The zeros of random sections of real vector bundles

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Abstract. We define integral geometric analogues of the Chern classes for real vector bundle on a smooth real variety. More precisely, we define the Chern densities of a real bundle. These densities are analogues of the Chern forms of a complex vector bundle and inherit some of their properties.
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The zeros of random systems of functions

We begin with a theorem on the number of common zeros of random systems of functions from [1]. Let V be a finite-dimensional space of smooth functions on an n-dimensional differentiable manifold X. Consider a random system of equations

$$f_1 = \ldots = f_n = 0, \ 0 \neq f_i \in V \tag{1}$$

Denote by $N(f_1, \ldots, f_n)$ the number of solutions of the system (1). We define the randomness of the system using a certain scalar product in V as follows: we consider the functions f_i as independent random vectors in V with respect to the Gaussian measure chosen in V according to the chosen scalar product. The situation with a more general choice of probability distribution in V is described in [2]. Let $\mathfrak{M}(V)$ denote the expected value of the random variable $N(f_1, \ldots, f_n)$. Next, for the calculation of $\mathfrak{M}(V)$, we will need the notion of a Banach set on X, as well as the notion of a volume of the Banach set.

Definiton 1. Let T^*X be a cotangent bundle of X, and $\mathcal{E}(x)$ be a convex centrally symmetric compact set in the cotangent space T_x^*X of X at the point x. The collection $\mathcal{E} = \{\mathcal{E}(x) \subset T_x^*X \mid x \in X\}$ is called a *Banach set* in X.

Definiton 2. Consider the domain $\bigcup_{x \in X} \mathcal{E}(x) \subset T^*(X)$. It's volume relative to the standard symplectic structure in T^*X is called the *volume of Banach set* and is denoted by $\mathrm{vol}(\mathcal{E})$.

For $x \in X$ let's define the linear functional $\theta(x)$ on V as $\theta(x)(f) = f(x)$. Next we assume that $\forall x \in X$, $\exists f \in V\colon f(x) \neq 0$. That's why the set $\theta(X)$ in V^* does not contain 0.

Definiton 3. Let's define the mapping $\Theta\colon X \to V^*$ as $\Theta(x) = \theta(x)/\sqrt{\langle\theta(x),\theta(x)\rangle_*}$, where $\langle *, * \rangle_*$ is the scalar product in the space V^* associated with the scalar product $\langle *, * \rangle$ in V. Let $d\Theta_x\colon T_x X \to V^*$ be a differential of Θ at the point x. Denote by $d^*\Theta_x\colon V \to T_x^* X$ the adjoint linear operator, and define the Banach set $\mathcal{E}_V$ by $\mathcal{E}_V(x) = d^*\Theta_x(B)$, where B is the unit ball in V centered at the origin. The compact set $\mathcal{E}_V(x)$ is an ellipsoid. We note that in a more general context discussed in [Ka1], arbitrary Banach sets on X can arise.

Theorem 1. $\mathfrak{M}(V) = n!/(2\pi)^n \operatorname{vol}(\mathcal{E}_V)$

Example 1. Let X be the unit circle S^1, V_m the space of trigonometric polynomials $f(\theta) = \sum_{k \leq m} a_k \cos(k\theta) + b_k \sin(k\theta)$ of degree m. Then (see [6])

$$\mathfrak{M}(V_m) = \sqrt{\frac{m(m+1)}{3}}$$

For trigonometric polynomials in many variables see [7]; see also [8].

Now let's state a similar theorem in the case where we consider n spaces V_i and equations $f_1 = \ldots = f_n = 0$, where $f_i \in V_i$. For this, we will need the concept of the mixed volume of Banach sets. Using Minkowski sum and homotheties, we can form linear combinations of convex sets with non-negative coefficients. The linear combination of Banach sets is defined by

$$\left(\sum_i \lambda_i \mathcal{E}_i\right)(x) = \sum_i \lambda_i \mathcal{E}_i(x).$$

For n Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ the volume of $\lambda_1 \mathcal{E}_1 + \ldots + \lambda_n \mathcal{E}_n$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$. Its coefficient at $\lambda_1 \cdot \ldots \cdot \lambda_n$ divided by $n!$ is called the mixed volume of Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$. The mixed volume of Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is denoted by $\operatorname{vol}(\mathcal{E}_1, \ldots, \mathcal{E}_n)$.

