The H-Transform in Wolfram Mathematica and Its Particular Cases

Oleg Marichev and Elina Shishkina

Abstract. Nowadays, the Fox H-function is the most important function, having accumulated almost all named functions, and yet it is widely unknown. This complicated function includes four groups of parameters inside of gamma functions, which allows it to accumulate about 150 named functions of very different types: power, exponential, logarithmic, discontinuous, etc. Each of these functions or their combinations can be a kernel of integral transform. The kernels of classical integral transforms (Laplace, Mellin, Fourier, Hilbert, Hankel and others) are the cases of Fox H-function. Each transform can be applied to the Fox H-function or its particular cases, which allows us to evaluate approximately 80% of integrals, presented in handbooks nowadays. Our talk is devoted to the Fox H-transform and its particular cases.

1. Fox H-Function in Wolfram Mathematica

Fox H-functions (introduced in [1]) are versatile special functions that enable a unified, coherent approach to various areas, such as integral transforms and fractional calculus. The Fox H-function is defined by a Mellin-Barnes type integral with an integrand involving products and quotients of Euler gamma functions. It generalizes most known elementary and special functions, allowing nearly all integral transforms to be expressed as H-transforms. Detailed information about Fox H-functions can be found in [2, 3, 4, 5]. Some applications of H-functions are given in [6, 7].

H-function is defined by

$$\mathrm{H}_{p,\,q}^{m,n}[z] \equiv \mathrm{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right] = \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,\ \ z \neq 0, \tag{1}$$

where

$$\mathcal{H}_{p,q}^{m,n}(s) \equiv \mathcal{H}_{p,q}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\middle| s\right] = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \quad (2)$$

$m,n,p,q \in \mathbb{N}$, $m \leq q$, $n \leq p$, $\alpha_i, \beta_j \in \mathbb{R}$, $\alpha_i > 0$, $1 \leq i \leq p$, $\beta_j > 0$, $1 \leq j \leq q$.

Here $\mathcal{L}$ is a specially chosen infinite contour, described in [2, 3, 4, 5]. In (2) an empty product, if it occurs, being taken to be one.

Fox H-function was implemented in the Wolfram Mathematica system as

$$\texttt{FoxH[\{\{\{}a_1\texttt{,}\alpha_1\texttt{\},...,\{}a_n\texttt{,}\alpha_n\texttt{\},\{\{}a_{n+1}\texttt{,}\alpha_{n+1}\texttt{\},...,\{}a_p\texttt{,}\alpha_p\texttt{\}\}\},}$$
$$\texttt{\{\{\{}b_1\texttt{,}\beta_1\texttt{\},...,\{}b_m\texttt{,}\beta_m\texttt{\},\{\{}b_{m+1}\texttt{,}\beta_{m+1}\texttt{\},...,\{}b_q\texttt{,}\beta_q\texttt{\}\}\},z].}$$

and introduced commands that allow users to transform many given function into an H-function or G-functions and back (if possible) [8, 9]. For example, commands `FoxHReduce[expr,z]`, `MeijerGReduce[expr,z]`, `FunctionExpand` can be used for these purposes, but currently it is better to use functions associated with the specified resource: `ResourceFunction["MeijerGForm"][expr,z]` and `ResourceFunction["FoxHForm"][expr,z]`.

2. Calculation of Mellin transform of product of two H-functions

For calculation of integrals by method described in the book of O.I.Marichev [10] we can use Mellin transform defined by the formula

$$K^*(s) = \int\limits_0^\infty K(x)x^{s-1}dx, \qquad s = \gamma + i\tau$$

is used. Then inverse Mellin transform is presented by integral

$$K(x) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} K^*(s)x^{-s}ds, \qquad x>0, \qquad \mathrm{Re}(s)=\gamma$$

and the Mellin convolution of functions $K_1(x)$ and $K_2(x)$ for $x > 0$ can be written thought relation

$$(K_1 \circ K_2)(x) = \int\limits_0^\infty K_1(t)K_2\left(\frac{x}{t}\right)\frac{dt}{t} = K(x).$$

We have

$$(K_1 \circ K_2)^*(s) = K_1^*(s)K_2^*(s).$$

In 1966, Gupta K.C. and Jain U.C. studied two integrals of Mellin convolution type derived from the product of two H-functions, and provided conditions for their convergence. These integrals are studied on pages 46-47 of the book [3] and on page 60 of the book [11]. Here, we consider a more generic integral along with a more comprehensive set of conditions for convergence, which encompass those two integrals as specific cases.

