The H-Transform in Wolfram Mathematica and Its Particular Cases

Oleg Marichev[1], Elina Shishkina[2]

[1] Wolfram Research Inc, Champaign, USA;
[2]Voronezh State University, Voronezh, Russia

Polynomial Computer Algebra'2024

Introduction

For computer algebra systems, it is necessary to have the most possible universal methods for writing functions and calculating integrals. One of the way is to write functions using Taylor series. Another way is to use Mellin-Barnes integral.

If we take a look to the world of mathematical functions then we can notice that a lot of them can be constructed with the help of only two building blocks: gamma function and power function with natural power.

Euler's Gamma function

Euler's Gamma function $\Gamma(z)$ is defined via a convergent improper integral

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy, \tag{1}$$

which converges for all $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$. Function (1) is extended by analytic continuation to all complex numbers except the non-positive integers (where the function has simple poles). Integration by parts of the expression (1) tends to the recurrent formula

$$\Gamma(z+1) = z\Gamma(z), \tag{2}$$

so

$$\Gamma(n+1) = n!, \qquad n \in \mathbb{N}.$$

Euler's Gamma function

The Bohr–Mollerup theorem shows that the gamma function is the only function that satisfies the properties

1. $f(1) = 1$,
2. $f(x+1) = xf(x)$,
3. for every $x \geq 0$ $\ln f$ is a convex function.

So the Euler's Gamma function the "best" extension of the factorial function to the reals (complex) numbers.

- H. Bohr und J. Mollerup, Laerebogri matematisk Analyse (Copenhagen 1922), v. III, p. 149—164.

Elementary functions

Let us start from elementary functions, for example,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)} x^n, \qquad x \in \mathbb{R},$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(2n+2)} (-1)^n x^{2n+1}, \qquad x \in \mathbb{R},$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{1}{\Gamma(2n+1)} (-1)^n x^{2n}, \qquad x \in \mathbb{R},$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+1)} (-1)^{n-1} (x-1)^n,$$

$$|x-1| \le 1 \text{ and } x \neq 0.$$

Elementary functions

$$\operatorname{arctg} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{\Gamma(2n+1)}{\Gamma(2n+2)} (-1)^n x^{2n+1}, \qquad |x| \leq 1,$$

$$x^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} (x-1)^n = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{1}{\Gamma(\alpha - n + 1)\Gamma(n+1)} (x-1)^n,$$

$$|x-1| \leq 1.$$

The exception is, for example,

$$\operatorname{tg} x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n \left(1 - 4^n\right)}{(2n)!} x^{2n-1},$$

where $B_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^n$ are Bernoulli numbers.

Elementary functions

In Wolfram Mathematica

`Series[f,`x`,`x_0`,n]`

generates a power series expansion for f about the point $x = x_0$ to order $(x - x_0)^n$, where n is an explicit integer.

For example,

Series[e^x, x, 0, 5]$=1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O\left(x^6\right)$

Series[$Tan[x]$, x, 0, 10]$=x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + O\left(x^{11}\right)$

Series[$\sqrt[3]{x}$, x, 1, 5]$=$

$$= 1 + \frac{x-1}{3} - \frac{1}{9}(x-1)^2 + \frac{5}{81}(x-1)^3 - \frac{10}{243}(x-1)^4 + O\left((x-1)^5\right)$$

Special functions

Next, a lot of special functions have the same structure, for example, Bessel function of the first kind

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\alpha+1)\Gamma(n+1)} (-1)^n \left(\frac{x}{2}\right)^{2n+\alpha}, \qquad x \in \mathbb{R},$$

parabolic cylinder function

$$D_\alpha(x) = \frac{e^{-\frac{x^2}{4}}}{2^{\frac{\alpha}{2}+1}\Gamma(-\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma(n+1)} (-1)^n \left(\sqrt{2}x\right)^n, \qquad x \in \mathbb{R},$$

hypergeometric Gauss function

$$_2F_1(a,b;c;x) = F(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} x^n,$$

$$|x| < 1, \qquad c \neq 0, -1, -2, \ldots.$$

Orthogonal polynomials

Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n}\sum_{m=0}^{n}\frac{\Gamma(n+\alpha+1)}{\Gamma(n-m+\alpha+1)\Gamma(m+1)}\times$$

$$\times\frac{\Gamma(n+\beta+1)}{\Gamma(m+\beta+1)\Gamma(n-m+1)}(x-1)^{n-m}(x+1)^m,$$

Legendre polynomials

$$P_n(x) = \frac{1}{2^n}\sum_{m=0}^{[\frac{n}{2}]}\frac{(-1)^m\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}\frac{\Gamma(2n-2m+1)}{\Gamma(n-2m+1)\Gamma(n+1)}x^{n-2m},$$

Chebyshev polymomials

$$T_n(x) = \frac{n}{2}\sum_{m=0}^{[\frac{n}{2}]}(-1)^m\frac{\Gamma(n-m)}{\Gamma(n-2m+1)\Gamma(m+1)}x^{n-2m}.$$

Mellin transform

A powerful tool which open structure of function and show how many gamma functions it involves is Mellin transform. This transform can be used for asymptotic expansions of functions, calculating integrals and solving of differential equations.
Mellin transform is

$$\mathcal{K}^*(s) = \int_0^\infty \mathcal{K}(x) x^{s-1} dx, \qquad s = \gamma + i\tau.$$

