Solution of Tropical Best Approximation Problems

Nikolai Krivulin

Abstract. We consider discrete best approximation problems in the framework of tropical algebra, which focuses on semirings and semifields with idempotent addition. Given a set of samples from input and output of an unknown function defined on an idempotent semifield, the problem is to find a best approximation of the function by tropical Puiseux polynomial and rational functions. We describe a solution approach that transforms the problem into the best approximation of linear vector equations. Application of this approach yields a direct analytical solution for the polynomial approximation problem and an iterative algorithmic solution for approximation by rational functions. As an illustration, we present results of the best Chebyshev approximation by piecewise linear functions.

Introduction

We consider a discrete approximation problem where an unknown function $f(x)$ is approximated given a set of samples (x_i, y_i) of function values $y_i = f(x_i)$ at some points x_i. Let $F(x; \boldsymbol{\theta})$ be an approximating function that depends on the vector $\boldsymbol{\theta}$ of unknown parameters. A minimax best approximate solution to the problem is defined in the sense of a distance function d to find

$$\boldsymbol{\theta}_* = \arg\min_{\boldsymbol{\theta}} \max_i d(F(x_i; \boldsymbol{\theta}), y_i). \tag{1}$$

In this paper, we outline recent results concerning the investigation of the best approximation problem in the framework of tropical algebra, which deals with the theory and methods of semirings and semifields with idempotent addition [1, 2, 3, 4, 5, 6]. An example of the tropical semifield is an extended set of reals, where the addition is defined as maximum and the multiplication as arithmetic addition (max-plus algebra).

We formulate problem (1) to approximate functions defined on a tropical semifield (a semiring with idempotent addition and invertible multiplication). As approximating functions, we use tropical analogues of Puiseux polynomials and rational functions. The approximation error is defined by a generalized metric on the tropical vector space over the semifield. We note that in the case of max-plus algebra, the Puiseux polynomials and rational functions are real piecewise linear functions, whereas the metric coincides with the Chebyshev metric.

Tropical Puiseux polynomials arise in a range of research contexts from tropical algebraic geometry to optimization problems in operations research [4, 6, 7, 8, 9]. Thus, the development of approximation techniques using tropical Puiseux polynomial and rational functions can be considered of benefit to both tropical algebra and its applications.

To solve the best approximation problems under study, we transform them into solving tropical linear vector equations with an unknown vector on one side (one-sided equations) or on both sides (two-sided equations). We handle the one-sided equation by applying the results in [10, 11], which offer a direct analytical solution to the problem. A best approximate solution of the two-sided equation is obtained by using the iterative alternating algorithm proposed in [12]. Further details on the solution approach and its implementation can be found in [13].

1. Definitions, Notation and Preliminary Results

In this section we outline basic definitions, notations and preliminary results that provide a framework for the description of the solutions of tropical approximation problems presented below. For more details on tropical (idempotent) algebra, one can consult several works, including [1, 2, 3, 4, 5, 6].

1.1. Idempotent Semifield

Let $\mathbb{X}$ be a non-empty set that is equipped with binary operations $\oplus$ (addition) and $\otimes$ (multiplication), and contains distinct elements $\mathbb{0}$ (zero) and $\mathbb{1}$ (unit). Assume that $(\mathbb{X}, \oplus, \mathbb{0})$ is an idempotent commutative monoid, $(\mathbb{X} \setminus \{\mathbb{0}\}, \otimes, \mathbb{1})$ is an Abelian group and multiplication $\otimes$ distributes over addition $\oplus$. The algebraic system $(\mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is commonly referred to as the tropical (idempotent) semifield.

The semifield has idempotent addition: for each $x \in \mathbb{X}$ the equality $x \oplus x = x$ holds, and invertible multiplication: for each $x \neq \mathbb{0}$, there exists x^{-1}, such that $xx^{-1} = \mathbb{1}$ (here and hereafter the multiplication sign $\otimes$ is omitted for brevity). It is assumed that the equation $x^p = a$ has a unique solution x for any $a \in \mathbb{X}$ and integer $p > 0$, which makes powers with rational exponents well defined.

