The zeros of random sections of real vector bundles

Boris Kazarnovskii

Polynomial Computer Algebra 2024
April 15-20
Euler International Mathematical Institute, St. Petersburg, Russia

Zeros of random functions, 1

Everywhere below we use the notation

- X – n-dimensional differentiable manifold
- V – finite dimensional vector space of smooth functions on X, $\langle *, * \rangle$
- $f_1, \ldots, f_n$ – random system of independent normally distributed random vectors in V with respect to the Gaussian measure in V associated with $\langle *, * \rangle$
- $\mathfrak{M}(V)$ – expectation of the number of common zeros of $f_1, \ldots, f_n$

Our first theorem is some geometric formula for $\mathfrak{M}(V)$. To give it, we need the notion of a **Banach set** on X, as well as the notion of a **volume of the Banach set**.

Zeros of random functions, 1

Everywhere below we use the notation

- X – n-dimensional differentiable manifold
- V – finite dimensional vector space of smooth functions on X, $\langle *, * \rangle$
- $f_1, \ldots, f_n$ – random system of independent normally distributed random vectors in V with respect to the Gaussian measure in V associated with $\langle *, * \rangle$
- $\mathfrak{M}(V)$ – expectation of the number of common zeros of $f_1, \ldots, f_n$

Our first theorem is some geometric formula for $\mathfrak{M}(V)$. To give it, we need the notion of a **Banach set** on X, as well as the notion of a **volume of the Banach set**.

Definition 1. Let $T_x^* X$ be a cotangent space of X at x, and $\mathcal{E}(x)$ be a convex centrally symmetric compact set in $T_x^* X$. By definition the collection $\mathcal{E} = \{\mathcal{E}(x) \subset T_x^* X \mid x \in X\}$ is called a *Banach set* on X.

Zeros of random functions, 1

Everywhere below we use the notation

• X – n-dimensional differentiable manifold

• V – finite dimensional vector space of smooth functions on X, $\langle *, * \rangle$

• $f_1, \ldots, f_n$ – random system of independent normally distributed random vectors in V with respect to the Gaussian measure in V associated with $\langle *, * \rangle$

• $\mathfrak{M}(V)$ – expectation of the number of common zeros of $f_1, \ldots, f_n$

Our first theorem is some geometric formula for $\mathfrak{M}(V)$. To give it, we need the notion of a **Banach set** on X, as well as the notion of a **volume of the Banach set**.

Definition 1. Let $T^*_x X$ be a cotangent space of X at x, and $\mathcal{E}(x)$ be a convex centrally symmetric compact set in $T^*_x X$. By definition the collection $\mathcal{E} = \{\mathcal{E}(x) \subset T^*_x X \mid x \in X\}$ is called a *Banach set* on X.

Definition 2. Consider the domain $\bigcup_{x \in X} \mathcal{E}(x)$ in the manifold $T^*(X) = \bigcup_{x \in X} T^*_x X$. It's volume relative to the standard symplectic structure in T^*X is called the *volume of Banach set* and is denoted by $\mathrm{vol}(\mathcal{E})$.

Zeros of random functions, 2

Below we will define the Banach set $\mathcal{E}_V$ corresponding to the Euclidean space V, participating in the following formulation.

Theorem 1. $\mathfrak{M}(V) = n!/(2\pi)^n \operatorname{vol}(\mathcal{E}_V)$

Zeros of random functions, 2

Below we will define the Banach set $\mathcal{E}_V$ corresponding to the Euclidean space V, participating in the following formulation.

Theorem 1. $\mathfrak{M}(V) = n!/(2\pi)^n \operatorname{vol}(\mathcal{E}_V)$

For $x \in X$ we define the linear functional $\theta(x)$ on V as $\theta(x)(f) = f(x)$. Assume that $\forall x \in X,\ \exists f \in V:\ f(x) \neq 0$. So the set $\theta(X)$ in V^* does not contain 0.

Zeros of random functions, 2

Below we will define the Banach set $\mathcal{E}_V$ corresponding to the Euclidean space V, participating in the following formulation.