Inverse Mellin transform is

$$\mathcal{K}(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathcal{K}^*(s) x^{-s} ds, \qquad x > 0, \qquad \mathrm{Re}\, s = \gamma.$$

Mellin transform

Mellin convolution of functions $K_1(x)$ and $K_2(x)$, defined for $x > 0$ is

$$(\mathcal{K}_1 \circ \mathcal{K}_2)(x) = \int_0^\infty \mathcal{K}_1(t)\mathcal{K}_2\left(\frac{x}{t}\right)\frac{dt}{t}.$$

We have

$$(\mathcal{K}_1 \circ \mathcal{K}_2)^*(s) = \mathcal{K}_1^*(s)\mathcal{K}_2^*(s).$$

In Wolfram Mathematica

```
MellinTransform[expr,x,s]
```

gives the Mellin transform of `expr`.

Mellin transform

For example,

$$\int\limits_0^\infty e^{-x}x^{s-1}dx = \texttt{MellinTransform[}e^{-x}\texttt{, x, s]} = \Gamma(s)$$

$$\int\limits_0^\infty \mathrm{arctg}(x)x^{s-1}dx = \texttt{MellinTransform[ArcTan[x], x, s]} =$$

$$= -\frac{\pi}{2s\cos\left(\frac{\pi s}{2}\right)} = \frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{2s}$$

$$\int\limits_0^\infty J_\alpha(x)x^{s-1}dx =$$

$$= \texttt{MellinTransform[BesselJ[}\alpha\texttt{], x], x, s]} = 2^{s-1}\frac{\Gamma\left(\frac{s}{2}+\frac{\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}-\frac{s}{2}+1\right)}$$

Mellin transform

By analogy with Mellin transform we can write iterated Mellin transform

$$\mathcal{K}^*(s_1, ..., s_n) = \int_0^\infty ... \int_0^\infty \mathcal{K}(x_1, ..., x_n) x_1^{s_1 - 1} ... x_n^{s_n - 1} dx_1 ..., dx_n,$$

$$s = \gamma + i\tau.$$

Inverse iterated Mellin transform is

$$\mathcal{K}(x_1, ..., x_n) = \frac{1}{(2\pi i)^n} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} ... \int_{\gamma_n - i\infty}^{\gamma_n + i\infty} \mathcal{K}^*(s_1, ..., s_n) x_1^{-s_1} ... x_n^{-s_n} ds_1 ... ds_n,$$

$$\operatorname{Re} s_k = \gamma_k, \qquad k = 1, 2, ..., n.$$

Mellin transform

Iterated Mellin convolution is

$$\int_0^\infty \mathcal{K}_1(t) \prod_{j=2}^{n+1} \mathcal{K}_j\left(\frac{x_{j-1}}{t}\right) \frac{dt}{t} = \mathcal{K}(x_1, ..., x_n).$$

We have

$$\mathcal{K}^*(s_1, ..., s_n) = \mathcal{K}_1^*(s_1 + ... + s_n) \prod_{j=2}^{n+1} \mathcal{K}_j^*(s_{j-1}).$$

Mellin transform

Oleg Marichev used Mellin transform and Slater theorem in order to get huge amount of integrals and integral transforms. See

- Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I., 1992. Integrals and series. 1, Elementary Functions. Gordon & Breach Sci. Publ., New York.
- Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I., 1990. Integrals and Series, Vol. 2, Special Functions. Gordon & Breach Sci. Publ., New York.
- Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I., 1990. Integrals and Series, Vol. 3, More Special Functions. Gordon & Breach Sci. Publ., New York.
- Wolfram Mathematica

Marichev's method

The base of Marichev's method are

- Mellin convolution theorem $(K_1 \circ K_2)^*(s) = K_1^*(s) K_2^*(s)$,
- properties of gamma functions $\mathrm{res}_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}$, $n = 0, 1, 2, ...$
- residues theory.

Theorem 1 (Residue theorem)

Let f be analytic, up to isolated singularities at the points $z_1, , ..., z_m$, in the simply connected domain R, and let $\mathcal{L}$ is a positively oriented Jordan curve and the points $z_1 ..., z_m$ lie in the interior of $\mathcal{L}$, then

$$\int_{\mathcal{L}} f(z) dz = 2\pi i \sum_{k=1}^{m} \mathrm{res}\, f(z_k).$$

Marichev's method: algorithm

Let we have an integral

$$\int_a^b \prod_{i=1}^{N-1} f_i(c_i\tau)d\tau = I(c_1, ..., c_{N-1}, a, b), \qquad N = 2, 3, ...$$

In order to calculate it we can use the algorithm.