Theorem 2. *Let $\mathfrak{M}(V_1, \ldots, V_n)$ denote the expectation of the random variable $N(f_1, \ldots, f_n)$. Then it holds that*

$$\mathfrak{M}(V_1, \ldots, V_n) = \frac{n!}{(2\pi)^n} \operatorname{vol}(\mathcal{E}_{V_1}, \ldots, \mathcal{E}_{V_n})$$

The ring of Banach sets

Next we need a concept of the ring of Banach sets. It arises as an analogue of the well-known concept of a ring of convex bodies, first defined in [3]. There are several different versions of this concept. Here we construct an analogue of the definition from [4]. We call the formal difference $\mathcal{E} - \mathcal{B}$ of Banach sets the *virtual Banach set*. Virtual Banach sets form a vector space, where multiplication by negative numbers is defined by $(-1) \cdot (\mathcal{E} - \mathcal{B}) = \mathcal{B} - \mathcal{E}$.

The following notations are used below

• $S = \bigoplus_{0 \le i} S_i$ – the graded symmetric algebra of the space of virtual Banach sets on the manifold X

• I – the linear functional on the space S defined by 1) $I_{S_k} = 0$ for $k \ne n$, and 2) $I(\mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_n) = \mathrm{vol}(\mathcal{E}_1, \ldots, \mathcal{E}_n)$

• $L(x, y) = I(x \cdot y)$ – the symmetric bilinear form on the vector space S

• J – the kernel of the form L.

Lemma 1. *J is a homogeneous ideal of the graded ring S.*

We will call ring $\mathfrak{S} = S/J$ the *ring of virtual Banach sets*.

Corollary 1. *The following statements hold:*

(i) $\mathfrak{S}_0 = \mathbb{R}$

(ii) $\dim \mathfrak{S}_n = 1$

(iii) *The graded ring $\mathfrak{S}$ is generated by elements of degree 1*

(iv) *The mappings $\mathfrak{S}_p \times \mathfrak{S}_{n-p} \to \mathbb{R}$, defined as $(\eta, \xi) \mapsto L(\eta, \xi)$, are non-degenerate pairings.*

Next for n virtual Banach sets $\mathcal{B}_1, \ldots, \mathcal{B}_n$, we use the notation

$$\mathrm{vol}(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_n) = I(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_n) = \mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

Zeros of random sections

Transitioning to zeros of random sections of vector bundles, without formulating precise theorems, we will first briefly describe the situation in the case when considering zeros of sections of an n-dimensional vector bundle $\mathcal{F}$ on X. Just as in the case of functions we consider a finite dimensional space V of smooth sections of $\mathcal{F}$. Here we denote by $\mathfrak{M}(V; U)$ the expectation of the number of zeros of random section $s \in V$ contained in the open set $U \subset X$. By $\mathrm{Res}_U \mathcal{B}$ we denote the constraint of $\mathcal{B} \in \mathfrak{S}$ on the subvariety U.

Theorem 3. *There exists the unique element $\mathcal{B} \in \mathfrak{S}_n$, such that for any $U \subset X$*

$$\mathfrak{M}(V; U) = \frac{n!}{(2\pi)^n} \mathrm{vol}(\mathrm{Res}_U \mathcal{B})$$

Further results can be approximately described as follows. We associate to an element $\mathfrak{s}$ of degree k of the ring of Banach sets $\mathfrak{S}$ a certain k-density $d_k(\mathfrak{s})$ on X and interpret the ring $\mathfrak{S}$ as a ring of these densities. Such densities serve as analogues of Chern forms, representing Chern classes of complex vector bundles, and inherit some properties of Chern forms.

In conclusion, let us define the density $d_k(\mathfrak{s})$. An alternative construction of multiplication in the density ring is given in [1]; see also [5].

Definiton 4. Let H be the subspace of $T_x X$ generated by tangent vectors $\xi_1, \ldots, \xi_k$, $H^\perp \subset T_x^* X$ the orthogonal complement to H, and $\pi_H : T_x^* X \to T_x^* X / H^\perp$ the projection map. The volume form on $T_x^* X / H^\perp$ is defined by $\omega(x) = \xi_1 \wedge \ldots \wedge \xi_m$. Let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be Banach sets on X. Then $d_k(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_k)(\xi_1, \ldots, \xi_m)$ is the mixed k-dimensional volume of convex k-dimensional sets $\pi_H \mathcal{B}_1(x), \ldots, \pi_H \mathcal{B}_k(x)$ in the sense of the volume form $\omega(x)$.

Lemma 2. *For different elements $\mathfrak{s}$ and $\mathfrak{t}$ of degree k in the ring $\mathfrak{S}$, the densities $d_k(\mathfrak{s})$ and $d_k(\mathfrak{t})$ are different.*

Thus, the ring of virtual Banach sets $\mathfrak{S}$ can be considered as a certain ring of densities on the manifold X.

The following statement is an analogue of the BKK formula for Banach sets and for densities d_i.

Theorem 4. *For any Banach sets $\mathcal{B}_1, \ldots, \mathcal{B}_k$ the equality*

$$d_1(\mathcal{B}_1) \cdot \ldots \cdot d_1(\mathcal{B}_k) = k! \, d_k(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_k)$$

holds.

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