Theorem 1. $\mathfrak{M}(V) = n!/(2\pi)^n \operatorname{vol}(\mathcal{E}_V)$

For $x \in X$ we define the linear functional $\theta(x)$ on V as $\theta(x)(f) = f(x)$. Assume that $\forall x \in X$, $\exists f \in V\colon\ f(x) \neq 0$. So the set $\theta(X)$ in V^* does not contain 0.

Definition 3. Let's define the mapping $\Theta\colon X \to V^*$ as

$$\Theta(x) = \theta(x)/\sqrt{\langle \theta(x), \theta(x) \rangle_*}$$

where $\langle *, * \rangle_*$ is the scalar product in the space V^* associated with the scalar product $\langle *, * \rangle$ in V. Let $d\Theta_x\colon T_x X \to V^*$ be a differential of Θ at the point x. Denote by $d^*\Theta_x\colon V \to T_x^* X$ the adjoint linear operator, and define the Banach set $\mathcal{E}_V$ by $\mathcal{E}_V(x) = d^*\Theta_x(B)$, where B is the unit ball in V centered at the origin. The compact set $\mathcal{E}_V(x)$ is an ellipsoid.

Zeros of random functions, 2

Below we will define the Banach set $\mathcal{E}_V$ corresponding to the Euclidean space V, participating in the following formulation.

Theorem 1. $\mathfrak{M}(V) = n!/(2\pi)^n \operatorname{vol}(\mathcal{E}_V)$

For $x \in X$ we define the linear functional $\theta(x)$ on V as $\theta(x)(f) = f(x)$. Assume that $\forall x \in X$, $\exists f \in V\colon\ f(x) \neq 0$. So the set $\theta(X)$ in V^* does not contain 0.

Definition 3. Let's define the mapping $\Theta\colon X \to V^*$ as

$$\Theta(x) = \theta(x)/\sqrt{\langle \theta(x), \theta(x)\rangle_*}$$

where $\langle *, * \rangle_*$ is the scalar product in the space V^* associated with the scalar product $\langle *, * \rangle$ in V. Let $d\Theta_x\colon T_xX \to V^*$ be a differential of Θ at the point x. Denote by $d^*\Theta_x\colon V \to T^*_xX$ the adjoint linear operator, and define the Banach set $\mathcal{E}_V$ by $\mathcal{E}_V(x) = d^*\Theta_x(B)$, where B is the unit ball in V centered at the origin. The compact set $\mathcal{E}_V(x)$ is an ellipsoid.

Theorem 1 survives for probability distributions more general than Gaussians. For such distributions, arbitrary Banach sets on X can arise.

Zeros of random functions, 3

Now we state a similar theorem in the case $f_i \in V_i$ for the spaces $V_1, \ldots, V_n$. For this, we will need the concept of the mixed volume of Banach sets. Using Minkowski sum and homotheties, we can form linear combinations of convex sets with non-negative coefficients. The linear combination of Banach sets is defined by

$$(\sum_i \lambda_i \mathcal{E}_i)(x) = \sum_i \lambda_i \mathcal{E}_i(x).$$

Zeros of random functions, 3

Now we state a similar theorem in the case $f_i \in V_i$ for the spaces $V_1, \ldots, V_n$. For this, we will need the concept of the mixed volume of Banach sets. Using Minkowski sum and homotheties, we can form linear combinations of convex sets with non-negative coefficients. The linear combination of Banach sets is defined by

$$(\sum_i \lambda_i \mathcal{E}_i)(x) = \sum_i \lambda_i \mathcal{E}_i(x).$$

Lemma 1. For n Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ the volume of $\lambda_1 \mathcal{E}_1 + \ldots + \lambda_n \mathcal{E}_n$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$.