- Changing variables and functions

$$c_1\tau = t, \qquad f_1(t) = \frac{1}{t}\mathcal{K}_1(t), \qquad f_i(\eta) = \mathcal{K}_i\left(\frac{1}{\eta}\right), \qquad \eta = \frac{x_{i-1}}{t},$$

$$x_{i-1} = \frac{c_1}{c_i}, \qquad i = 2, 3, ..., N{-}1, \qquad x_{N-1} = bc_1, \qquad x_N = ac_1,$$

$$I(c_1, ..., c_{N-1}, a, b) = \frac{1}{c_1}\mathcal{K}(x_1, ..., x_N).$$

- If $a \neq 0$ and/or $b \neq \infty$ we add Heaviside functions

$$\mathcal{K}_N(\eta) = H(\eta - 1), \qquad \mathcal{K}_{N+1}(\eta) = H(1 - \eta),$$

where $H(t) = 1$ when $t \geq 0$ and $H(t) = 0$ when $t < 0$. Then integral $c_1 I(c_1, ..., c_{N-1}, a, b)$ becomes the iterated Mellin convolution

$$\int_0^\infty \mathcal{K}_1(t) \prod_{i=2}^{N+1} \mathcal{K}_i\left(\frac{x_{i-1}}{t}\right) \frac{dt}{t} = \mathcal{K}(x_1, ..., x_N).$$

Marichev's method: algorithm

- Applying the iterated Mellin transform to the $\mathcal{K}(x_1, ..., x_N)$ we get
$$\mathcal{K}^*(s_1, ..., s_N) = \mathcal{K}_1^*(s_1 + ... + s_N) \prod_{i=2}^{N+1} \mathcal{K}_i^*(s_{i-1})$$
and now we can find $\mathcal{K}^*(s_1, ..., s_N)$ as a multiplication of $\mathcal{K}_i^*$, $i = 1, 2, ..., N + 1$.
- Using inverse iterated Mellin transform we can find $\mathcal{K}(x_1, ..., x_N)$. Next we should do inverse substitutions and get $I(c_1, ..., c_{N-1}, a, b)$.

Example 1

Let consider

$$I = \int\limits_0^\infty t^{\alpha-1} e^{-t-\frac{t}{x}} dt, \qquad \operatorname{Re}\alpha > 0, \qquad \operatorname{Re}\left(1+\frac{1}{x}\right).$$

Putting

$$\mathcal{K}_1(t) = t^{\alpha} e^{-t}, \qquad \mathcal{K}_2(\eta) = e^{-\frac{1}{\eta}}, \qquad \eta = \frac{x}{t},$$

we get

$$I = (\mathcal{K}_1 \circ \mathcal{K}_2)(x) = \int\limits_0^\infty \mathcal{K}_1(t)\mathcal{K}_2\left(\frac{x}{t}\right)\frac{dt}{t},$$

$$I^* = (\mathcal{K}_1 \circ \mathcal{K}_2)^*(s) = \mathcal{K}_1^*(s)\mathcal{K}_2^*(s).$$

Example 1

Mellin transforms are

$$\mathcal{K}_1^*(s) = \Gamma(\alpha + s), \qquad \mathrm{Re}(\alpha + s) > 0;$$

$$\mathcal{K}_2^*(s) = \Gamma(-s), \qquad \mathrm{Re}\, s < 0.$$

Then

$$I^* = (\mathcal{K}_1 \circ \mathcal{K}_2)^*(s) = \Gamma(\alpha + s)\Gamma(-s),$$

$$-\mathrm{Re}\,\alpha < \mathrm{Re}\, s = \gamma < 0.$$

Example 1

Now

$$I = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(\alpha + s)\Gamma(-s) x^{-s} ds =$$

$$= -\sum_{k=0}^{\infty} \mathrm{res}_{s=k} (\Gamma(\alpha + s)\Gamma(-s) x^{-s}) =$$

$$= -\sum_{k=0}^{\infty} \Gamma(\alpha + k) x^{-k} \frac{(-1)^{k-1}}{k!} = \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} \left(-\frac{1}{x}\right)^k =$$

$$= \Gamma(\alpha) \left(1 + \frac{1}{x}\right)^{-\alpha}, \qquad |x| > 1.$$

Hypergeometric type functions

Marichev's method is the most effective in the case when $\mathcal{K}_i$ is so called **hypergeometric type function** for $i = 1, ..., N-1$ which are linear combinations of Mellin-Barnes integrals

$$\mathfrak{R}(z) = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - \alpha_j s) \prod\limits_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)} z^s ds. \quad (3)$$

Here $\mathcal{L}$ is a specially chosen infinite contour.

Hypergeometric type functions

Integral (3) is H-function

$$\mathrm{H}_{p,q}^{m,n}(z) = \mathrm{H}_{p,q}^{m,n}\left(\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z \right) =$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s) \prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s)} z^s ds, \qquad (4)$$

where $m, n, p, q \in \mathbb{N}$, $m \leq q$, $n \leq p$, $\alpha_i, \beta_j \in \mathbb{R}$, $\alpha_i > 0$, $1 \leq i \leq p$, $\beta_j > 0$, $1 \leq j \leq q$.

Fox H-function was implemented in the Wolfram Mathematica system as

$$\texttt{FoxH[\{\{\{}a_1,\alpha_1\},\ldots,\{a_n,\alpha_n\},\{\{a_{n+1},\alpha_{n+1}\},\ldots,\{a_p,\alpha_p\}\}\},$$
$$\{\{\{b_1,\beta_1\},\ldots,\{b_m,\beta_m\},\{\{b_{m+1},\beta_{m+1}\},\ldots,\{b_q,\beta_q\}\}\},z\texttt{]}.$$

and introduced commands that allow users to transform many given function into an H-function or G-functions and back (if possible).For example, commands `FoxHReduce[expr,z]`, `MeijerGReduce[expr,z]`, `FunctionExpand` can be used for these purposes, but currently it is better to use functions associated with the specified resource:
`ResourceFunction["MeijerGForm"][expr,z]` and
`ResourceFunction["FoxHForm"][expr,z]`.

