma_1, \cdots, \gamma_l)$ and $k \in [1, l]$, we put

$$d_k(\pi) := \frac{\gamma_{k-1} + \gamma_k}{2}(h_{i_k}). \tag{1.3}$$

One obtains $d_k(\pi) = c_k + \gamma_k(h_{i_k}) \in \mathbb{Z}$ by (1.2). If $\gamma_{k-1} = s_{i_k}\gamma_k$ then $d_k(\pi) = 0$.

Lemma 1.5 ([8]). *Let γ, δ be weights of a finite dimensional representation V of $\mathfrak{g}$. Let $\mathbf{i} = (i_1, \cdots, i_l)$ be a sequence of indices from I and $\pi = (\gamma_0, \gamma_1, \cdots, \gamma_l)$, $\pi' = (\gamma'_0, \gamma'_1, \cdots, \gamma'_l)$ be two pre-$\mathbf{i}$-trails from γ to δ. If $d_k(\pi) = d_k(\pi')$ for all $k \in [1, l]$ then $\pi = \pi'$.*

For a sequence $\mathbf{i} = (i_1, \cdots, i_l)$ of indices from I and $t_1, \cdots, t_l \in \mathbb{C}^\times$, just as in (1.1), we set

$$\theta_{\mathbf{i}}^-(t_1, \cdots, t_l) := x_{-i_1}(t_1) \cdots x_{-i_l}(t_l) \in G.$$

Then the following theorem holds:

Theorem 1.6 ([2]). *For u, $v \in W$ and $i \in I$, it holds*

$$\Delta_{u\Lambda_i, v\Lambda_i}(\theta_{\mathbf{i}}^-(t_1, \cdots, t_l)) = \sum_\pi C_\pi t_1^{d_1(\pi)} \cdots t_l^{d_l(\pi)},$$

where C_π is a positive integer and π runs over all $\mathbf{i}$-trails from $-u\Lambda_i$ to $-v\Lambda_i$ in $V(-w_0\Lambda_i)$.

By this theorem and Lemma 1.5, for each monomial M in $\Delta_{u\Lambda_i, v\Lambda_i}(\theta_{\mathbf{i}}^-(t_1, \cdots, t_l))$, there uniquely exists a corresponding $\mathbf{i}$-trail π from $-u\Lambda_i$ to $-v\Lambda_i$ satisfying $M = t_1^{d_1(\pi)} \cdots t_l^{d_l(\pi)}$.

2. The Berenstein-Kazhdan decoration functions and i-trails

2.1. Geometric crystal structure on $B_{w_0}^-$

Defining maps

$$\gamma_i : B_{w_0}^- \to \mathbb{C}^\times, \quad \varepsilon_i : B_{w_0}^- \to \mathbb{C}^\times, \quad \overline{e}_i : \mathbb{C}^\times \times B_{w_0}^- \to B_{w_0}^-$$

on $B_{w_0}^- = B^- \cap U\overline{w_0}U$, we get a $\mathfrak{g}$-geometric crystal $(B_{w_0}^-, \{\overline{e}_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ [1]. For the definition of maps, refer to Sect.3 of the paper [8].

The variety $T \cdot B_{w_0}^-$ has a positive structure $\theta_{\mathbf{i}} : T \times (\mathbb{C}^\times)^{l(w_0)} \to T \cdot B_{w_0}^-$ associated with each reduced word $\mathbf{i}$ of w_0 so that we obtain a crystal $X_*(T \times (\mathbb{C}^\times)^{l(w_0)})$

by the tropicalization functor. The *Berenstein-Kazhdan decoration function* Φ_{BK} on $T \cdot B^-_{w_0}$ is defined as

$$\Phi_{BK} = \sum_{i \in I} \frac{\Delta_{w_0 \Lambda_i, s_i \Lambda_i}}{\Delta_{w_0 \Lambda_i, \Lambda_i}} + \sum_{i \in I} \frac{\Delta_{w_0 s_i \Lambda_i, \Lambda_i}}{\Delta_{w_0 \Lambda_i, \Lambda_i}}. \tag{2.1}$$

Here, Λ_i is the i-th fundamental weight, for u, $v \in W$, the function $\Delta_{u\Lambda_i, v\Lambda_i}$ is a generalized minor.

Let us define a regular function Φ^h_{BK} on $B^-_{w_0}$ as follows:

$$\Phi^h_{\mathrm{BK}} := \sum_{i \in I} \Delta_{w_0 \Lambda_i, s_i \Lambda_i}.$$

In [13], Kanakubo and Nakashima proved that the function Φ^h_{BK} is an upper half-decoration on the geometric crystal $B^-_{w_0}$.