Idempotent addition induces a partial order relation: $x \leq y$ if and only if $x \oplus y = y$. The corresponding partial order is assumed to extend to a total order.

An example of the idempotent semifield under consideration is the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, also known as max-plus algebra. In this semifield, we have $\oplus = \max$, $\otimes = +$, $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. The power x^y coincides with the product $x \times y$. The inverse x^{-1} of any $x \in \mathbb{R}$ corresponds to the opposite number $-x$. The order relation agrees with the usual linear order on $\mathbb{R}$.

1.2. Algebra of Matrices and Vectors

Matrix algebra over over a semifield is introduced in the usual way. Addition, multiplication and scalar multiplication of matrices follow the standard entrywise rules, where addition and multiplication are replaced by $\oplus$ and $\otimes$. A matrix without zero rows and columns is called regular.

A matrix that consists of a single column (row) is a column (row) vector. If a vector has no zero elements, it is called regular.

For any nonzero column vector $\boldsymbol{x} = (x_j)$, the multiplicative conjugate is the row vector $\boldsymbol{x}^- = (x_j^-)$ where $x_j^- = x_j^{-1}$ if $x_j \neq \mathbb{0}$, and $x_j^- = \mathbb{0}$ otherwise.

For any regular vectors $\boldsymbol{x} = (x_j)$ and $\boldsymbol{y} = (y_j)$, we define a distance function

$$d(\boldsymbol{x}, \boldsymbol{y}) = \bigoplus_j \left(x_j y_j^{-1} \oplus x_j^{-1} y_j\right) = \boldsymbol{y}^- \boldsymbol{x} \oplus \boldsymbol{x}^- \boldsymbol{y}.$$

In the context of $\mathbb{R}_{\max,+}$, this function coincides with the Chebyshev metric

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_j |x_j - y_j| = \max_j \max(x_j - y_j, y_j - x_j).$$

In the case of the arbitrary idempotent semifield $\mathbb{X}$, the distance function d can be considered as a generalized metric that takes values in the interval $[\mathbb{1}, \infty)$.

1.3. Tropical Puiseux Polynomials and Rational Functions

We consider a tropical Puiseux polynomial of n monomials in one variable $x \in \mathbb{X}$, which is written in the following form:

$$P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j}, \qquad x \neq \mathbb{0},$$

where $p_j \in \mathbb{Q}$ are exponents and $\theta_j \in \mathbb{X}$, $\theta_j \neq \mathbb{0}$, are coefficients for all $j = 1, \ldots, n$.

We note that a polynomial defined in the context of the semifield $\mathbb{R}_{\max,+}$ (max-plus algebra) is represented in terms of the usual operations as

$$P(x) = \max_{1 \leq j \leq n} (p_j x + \theta_j),$$

and therefore defines a piecewise-linear convex function on $\mathbb{R}$.

Now consider a tropical rational function that is given by

$$R(x) = \frac{P(x)}{Q(x)}, \qquad P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j}, \qquad Q(x) = \bigoplus_{k=1}^{l} \sigma_k x^{q_k}, \qquad x \neq \mathbb{0}.$$

When defined in terms of $\mathbb{R}_{\max,+}$, the rational function can be written as

$$R(x) = P(x) - Q(x) = \max_{1 \leq j \leq n} (p_j x + \theta_j) - \max_{1 \leq k \leq l} (q_k x + \sigma_k),$$

which is a difference of convex functions. We observe that any arbitrary continuous function can be represented as the difference of two convex functions [14].

1.4. Best Approximate Solution of Vector Equations

Given an $(m \times n)$-matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{b}$, consider the problem to find regular n-vectors $\boldsymbol{x}$ that solve the one-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{2}$$

Since the problem may have no solution, we concentrate on finding a best approximate solution to the equation in the sense of the metric d. The next statement is a consequence of the results in [10] (see also [11]).