Hypergeometric type functions

H-function is described in

- A.M. Mathai and R.K. Saxena, *The H-Function with Applications in Statistics and other Disciplines*, Halsted Press, John Wiley and Sons, New York, 1978.
- H.M. Srivastava, K.C. Gupta and S.L. Goyal, *The H-function of One and Two Variables with Applications*, South Asian Publishers, New Delhi, 1982.
- A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series, Vol. 3, More Special Functions*, Gordon & Breach Sci. Publ., New York, 1990.
- A.A. Kilbas and M. Saigo, *H–Transforms. Theory and Applications*, Boca Raton, Florida, Chapman and Hall, 2004.

If $\alpha_j = 1,\ j = 1, ..., p;\ \beta_j = 1,\ j = 1, ..., q$ we obtain the Meijer G-function

$$G^{m,n}_{p,q}(z) = G^{m,n}_{p,q}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z \right) =$$

$$= \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m} \Gamma(b_j - s) \prod\limits_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - s) \prod\limits_{j=m+1}^{q} \Gamma(1 - b_j + s)}\, z^s\, ds,$$

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\middle|\, z \right) = \mathrm{H}^{m,n}_{p,q}\left(\begin{matrix} (a_i, 1)_{1,p} \\ (b_j, 1)_{1,q} \end{matrix} \middle| z \right).$$

Slater theorem

In the case when $\mathcal{K}_i$, $i = 1, ..., N$ in integral which we want to calculate

$$\int_0^\infty \mathcal{K}_1(t) \prod_{i=2}^{N+1} \mathcal{K}_i \left(\frac{x_{i-1}}{t} \right) \frac{dt}{t} = \mathcal{K}(x_1, ..., x_N)$$

are Meijer G-functions (have forms $G_{p,q}^{m,n}(z)$) we can apply Slater theorem.

Slater theorem

Let the function $\mathcal{K}^*(s)$ has the form

$$\mathcal{K}^*(s) = \frac{\prod\limits_{j=1}^{A} \Gamma(a_j + s) \prod\limits_{k=1}^{B} \Gamma(b_k - s)}{\prod\limits_{l=1}^{C} \Gamma(c_l + s) \prod\limits_{m=1}^{D} \Gamma(d_m - s)}.$$

Let the conditions

$$-\mathrm{Re}\, a_j < \mathrm{Re}\, s < \mathrm{Re}\, b_k,\ \ j = 1, 2, ..., A,\ \ k = 1, 2, ..., B$$

are valid.

Slater theorem

And one of the conditions

$$A + B > C + D,$$

$$A + B = C + D,\ A + D \neq B + C,\ \mathrm{Re}\, s(A + D - B - C) < \frac{1}{2} - \mathrm{Re}\, \nu,$$

$$A = C,\ B = D,\ \mathrm{Re}\nu < 0,$$

where

$$\nu = \sum_{j=1}^{A} a_j + \sum_{k=1}^{B} b_k - \sum_{l=1}^{C} c_l - \sum_{m=1}^{D} d_m,$$

is valid, then for such s there is an inverse transform $\mathcal{K}(x)$ and

$$\mathcal{K}(x) = \begin{cases} \Sigma_A(x), & x > 0, A + D > B + C; \\ \Sigma_A(x), & 0 < x < 1, A + D = B + C; \\ \Sigma_B(1/x), & x > 1, A + D = B + C; \\ \Sigma_B(1/x), & x > 0, A + D < B + C. \end{cases}$$

Slater theorem

Moreover

$$\mathcal{K}(1) = \Sigma_A(1) = \Sigma_B(1),$$

if $\mathrm{Re}\,\nu + C - A + 1 < 0$, $A \geq C$.
In Slater theorem $\Sigma_A(x)$ and $\Sigma_B(1/x)$ are sum of residues of $K^*(s)x^{-s}$ at poles $-a_j - k$ and $b_i + l$, respectively,

$$\Sigma_A(x) = \sum_{j=1}^{A} \sum_{k=0}^{\infty} res_{s=-a_j-k}\{K^*(s)x^{-s}\},$$

$$\Sigma_B(1/x) = -\sum_{i=1}^{B} \sum_{l=0}^{\infty} res_{s=b_i+l}\{K^*(s)x^{-s}\},$$

under the assumption that these series converge.

Slater theorem

These residues have the form

$$res_{s=-a_j-k}\{K^*(s)z^{-s}\} = K^{*\prime}(-a_j - k)z^{a_j+k}\frac{(-1)^k}{k!},$$

$$res_{s=b_i+l}\{K^*(s)z^{-s}\} = K^{*\prime}(b_i + l)z^{-b_i-l}\frac{(-1)^{l-1}}{l!},$$

where the prime in $K^{*\prime}$ means that K^* has had its factors removed $\Gamma(-k)$ or $\Gamma(-l)$ respectively.