An open embedding $\theta^-_{\mathbf{i}} : (\mathbb{C}^\times)^N \hookrightarrow B^-_{w_0}$ in Proposition 1.1, which gives a positive structure on $(B^-_{w_0}, \Phi^h_{\mathrm{BK}})$. Thus, one obtains a crystal $\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}$

$$\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}} := \{z \in X_*((\mathbb{C}^\times)^N) | \mathrm{Trop}(\Phi^h_{\mathrm{BK}} \circ \theta^-_{\mathbf{i}})(z) \geq 0\},$$

$$\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}} = (\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}, \{\tilde{e}_i\}_{i \in I}, \{\tilde{f}_i\}_{i \in I}, \{\tilde{\varepsilon}_i\}_{i \in I}, \{\tilde{\varphi}_i\}_{i \in I}, \{\tilde{\gamma}_i\}_{i \in I}). \tag{2.2}$$

Here, we omitted the notation of restrictions $|_{\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}}$ for $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$, $\tilde{\varphi}_i$ and $\tilde{\gamma}_i$.

Theorem 2.1 ([13]). *For each reduced word* $\mathbf{i}$ *of the longest element* w_0*, the set* $\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}$ *is a* ${}^L\mathfrak{g}$*-crystal isomorphic to the crystal* $B(\infty)$*.*

2.2. i-trails and BK decoration functions

The main result of ([9], Theorem 4.4) allows us, for all reduced words $\mathbf{i}$, to get all monomials in $\Delta_{w_0 \Lambda_i, s_i \Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$ explicitly in the following cases (the numbering of Dynkin diagram is same as in [6]), which covers a significantly wide range of indices $i \in I$ comparing with [8]. Due to this theorem is computed an

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
i	all $i \in I$	all $i \in I$	all $i \in I$	all $i \in I$	$1, 2, 4, 5, 6$	$1, 5, 6, 7$	$1, 7$	$1, 4$	all $i \in I$

edge-colored directed graph $\overline{DG}$ whose vertices are labelled by the monomials in $\Delta_{w_0 \Lambda_i, s_i \Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$, and edges are colored by letters of $\{1, 2, \cdots, N\}$. We only use easy computations of the Weyl group action on simple roots and weights and multiplications of Laurent monomials. In particular, in case of $\mathfrak{g}$ is of classical type (A_n, B_n, C_n or D_n) or type G_2, by the tropicalization, we get an explicit form of the crystal

$$\{z \in X_*((\mathbb{C}^\times)^N) | \mathrm{Trop}(\Phi^h_{BK} \circ \theta^-_{\mathbf{i}})(z) \geq 0\}, \tag{2.3}$$

for any reduced word $\mathbf{i}$.

2.3. Algorithm for computing coefficients of BK decoration functions for classical types ABCD

A problem on computing coefficients of Berenstein-Kazhdan decoration arises in study of redundant inequalities defining the cone (2.3) (see [10]).

In case of $\mathfrak{g}$ is of classical type A_n, all weights are minuscule, and because of that all coefficients are equal one [8], and there are no redundancy [10].

In case of $\mathfrak{g}$ is of classical type B_n, C_n or D_n this is not the case. We have the following

Theorem 2.2. *In case of $\mathfrak{g}$ is of classical type B_n, C_n or D_n, coefficients with $\mathbf{i}$-trails take the form 2^k, $k \geq 0$.*

For any reduced decomposition $\mathbf{i}$, the portion of $\mathbf{i}$-trails π with coefficients bigger than 1 is rather small, and we have an algorithm for computing such cases.

For the proof of Theorem 3.2 we provide an algorithm to compute all coefficients in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^{-}(t_1, \cdots, t_N)$. Firstly, we use the algorithm of [9] (see also [11]) to get monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^{-}(t_1, \cdots, t_N)$, and edge-colored directed graph $\overline{DG}$. Then we apply the following procedure:

```
set S=all monomials
set k=1
while S is not empty
      S1=get all pairs (a,b) of S,
         such that a*b is perfect square Laurent monomial
      for each pair (a,b)∈S1 set coefficient of √(a*b) to 2^k
      set S=S1
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The proof of correctness of this algorithm is essentially the proof of Theorem 3.2. To elaborate why this algorithm always halts we use correspondence between monomials with coefficients 2^k and k-dimensional faces of Newton polytope of $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^{-}(t_1, \cdots, t_N)$ ([3]), so it runs no more than length of w_0 cycles.

This procedure can also be used to compute Gross-Hacking-Keel-Kontsevich potential with proper coefficients [11] (set same coefficients for corresponding monomials) and prove that coefficients of Gross-Hacking-Keel-Kontsevich potential take the form 2^k, $k \geq 0$.

2.4. Algorithm complexity

From [11] we know that complexity of computing $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^{-}(t_1, \cdots, t_N)$ consisting of K monomials is

$$O(r^4 K) \sim O(r^2 * \textbf{length of string representation})$$

where **length of string representation** $\sim O(r^2 K)$. Overall complexity of computing coefficients is bounded by product of number of cycles (length $w_0 \sim r^2$) and square of number of monomials

$$O(r^2 * K^2) \leq O(\textbf{length of string representation}^2).$$

This means that whole Berenstein-Kazhdan decoration function and Gross-Hacking-Keel-Kontsevich potential computation algorithm is polynomial (square) in length of string representation of answer.

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