Zeros of random functions, 3

Now we state a similar theorem in the case $f_i \in V_i$ for the spaces $V_1, \dots, V_n$. For this, we will need the concept of the mixed volume of Banach sets. Using Minkowski sum and homotheties, we can form linear combinations of convex sets with non-negative coefficients. The linear combination of Banach sets is defined by

$$(\sum_i \lambda_i \mathcal{E}_i)(x) = \sum_i \lambda_i \mathcal{E}_i(x).$$

Lemma 1. For n Banach sets $\mathcal{E}_1, \dots, \mathcal{E}_n$ the volume of $\lambda_1 \mathcal{E}_1 + \ldots + \lambda_n \mathcal{E}_n$ is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_n$.

Definition 4. The coefficient of polynomial $\mathrm{vol}(\lambda_1 \mathcal{E}_1 + \ldots + \lambda_n \mathcal{E}_n)$ at the monomial $\lambda_1 \cdot \ldots \cdot \lambda_n$ divided by $n!$ is called the *mixed volume of Banach sets* $\mathcal{E}_1, \dots, \mathcal{E}_n$. The mixed volume of Banach sets $\mathcal{E}_1, \dots, \mathcal{E}_n$ is denoted by $\mathrm{vol}(\mathcal{E}_1, \dots, \mathcal{E}_n)$.

Zeros of random functions, 4

Theorem 2. Let $\mathfrak{M}(V_1, \dots, V_n)$ be the expectation of the number of roots of random system $f_1 = \ldots = f_n = 0$. Then

$$\mathfrak{M}(V_1, \dots, V_n) = \frac{n!}{(2\pi)^n} \operatorname{vol}(\mathcal{E}_{V_1}, \dots, \mathcal{E}_{V_n})$$

Zeros of random functions, 4

Theorem 2. Let $\mathfrak{M}(V_1, \ldots, V_n)$ be the expectation of the number of roots of random system $f_1 = \ldots = f_n = 0$. Then

$$\mathfrak{M}(V_1, \ldots, V_n) = \frac{n!}{(2\pi)^n} \mathrm{vol}(\mathcal{E}_{V_1}, \ldots, \mathcal{E}_{V_n})$$

Trigonometric polynomials; example

Let X be the unit circle S^1, V_m the space of trigonometric polynomials $f(\theta) = \sum_{k \le m} a_k \cos(k\theta) + b_k \sin(k\theta)$ of degree m. Then $\mathfrak{M}(V_m) = \sqrt{\frac{m(m+1)}{3}}$.

Corollary. $\lim_{m\to\infty} \mathfrak{M}(V_m)/(2m) = 1/\sqrt{3}$,

The ring of Banach sets, 1

Next we need a concept of the ring of Banach sets. It arises as an analogue of the well-known concept of a ring of convex sets. There are several different versions of this concept. Here we construct an analogue of the definition given by A. Khovanskii in Russ. Math. Surv, 2021, (76:1).
We call the formal difference $\mathcal{E} - \mathcal{B}$ of Banach sets the *virtual Banach set*. Virtual Banach sets form a vector space, where multiplication by negative numbers is defined by $(-1) \cdot (\mathcal{E} - \mathcal{B}) = \mathcal{B} - \mathcal{E}$.

The following notations are used below
• $S = \bigoplus_{i \leq 0} S_i$ – the graded symmetric algebra of the space of virtual Banach sets on X
• I – the linear functional on the space S defined by 1) $I_{S_k} = 0$ for $k \neq n$, and 2) $I(\mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_n) = \mathrm{vol}(\mathcal{E}_1, \ldots, \mathcal{E}_n)$.
• $L(x, y) = I(x \cdot y)$ – the symmetric bilinear form $I(x \cdot y)$ on the vector space S
• J – the kernel of the form L

Lemma 1. The kernel J of the form L is a homogeneous ideal of the graded ring S.

The ring of Banach sets, 2

Definition 4. The ring $\mathfrak{S} = S/J$ is said to a ring of virtual Banach sets.

Corollary 1. The following statements hold:
(i) $\mathfrak{S}_0 = \mathbb{R}$
(ii) $\dim \mathfrak{S}_n = 1$
(iii) The graded ring $\mathfrak{S}$ is generated by elements of degree 1
(iv) The mappings $\mathfrak{S}_p \times \mathfrak{S}_{n-p} \to \mathbb{R}$, defined as $(\eta, \xi) \mapsto L(\eta, \xi)$, are non-degenerate pairings.