Example 2

Let us calculate an integral

$$I = g^h \int\limits_0^{+\infty} F(a, b, c, kt) F(d, e, f, gt) t^h dt, \qquad (5)$$

where $a, b, c, d, e, f, g, h, k \in \mathbb{C}$ don't depend from $t \in \mathbb{C}$. Let's reduce this integral to the Mellin convolution. To do this, we introduce the notation

$$K_1(t) = F\left(a, b, c; kt\right), \qquad K_2\left(\frac{1}{gt}\right) = (gt)^{h+1} F(d, e, f, gt),$$

$$K_2\left(\eta\right) = (gt)^{-h-1} F(d, e, f, 1/gt), \; \eta = gt.$$

Example 2

We obtain

$$I = K(x) = \int\limits_{0}^{+\infty} K_1(t) K_2\left(\frac{x}{t}\right) \frac{dt}{t}, \qquad x = \frac{1}{g}.$$

Using formula

$$\int\limits_{0}^{\infty} x^{\alpha-1} F(a, b, c, -\omega x) dx = \omega^{-\alpha} \Gamma \left[\begin{array}{c} c, \alpha, a-\alpha, b-\alpha \\ a, b, c-\alpha \end{array} \right],$$

where $0 < \mathrm{Re}\,\alpha < \mathrm{Re}\,a, \mathrm{Re}\,b$, $c \neq 0, -1, -2, \ldots,$ $|\arg \omega| < \pi$, we can find $K_1^*(s)$ and $K_2^*(s)$.

Example 2

We get

$$K_1^*(x) = \int_0^\infty F\left(a, b, c; kt\right) t^{s-1} dt = (-k)^s \Gamma\left[\begin{array}{c} c, s, a-s, b-s \\ a, b, c-s \end{array}\right],$$

$$0 < \mathrm{Re}\, s < \mathrm{Re}\, a, \mathrm{Re}\, b;\ \ c \neq 0, -1, -2, ...,\ \ |\arg(-k)| < \pi,$$

$$K_2^*(s) = g^{s-h-1} \int_0^\infty F\left(d, e, f; \frac{1}{gt}\right) t^{s-h-2} dt =$$

$$= \left(-\frac{1}{g}\right)^{h-s+1} g^{s-h-1} \Gamma\left[\begin{array}{c} f, h-s+1, d-h+s-1, e-h+s-1 \\ d, e, f-h+s-1 \end{array}\right],$$

$$0 < \mathrm{Re}\,(h-s+1) < \mathrm{Re}\, d, \mathrm{Re};\ \ e, f \neq 0, -1, -2, ...,\ \ |\arg(-1/g)| < \pi.$$

Example 2

Then

$$K^*(s) = K_1^*(s) K_2^*(s) = \frac{(-1)^{h+1} k^s}{g^{2(h-s+1)}} \Gamma \left[\begin{array}{c} c, f \\ a, b, d, e \end{array} \right] \times$$

$$\times \Gamma \left[\begin{array}{c} h-s+1, d-h+s-1, e-h+s-1, s, a-s, b-s \\ f-h+s-1, c-s \end{array} \right]$$

and

$$K(x) = \frac{(-1)^{h+1}}{2\pi i} \Gamma \left[\begin{array}{c} c, f \\ a, b, d, e \end{array} \right] \times$$

$$\times \int_{\beta - i\infty}^{\beta + i\infty} \Gamma \left[\begin{array}{c} h-s+1, d-h+s-1, e-h+s-1, s, a-s, b-s \\ f-h+s-1, c-s \end{array} \right] \times$$

$$\times x^{-s} ds.$$

Example 2

Using Slater theorem we get

$$A = B = 3, \qquad C = D = 1$$

and

$$\mathrm{Re}\, d > 0, \qquad \mathrm{Re}\, l > 0, \qquad \mathrm{Re}\, a + \mathrm{Re}\, d > \mathrm{Re}\, h + 1,$$

$$\mathrm{Re}\, b + \mathrm{Re}\, d > \mathrm{Re}\, h + 1, \qquad \mathrm{Re}\, a + \mathrm{Re}\, l > \mathrm{Re}\, h + 1, \qquad \mathrm{Re}\, b + \mathrm{Re}\, l > \mathrm{Re}\, \mathrm{h} + 1.$$

Since

$$A + B = 6 > 2 = C + D,$$

then inverse Mellin transform $K(x)$ exists.

Example 2

Since $A + D = 4 = 4 = B + C$, then for $0 < x < 1$ we get

$$K(x) = \Sigma_3(x),$$

where

$$\Sigma_3(x) = \sum_{k=0}^{\infty} res_{s=k} K^*(s) x^{-s} +$$

$$+ res_{s=-k-d+h+1} K^*(s) x^{-s} + res_{s=-k-e+h+1} K^*(s) x^{-s}.$$

Example 2

Let us calculate residues

$$res_{s=k} K^*(s) x^{-s} =$$

$$= \frac{(-1)^k}{k!} \Gamma \left[\begin{array}{c} h+k+1, d-h-k-1, e-h-k-1, a+k, b+k \\ f-h-k-1, c+k \end{array} \right] x^k =$$

$$= \frac{(-1)^k}{k!} x^k \mathbf{G}_1,$$

$$res_{s=-k-d+h+1} K^*(s) x^{-s} =$$

$$= \frac{(-1)^k}{k!} \Gamma \left[\begin{array}{c} k+d, e-k-d, -k-d+h+1, a+k+d-h-1, b+k+d-h-1 \\ f-k-d, c+k+d-h-1 \end{array} \right] \times$$