Theorem 1. *Let $\boldsymbol{A}$ be a regular matrix, $\boldsymbol{b}$ a regular vector and $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}$. Then the following statements hold:*

1. *The best approximate error for equation (2) is equal to $\sqrt{\Delta}$;*
2. *The best approximate solution of the equation is given by*

$$\boldsymbol{x}_{*} = \sqrt{\Delta}(\boldsymbol{b}^{-}\boldsymbol{A})^{-};$$

3. *If $\Delta = \mathbb{1}$, there are exact solutions; $\boldsymbol{x}_{*} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is the maximum solution.*

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are given $(m \times n)$- and $(m \times l)$-matrices. The problem is to find regular $\boldsymbol{x}$ and $\boldsymbol{y}$ of order n and m to satisfy the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{y}. \tag{3}$$

To obtain a best approximate solution to the equation, we apply the alternating algorithm proposed in [12]. The algorithm implements the solution offered by Theorem 1 to solve a series of one-sided equations obtained from (3) in which the left and right sides are alternately replaced by constant vectors.

2. Discrete Best Approximation of Functions

We now describe an algebraic technique to solve the data-fitting problems of approximating an unknown function $y = f(x)$ from finitely many samples (x_i, y_i) in the tropical algebra setting. Both tropical polynomials and rational functions are used as approximants. The problems are handled by transforming them into best approximation of vector equations obtained from the sample data.

Suppose there are m samples (x_i, y_i) where x_i and y_i for $i = 1, \ldots, m$ are corresponding input and output of an unknown function $f : \mathbb{X} \to \mathbb{X}$. Consider the problem of approximating this function by polynomials of n monomials, given by

$$P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j},$$

where we assume for all $j = 1, \ldots, n$ that $p_j \in \mathbb{Q}$ are known exponents and $\theta_j \in \mathbb{X}$ are unknown coefficients. The problem consists in the determination of the unknown coefficients that make the equations

$$P(x_i) = y_i \qquad i = 1, \ldots, m,$$

hold exactly or approximately by minimizing the deviation between both sides.

With the matrix-vector notation

$$\boldsymbol{X} = \begin{pmatrix} x_1^{p_1} & \dots & x_1^{p_n} \\ \vdots & & \vdots \\ x_m^{p_1} & \dots & x_m^{p_n} \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1, \\ \vdots \\ y_m \end{pmatrix}, \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix},$$

we combine the scalar equations into the one-sided vector equation

$$\boldsymbol{X\theta} = \boldsymbol{y},$$

where $\boldsymbol{X}$ and $\boldsymbol{y}$ are a known matrix and vector, and $\boldsymbol{\theta}$ is an unknown vector.

We find a best approximate solution of the equation by applying Theorem 1 to obtain the squared error Δ^* and vector $\boldsymbol{\theta}_* = (\theta_1^*, \dots, \theta_n^*)^T$ of coefficients,

$$\Delta_* = (\boldsymbol{X}(\boldsymbol{y}^- \boldsymbol{X})^-)^- \boldsymbol{y}, \qquad \boldsymbol{\theta}_* = \sqrt{\Delta_*}(\boldsymbol{y}^- \boldsymbol{X})^-.$$

The best approximating polynomial is then given by

$$P_*(x) = \theta_1^* x^{p_1} \oplus \cdots \oplus \theta_n^* x^{p_n}.$$

Consider a rational function as an approximant, which is defined as

$$R(x) = \frac{P(x)}{Q(x)}, \qquad P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j}, \qquad Q(x) = \bigoplus_{k=1}^{l} \sigma_k x^{q_k}.$$