Next for n virtual Banach sets $\mathcal{B}_1, \ldots, \mathcal{B}_n$, we use the notation

$$\mathrm{vol}(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_n) = I(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_n) = \mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

Zeros of random sections, 1

Transitioning to zeros of random sections of vector bundles, without formulating precise theorems, we will first briefly describe the situation in the case when considering zeros of sections of an n-dimensional vector bundle $\mathcal{F}$ on X. Just as in the case of functions we consider a finite dimensional space V of smooth sections of $\mathcal{F}$.
Here we denote by $\mathfrak{M}(V;U)$ the expectation of the number of zeros of random section $s \in V$ contained in the open set $U \subset X$.
By $\mathrm{Res}_U \mathcal{B}$ we denote the constraint of $\mathcal{B} \in \mathfrak{S}$ on the subvariety U.

Zeros of random sections, 1

Transitioning to zeros of random sections of vector bundles, without formulating precise theorems, we will first briefly describe the situation in the case when considering zeros of sections of an n-dimensional vector bundle $\mathcal{F}$ on X. Just as in the case of functions we consider a finite dimensional space V of smooth sections of $\mathcal{F}$.
Here we denote by $\mathfrak{M}(V;U)$ the expectation of the number of zeros of random section $s \in V$ contained in the open set $U \subset X$.
By $\mathrm{Res}_U \mathcal{B}$ we denote the constraint of $\mathcal{B} \in \mathfrak{S}$ on the subvariety U.

Theorem 3. There exists the unique element $\mathcal{B} \in \mathfrak{S}_n$, such that for any open set $U \subset X$

$$\mathfrak{M}(V;U) = \frac{n!}{(2\pi)^n} \mathrm{vol}(\mathrm{Res}_U \mathcal{B})$$

Zeros of random sections, 2

Further results can be approximately described as follows. We associate to an element $\mathfrak{s}$ of degree k of the ring of Banach sets $\mathfrak{S}$ a certain k-density $d_k(\mathfrak{s})$ on X and interpret the ring $\mathfrak{S}$ as a ring of these densities. Such densities serve as analogues of Chern forms, representing Chern classes of complex vector bundles, and inherit some properties of Chern forms. In conclusion, let us define the density $d_k(\mathfrak{s})$. An alternative construction of density ring is given in (Geom. Funct. Anal., vol. 28, 2018) and in (Math. J. Volume (22:3), 2022).

Zeros of random sections, 2

Further results can be approximately described as follows. We associate to an element $\mathfrak{s}$ of degree k of the ring of Banach sets $\mathfrak{S}$ a certain k-density $d_k(\mathfrak{s})$ on X and interpret the ring $\mathfrak{S}$ as a ring of these densities. Such densities serve as analogues of Chern forms, representing Chern classes of complex vector bundles, and inherit some properties of Chern forms. In conclusion, let us define the density $d_k(\mathfrak{s})$. An alternative construction of density ring is given in (Geom. Funct. Anal., vol. 28, 2018) and in (Math. J. Volume (22:3), 2022).

Definition. Let the subspace H in T_xX is generated by $\xi_1, \ldots, \xi_k$, $H^\perp \subset T_x^*X$ the orthogonal complement to H, and $\pi_H : T_x^*X \to T_x^*X/H^\perp$ the projection map. The volume form on $T_x^*X/H^\perp$ is defined by $\omega(x) = \xi_1 \wedge \ldots \wedge \xi_m$. Let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be Banach sets on X. Then $d_k(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_k)(\xi_1, \ldots, \xi_m)$ is *the mixed k-dimensional volume of convex k-dimensional sets* $\pi_H\mathcal{B}_1(x), \ldots, \pi_H\mathcal{B}_k(x)$ in the sense of the volume form $\omega(x)$.

There exists an analogue of the BKK formula for Banach sets and densities d_i

$$d_1(\mathcal{B}_1) \cdot \ldots \cdot d_1(\mathcal{B}_k) = k!\, d_k(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_k)$$