$$\times x^{k+d-h-1} = \frac{(-1)^k}{k!} x^{k+d-h-1} \mathbf{G}_2.$$

Example 2

Next

$$res_{s=-k-e+h+1} K^*(s)x^{-s} =$$

$$= \frac{(-1)^k}{k!}\Gamma\left[\begin{array}{c} k+e, d-k-e, -k-e+h+1, a+k+e-h-1, b+k+e-h-1 \\ f-k-e, c+k+e-h-1 \end{array}\right]\times$$

$$\times x^{k+e-h-1} = \frac{(-1)^k}{k!} x^{k+e-h-1}\mathbf{G}_3.$$

Example 2

Let us reduce the gamma functions in each residue to such a form that the sum $\Sigma_3(x)$ could be written through the generalized hypergeometric function:

$$\mathbf{G}_1 = (-1)^k \frac{\Gamma(h+1)\Gamma(d-h-1)\Gamma(e-h-1)\Gamma(a)\Gamma(b)}{\Gamma(f-h-1)\Gamma(c)} \times$$

$$\times \frac{(h+1)_k (a)_k (b)_k (2-f-h)_k}{(2-d-h)_k (2-e-h)_k (c)_k},$$

$$\mathbf{G}_2 = (-1)^k \frac{\Gamma(d)\Gamma(e-d)\Gamma(-d+h+1)\Gamma(a+d-h-1)\Gamma(b+d-h-1)}{\Gamma(f-d)\Gamma(c+d-h-1)} \times$$

$$\times \frac{(d)_k (a+d-h-1)_k (b+d-h-1)_k (1-f+d)_k}{(1-e+d)_k (d-h)_k (c+d-h-1)_k},$$

$$\mathbf{G}_3 = (-1)^k \frac{\Gamma(e)\Gamma(d-e)\Gamma(-e+h+1)\Gamma(a+e-h-1)\Gamma(b+e-h-1)}{\Gamma(f-e)\Gamma(c+e-h-1)} \times$$

$$\times \frac{(e)_k (a+e-h-1)_1 (b+e-h-1)_k (1-f+e)_k}{(1-d+e)_k (1+e-h-1)_k (c+e-h-1)_k}.$$

Example 2

Then we get

$$K(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}\Bigg\{\frac{\Gamma(h+1)\Gamma(d-h-1)\Gamma(e-h-1)\Gamma(a)\Gamma(b)}{\Gamma(f-h-1)\Gamma(c)}\times$$

$$\times\frac{(h+1)_k(a)_k(b)_k(2-f-h)_k}{(2-d-h)_k(2-e-h)_k(c)_k}+$$

$$+\frac{\Gamma(d)\Gamma(e-d)\Gamma(-d+h+1)\Gamma(a+d-h-1)\Gamma(b+d-h-1)}{\Gamma(f-d)\Gamma(c+d-h-1)}\times$$

$$\times\frac{(d)_k(a+d-h-1)_k(b+d-h-1)_k(1-f+d)_k}{(1-e+d)_k(d-h)_k(c+d-h-1)_k}x^{d-h-1}+$$

$$+\frac{\Gamma(e)\Gamma(d-e)\Gamma(-e+h+1)\Gamma(a+e-h-1)\Gamma(b+e-h-1)}{\Gamma(f-e)\Gamma(c+e-h-1)}\times$$

$$\times\frac{(e)_k(a+e-h-1)_1(b+e-h-1)_k(1-f+e)_k}{(1-d+e)_k(1+e-h-1)_k(c+e-h-1)_k}x^{e-h-1}\Bigg\}.$$

Example 2

Then

$$K(x) = \frac{\Gamma(h+1)\Gamma(d-h-1)\Gamma(e-h-1)\Gamma(a)\Gamma(b)}{\Gamma(f-h-1)\Gamma(c)} \times$$

$$\times\, {}_4F_3(h+1, a, b, 2-f-h; 2-d-h, 2-e-h, c; x) +$$

$$+x^{d-h-1} \frac{\Gamma(d)\Gamma(e-d)\Gamma(-d+h+1)\Gamma(a+d-h-1)\Gamma(b+d-h-1)}{\Gamma(f-d)\Gamma(c+d-h-1)} \times$$

$$\times\, {}_4F_3(d, a+d-h-1, b+d-h-1, 1-f+d; 1-e+d, d-h, c+d-h-1; x) +$$

$$+x^{e-h-1} \frac{\Gamma(e)\Gamma(d-e)\Gamma(-e+h+1)\Gamma(a+e-h-1)\Gamma(b+e-h-1)}{\Gamma(f-e)\Gamma(c+e-h-1)} \times$$

$$\times\, {}_4F_3(e, a+e-h-1, b+e-h-1, 1-f+e; 1-d+e, e-h, c+e-h-1; x).$$

Example 2

We finally obtain that the improper integral of the product of hypergeometric functions and a power function has the form

$$\int_0^{+\infty} F(a,b,c,t)F(d,e,f,gt)t^h dt =$$

$$= g^{-h}\frac{\Gamma(h+1)\Gamma(d-h-1)\Gamma(e-h-1)\Gamma(a)\Gamma(b)}{\Gamma(f-h-1)\Gamma(c)}\times$$