We assume $p_j, q_k \in \mathbb{Q}$ to be known exponents and $\theta_j, \sigma_k \in \mathbb{X}$ unknown coefficients for $j = 1, \dots, n$ and $k = 1, \dots, l$. Given samples $x_i, y_i \in \mathbb{X}$ for $i = 1, \dots, m$ from input and output of an unknown function, the problem is to find the coefficients that achieve the best approximation of the equations

$$R(x_i) = y_i, \qquad i = 1, \dots, m.$$

To represent the problem in vector form, we introduce the notation

$$\boldsymbol{X} = \begin{pmatrix} x_1^{p_1} & \dots & x_1^{p_n} \\ \vdots & & \vdots \\ x_m^{p_1} & \dots & x_m^{p_n} \end{pmatrix}, \qquad \boldsymbol{Y} = \begin{pmatrix} y_1 & & \mathbb{0} \\ & \ddots & \\ \mathbb{0} & & y_m \end{pmatrix},$$

$$\boldsymbol{Z} = \begin{pmatrix} x_1^{q_1} & \dots & x_1^{q_l} \\ \vdots & & \vdots \\ x_m^{q_1} & \dots & x_m^{q_l} \end{pmatrix}, \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_l \end{pmatrix}.$$

The scalar equations can be represented as the two-sided vector equation

$$\boldsymbol{X\theta} = \boldsymbol{YZ\sigma},$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are known matrices, and $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ are unknown vectors.

We obtain a best approximate solution of the vector equation by using the alternating algorithm proposed in [12]. The algorithm yields a minimum squared error Δ_* and related coefficients $\theta_1^*, \dots, \theta_n^*$ and $\sigma_1^*, \dots, \sigma_l^*$ that define the function

$$R_*(x) = \frac{\theta_1^* x^{p_1} \oplus \cdots \oplus \theta_n^* x^{p_n}}{\sigma_1^* x^{q_1} \oplus \cdots \oplus \sigma_l^* x^{q_l}}.$$

We note that in the real problems, the exponents in the approximating polynomials $P(x)$ and $Q(x)$ may be unknown and thus need to be assessed along with the coefficients of monomials. Below we use a Monte Carlo random sampling technique to search for optimal values of exponents in the polynomials.

3. Numerical Examples

In this section, we offer examples in terms of the semifield $\mathbb{R}_{\max,+}$ (max-plus algebra), for which both tropical polynomials and rational functions can be represented as piecewise linear functions. We assume the polynomials to have a fixed number of monomials, while the exponents of these monomials are not given in advance.

We apply a two-level solution approach that combines direct random search to fix exponents with the best approximation by the polynomials with fixed exponents to evaluate the coefficients of monomials. To reduce the feasible set of exponents in random search, we consider only polynomials with integer exponents.

We start with a function defined on the interval $[0,2]$ as follows:

$$f(x) = x^2 - 3x^{1/3} + 5/2, \qquad 0 \leq x \leq 2.$$

The problem is to find an approximate tropical polynomial from a set of $m = 21$ samples (x_i, y_i), where $x_i = (i-1)/10$ and $y_i = f(x_i)$ for $i = 1, \dots, m$. We consider polynomials with $n = 7$ monomials where the exponents are produced by random sampling from the discrete uniform distribution over $[-15, 5]$.

For each sample set of exponents, we evaluate the coefficients that attain the minimum of the approximation error. After examining 10,000 sample sets of exponents, we arrive at the minimum squared error $\Delta_* = 0.0481$ and the polynomial, which in the conventional form is written as

$$\begin{aligned} P_*(x) = \max(&2.5240 - 15x, 1.4096 - 3x, 0.8736 - x, 0.3503, \\ &- 0.4760 + x, -1.6720 + 2x, -3.2853 + 3x). \end{aligned}$$

A graphical illustration of the solution is given in Figure 1.

Now suppose that $m = 21$ samples (x_i, y_i) are given from the function

$$g(x) = 3(x-1)^2 \sin(x) + 1/4, \qquad 0 \leq x \leq 2;$$

where $x_i = (i-1)/10$ and $y_i = g(x_i)$ for $i = 1, \dots, m$.

We approximate $g(x)$ by a tropical rational function $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials with $n = 6$ and $l = 4$ monomials.

