The rotation number integer quantization effect in groups acting on the circle

Andrei Malyutin

Abstract. V.M. Buchstaber, O.V. Karpov, and S.I. Tertychnyi initiated the study of the rotation number integer quantization effect for a class of dynamical systems on a torus that includes dynamical systems modeling the dynamics of the Josephson junction. Based on this approach, we study the rotation number integer quantization effect in Artin braid groups and other finitely generated groups acting on the circle. In this case, we find the following manifestation of the quantization effect. Assume that a finitely generated group G acts proximally on the circle by orientation-preserving homeomorphisms. Then for almost every path of any non-degenerate random walk on G, the proportion of elements with integer rotation number in the initial section of the path tends to 1 as the length of the section approaches infinity.

Introduction

We will discuss a new counterintuitive effect for groups acting on the circle. In order to describe this effect, we introduce a series of definitions.

We begin with the concepts of translation and rotation numbers introduced by Henri Poincaré [24]. Let $\mathbb{R}$ be the real line, $\mathbb{Z}$ be the set of integers, and S^1 be the circle $\mathbb{R}/\mathbb{Z}$. The quotient map $\pi\colon \mathbb{R} \to S^1$ is the universal covering map. If $f\colon S^1 \to S^1$ is an orientation-preserving autohomeomorphism, $F\colon \mathbb{R} \to \mathbb{R}$ is a lift of f to $\mathbb{R}$ (that is, $\pi \circ F = f \circ \pi$), and x is a point in $\mathbb{R}$, then the sequence

$$\frac{F(x)-x}{1}, \quad \frac{F^2(x)-x}{2}, \quad \frac{F^3(x)-x}{3}, \quad \dots$$

converges. The limit

$$\tau(F) \;=\; \lim_{k\to\infty} \frac{F^k(x)-x}{k} \;=\; \lim_{k\to\infty} \frac{F^k(x)}{k} \tag{1}$$

does not depend on x and is called the *Poincaré translation number* of F. The value

$$\rho(f) = \tau(F) \mod \mathbb{Z}$$

is independent of the choice of F and is called the *Poincaré rotation number* of f. The translation and rotation numbers are naturally defined for circle isotopies, torus foliations, etc. It is often the case that the term "rotation number" is used to refer to the translation number as well.

Another concept we use is that of proximal group actions. An action of a group G on a space X is said to be *proximal* if, for any two points x and y in X, there exists a sequence $\{g_k\}$ in G such that the sequences $\{g_k(x)\}$ and $\{g_k(y)\}$ converge to one and the same point.

Theorem 1. *Assume that a finitely generated group G acts proximally on the circle by orientation-preserving homeomorphisms. Then for almost every path of any non-degenerate random walk on G, the proportion of elements with integer rotation number in the initial section of the path tends to* 1 *as the length of the section approaches infinity.*

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Andrei Malyutin
St. Petersburg Department of
Steklov Institute of Mathematics
St. Petersburg, Russia
Steklov Mathematical Institute,
Moscow, Russia
e-mail: `malyutin@pdmi.ras.ru`