$$\times\, {}_4F_3(h+1,a,b,2-f-h;2-d-h,2-e-h,c;1/g)+$$

$$+g^{1-d}\frac{\Gamma(d)\Gamma(e-d)\Gamma(-d+h+1)\Gamma(a+d-h-1)\Gamma(b+d-h-1)}{\Gamma(f-d)\Gamma(c+d-h-1)}\times$$

$$\times\, {}_4F_3(d,a+d-h-1,b+d-h-1,1-f+d;1-e+d,d-h,c+d-h-1;1/g)+$$

$$+g^{1-e}\frac{\Gamma(e)\Gamma(d-e)\Gamma(-e+h+1)\Gamma(a+e-h-1)\Gamma(b+e-h-1)}{\Gamma(f-e)\Gamma(c+e-h-1)}\times$$

$$\times\, {}_4F_3(e,a+e-h-1,b+e-h-1,1-f+e;1-d+e,e-h,c+e-h-1;1/g).$$

Example 2

Also we get conditions

$$\operatorname{Re} d > 0,$$

$$\operatorname{Re} l > 0,$$

$$\operatorname{Re} a + \operatorname{Re} d > \operatorname{Re} h + 1,$$

$$\operatorname{Re} b + \operatorname{Re} d > \operatorname{Re} h + 1,$$

$$\operatorname{Re} a + \operatorname{Re} l > \operatorname{Re} h + 1,$$

$$\operatorname{Re} b + \operatorname{Re} l > \operatorname{Re} \mathrm{h} + 1, \qquad |\arg(-1/g)| < \pi.$$

Mittag-Leffler function

Slater theorem doesn't work when we deal with integrals of H-functions and its particular cases which are not Meijer functions.

The *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ is entire function of order $1/\alpha$ defined by the following series when the real part of α is strictly positive

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z \in \mathbb{C}, \ \alpha, \beta \in \mathbb{C}, \ \mathrm{Re}\, \alpha > 0, \ \mathrm{Re}\, \beta > 0. \tag{5}$$

The function (5) was introduced by Gesta Mittag–Leffler in 1903 for α=1 and A. Wimann in 1905 in the general case. The first applications of these functions by Mittag–Leffler and Wiman were applications in complex analysis (non-trivial examples of entire functions with non-integer orders of growth and generalized summation methods).

Mittag-Leffler function

In the USSR, these functions became mainly known after the publication of the famous monograph by M. M. Dzhrbashyan.

- Dzhrbashyan, M.M., 1966. Integral Transforms and Representations of Functions in Complex Plane. Moscow, Nauka. (in Russian)

The most famous application of the Mittag–Leffler functions in the theory of integro-differential equations and fractional calculus is the fact that through them the resolvent of the Riemann–Liouville fractional integral is explicitly expressed in accordance with the famous Hille–Tamarkin–Dzhrbashyan formula. In view of the numerous applications to the solution of fractional differential equations, this function was deservedly named "*Royal function of fractional calculus*".

The most general formula includes a lot of integrals was obtain by Oleg Marichev and has the form

$$\int\limits_{0}^{\infty} t^{c-1} \mathrm{H}_{p,q}^{m,n}\left[u_2 t^{r_2} \left| \begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \right.\right] \cdot \mathrm{H}_{P,Q}^{M,N}\left[u_1 t^{r_1} \left| \begin{matrix} (c_i, \gamma_i)_{1,P} \\ (d_j, \delta_j)_{1,Q} \end{matrix} \right.\right] dt = \theta\left(-\frac{r_1}{r_2}\right) \frac{u_2^{-\frac{c}{r_2}}}{|r_2|} \times$$

$$\times \mathrm{H}_{p+P,q+Q}^{m+M,n+N}\left[u_1 u_2^{-\frac{r_1}{r_2}} \left| \begin{matrix} (c_i, \gamma_i)_{1,N}, (\mathfrak{A}_i, \mathfrak{B}_i)_{N+1,N+n}, \left(\overline{\mathfrak{A}}_j, \overline{\mathfrak{B}}_j\right)_{n+1,p}, (c_{j+N-p}, \gamma_{j+N-p})_{p+1,p+P-N} \\ (d_{i-m}, \delta_{i-m})_{m+1,m+M}, (\mathfrak{C}_i, \mathfrak{D}_i)_{1,m}, \left(\overline{\mathfrak{C}}_j, \overline{\mathfrak{D}}_j\right)_{Q+1,Q+q-m}, (d_j, \delta_j)_{1+M,Q} \end{matrix} \right.\right] +$$

$$+ \theta\left(\frac{r_1}{r_2}\right) \frac{u_2^{-\frac{c}{r_2}}}{|r_2|} \times \tag{6}$$

$$\times \mathrm{H}_{q+P,p+Q}^{n+M,m+N}\left[u_1 u_2^{-\frac{r_1}{r_2}} \left| \begin{matrix} (c_i, \gamma_i)_{1,N}, (\mathfrak{E}_i, \mathfrak{F}_i)_{N+1,N+m}, \left(\overline{\mathfrak{E}}_j, \overline{\mathfrak{F}}_j\right)_{m+1,q}, (c_{j+N-q}, \gamma_{j+N-q})_{q+1,q+P-N} \\ (d_{i-n}, \delta_{i-n})_{n+1,n+M}, (\mathfrak{G}_i, \mathfrak{H}_i)_{1,n}, \left(\overline{\mathfrak{G}}_j, \overline{\mathfrak{H}}_j\right)_{Q+1,Q+p-n}, (d_j, \delta_j)_{1+M,Q} \end{matrix} \right.\right].$$