After random sampling of 10,000 pairs of sets of exponent and evaluating corresponding coefficients, we obtain a solution with $\Delta_* = 0.0701$. Figure 2 shows the obtained approximating function $R_*(x) = P_*(x) - Q_*(x)$, where

$$\begin{aligned} P_*(x) &= \max(6.9455 - 3x, 6.0860 - 2x, 4.9978 - x, 3.7461, \\ &\qquad 0.7639 + 2x, -2.6361 + 4x), \\ Q_*(x) &= \max(6.6880 - 5x, 6.2962 - 3x, 5.8009 - 2x, 2.4211). \end{aligned}$$

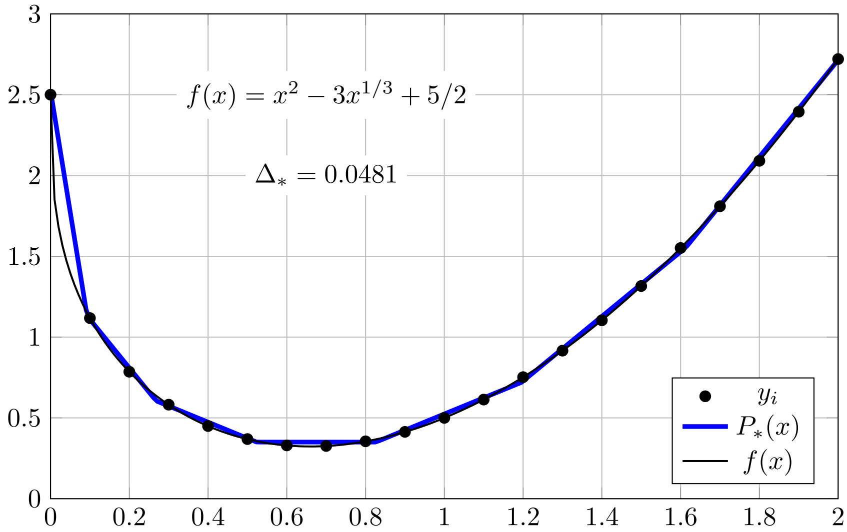

FIGURE 1. Approximation of $f(x)$ by a tropical polynomial $P_*(x)$ with $n = 7$ terms.

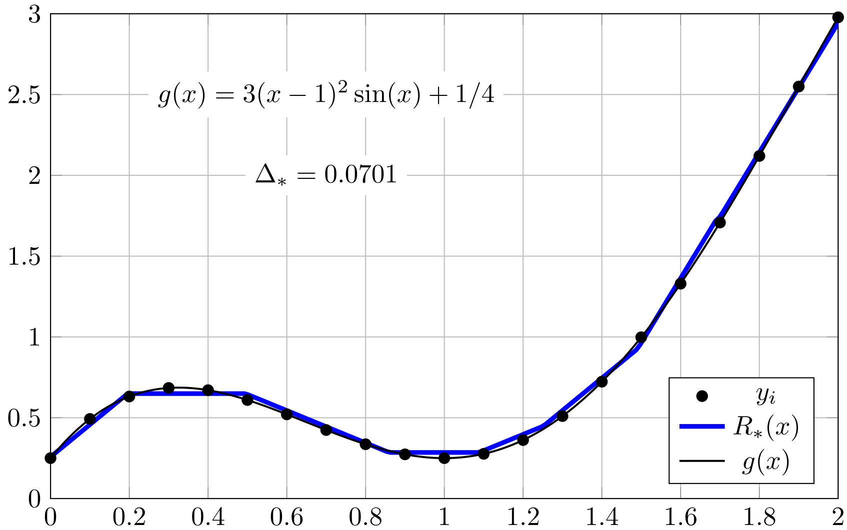

FIGURE 2. Approximation of $g(x)$ by a tropical rational function $R_*(x)$ with $n = 6$ and $l = 4$.

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Nikolai Krivulin
Faculty of Mathematics and Mechanics
St. Petersburg State University
St. Petersburg, Russia
e-mail: `nkk@math.spbu.ru`