Here θ is the Heaviside function

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

$\mathfrak{A}_i = a_{-N+i} + \frac{c\alpha_{-N+i}}{r_2}$, $\mathfrak{B}_i = -\frac{r_1\alpha_{-N+i}}{r_2}$, $\overline{\mathfrak{A}_j} = a_j + \frac{c\alpha_j}{r_2}$, $\overline{\mathfrak{B}_j} = -\frac{\alpha_j r_1}{r_2}$,
$\mathfrak{C}_i = b_i + \frac{c\beta_i}{r_2}$, $\mathfrak{D}_i = -\frac{r_1\beta_i}{r_2}$. $\overline{\mathfrak{C}}_j = b_{m-Q+j} + \frac{c\beta_{m-Q+j}}{r_2}$,
$\overline{\mathfrak{D}}_j = -\frac{r_1\beta_{m-Q+j}}{r_2}$, $\mathfrak{E}_i = 1 - b_{i-N} - \frac{c\beta_{i-N}}{r_2}$, $\mathfrak{F}_i = \frac{r_1\beta_{i-N}}{r_2}$,
$\overline{\mathfrak{E}}_j = 1 - b_j - \frac{c\beta_j}{r_2}$, $\overline{\mathfrak{F}}_j = \frac{r_1\beta_j}{r_2}$, $\mathfrak{G}_i = 1 - a_i - \frac{c\alpha_i}{r_2}$, $\mathfrak{H}_i = \frac{r_1\alpha_i}{r_2}$,
$\overline{\mathfrak{G}}_j = 1 - a_{n-Q+j} - \frac{c\alpha_{n-Q+j}}{r_2}$, $\overline{\mathfrak{H}}_j = \frac{r_1\alpha_{n-Q+j}}{r_2}$.

H-transform

The integral transform of the form

$$(\mathbf{H}f)(x) = \int_0^\infty \mathrm{H}_{p,q}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| xt\right] f(t)dt$$

where $\mathrm{H}_{p,q}^{m,n}$ is the H-function, is called the **H-transform** of a function $f(t)$. H-function is

$$\mathrm{H}_{p,q}^{m,n}(z) = \mathrm{H}_{p,q}^{m,n}\left(\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle| z\right) =$$

$$= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)} z^s ds,$$

where $m,n,p,q\in\mathbb{N}$, $m\le q$, $n\le p$, $\alpha_i,\beta_j\in\mathbb{R}$, $\alpha_i>0$, $1\le i\le p$, $\beta_j>0$, $1\le j\le q$.

H-transform

H-transform of H-function can be calculated by (see (6))

$$\int_0^\infty t^{c-1} \mathrm{H}^{m,n}_{p,q}\left[u_2 t^{r_2} \middle| \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] \cdot \mathrm{H}^{M,N}_{P,Q}\left[u_1 t^{r_1} \middle| \begin{matrix} (c_i,\gamma_i)_{1,P} \\ (d_j,\delta_j)_{1,Q} \end{matrix}\right] dt.$$

The H-Transform is one of the most general integral transforms today is the H-Transform, which uses the Fox's H-function as a kernel. Anatoly A. Kilbas and Megumi Saigo wrote a book which is fully devoted to the H-Transforms.

- A.A. Kilbas and M. Saigo, *H–Transforms. Theory and Applications*, Boca Raton, Florida, Chapman and Hall, 2004.

Examples

Formula (6) can be used for evaluation integral from product of power, Bessel J_ν and Mittag–Leffler functions with arbitrary power arguments. This result can be interpreted as values of Mellin transform from corresponding product of J_ν and Mittag–Leffler functions or as Hankel transform from product power and Mittag–Leffler functions or Mittag–Leffler transform from product power and Bessel J_ν functions. Below we give value of this integral:

$$2\int_0^\infty t^{c-1} J_\nu\left(2u_2 t^{r_2}\right) E_{\alpha,a}\left(-u_1 t^{r_1}\right) dt =$$

$$= \theta\left(-\frac{r_1}{r_2}\right)\frac{u_2^{-\frac{c}{r_2}}}{|r_2|}\mathrm{H}^{2,1}_{1,4}\left[u_1 u_2^{-\frac{r_1}{r_2}}\,\middle|\, \begin{matrix}(0,1)\\ \left(\frac{c}{2r_2}+\frac{\nu}{2},-\frac{r_1}{2r_2}\right),(0,1),(1-a,\alpha),\left(\frac{c}{2r_2}-\frac{\nu}{2},-\frac{r_1}{2r_2}\right)\end{matrix}\right]+$$

$$+\theta\left(\frac{r_1}{r_2}\right)\frac{u_2^{-\frac{c}{r_2}}}{|r_2|}\mathrm{H}^{1,2}_{3,2}\left[u_1 u_2^{-\frac{r_1}{r_2}}\,\middle|\, \begin{matrix}(0,1),\left(1-\frac{c}{2r_2}-\frac{\nu}{2},\frac{r_1}{2r_2}\right),\left(1-\frac{c}{2r_2}+\frac{\nu}{2},\frac{r_1}{2r_2}\right)\\ (0,1),(1-a,\alpha)\end{matrix}\right].$$

THANK YOU FOR ATTENTION.