The most general formula includes a lot of integrals is

$$\int_0^\infty t^{c-1} \mathrm{H}_{p,q}^{m,n}\left[u_2 t^{r_2} \middle| \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] \cdot \mathrm{H}_{P,Q}^{M,N}\left[u_1 t^{r_1} \middle| \begin{matrix} (c_i,\gamma_i)_{1,P} \\ (d_j,\delta_j)_{1,Q} \end{matrix}\right] dt = \theta\left(-\frac{r_1}{r_2}\right) \frac{u_2^{-\frac{c}{r_2}}}{|r_2|} \times$$

$$\times \mathrm{H}_{p+P,q+Q}^{m+M,n+N}\left[u_1 u_2^{-\frac{r_1}{r_2}} \middle| \begin{matrix} (c_i,\gamma_i)_{1,N}, (\mathfrak{A}_i,\mathfrak{B}_i)_{N+1,N+n}, \left(\overline{\mathfrak{A}_j},\overline{\mathfrak{B}_j}\right)_{n+1,p}, (c_{j+N-p},\gamma_{j+N-p})_{p+1,p+P-N} \\ (d_{i-m},\delta_{i-m})_{m+1,m+M}, (\mathfrak{C}_i,\mathfrak{D}_i)_{1,m}, \left(\overline{\mathfrak{C}}_j,\overline{\mathfrak{D}}_j\right)_{Q+1,Q+q-m}, (d_j,\delta_j)_{1+M,Q} \end{matrix}\right] +$$

$$+\theta\left(\frac{r_1}{r_2}\right) \frac{u_2^{-\frac{c}{r_2}}}{|r_2|} \times$$

$$\times \mathrm{H}_{q+P,p+Q}^{n+M,m+N}\left[u_1 u_2^{-\frac{r_1}{r_2}} \middle| \begin{matrix} (c_i,\gamma_i)_{1,N}, (\mathfrak{E}_i,\mathfrak{F}_i)_{N+1,N+m}, \left(\overline{\mathfrak{E}}_j,\overline{\mathfrak{F}}_j\right)_{m+1,q}, (c_{j+N-q},\gamma_{j+N-q})_{q+1,q+P-N} \\ (d_{i-n},\delta_{i-n})_{n+1,n+M}, (\mathfrak{G}_i,\mathfrak{H}_i)_{1,n}, \left(\overline{\mathfrak{G}}_j,\overline{\mathfrak{H}}_j\right)_{Q+1,Q+p-n}, (d_j,\delta_j)_{1+M,Q} \end{matrix}\right], \tag{3}$$

where θ is the Heaviside function

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases},$$

$\mathfrak{A}_i = a_{-N+i} + \frac{c\alpha_{-N+i}}{r_2}$, $\mathfrak{B}_i = -\frac{r_1\alpha_{-N+i}}{r_2}$, $\overline{\mathfrak{A}_j} = a_j + \frac{c\alpha_j}{r_2}$, $\overline{\mathfrak{B}_j} = -\frac{\alpha_j r_1}{r_2}$, $\mathfrak{C}_i = b_i + \frac{c\beta_i}{r_2}$, $\mathfrak{D}_i = -\frac{r_1\beta_i}{r_2}$. $\overline{\mathfrak{C}}_j = b_{m-Q+j} + \frac{c\beta_{m-Q+j}}{r_2}$, $\overline{\mathfrak{D}}_j = -\frac{r_1\beta_{m-Q+j}}{r_2}$, $\mathfrak{E}_i = 1 - b_{i-N} - \frac{c\beta_{i-N}}{r_2}$, $\mathfrak{F}_i = \frac{r_1\beta_{i-N}}{r_2}$, $\overline{\mathfrak{E}}_j = 1 - b_j - \frac{c\beta_j}{r_2}$, $\overline{\mathfrak{F}}_j = \frac{r_1\beta_j}{r_2}$, $\mathfrak{G}_i = 1 - a_i - \frac{c\alpha_i}{r_2}$, $\mathfrak{H}_i = \frac{r_1\alpha_i}{r_2}$, $\overline{\mathfrak{G}}_j = 1 - a_{n-Q+j} - \frac{c\alpha_{n-Q+j}}{r_2}$, $\overline{\mathfrak{H}}_j = \frac{r_1\alpha_{n-Q+j}}{r_2}$.

The integral transform of the form (see [5], p. 71, formula (3.1.1))

$$(\mathbf{H}f)(x) = \int_0^\infty \mathrm{H}_{p,q}^{m,n}\left[xt \middle| \begin{matrix} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix}\right] f(t)dt$$

where $\mathrm{H}_{p,q}^{m,n}$ is the H-function defined in (1), is called the H-transform of a function $f(t)$.

H-Transform can be calculated by (3). The H-Transform is one of the most general integral transforms today is the H-Transform, which uses the Fox's H-function as a kernel. Anatoly A. Kilbas and Megumi Saigo wrote a book [5] which is fully devoted to the H-Transforms.

3. Examples

Formula (3) can be used for evaluation integral from product of power, Bessel J_ν and Mittag–Leffler functions with arbitrary power arguments. This result can be interpreted as values of Mellin transform from corresponding product of J_ν and Mittag–Leffler functions or as Hankel transform from product power and Mittag–Leffler functions or Mittag–Leffler transform from product power and Bessel J_ν

functions. Below we give value of this integral:

$$2\int_0^\infty t^{c-1} J_\nu\left(2u_2 t^{r_2}\right) E_{\alpha,a}\left(-u_1 t^{r_1}\right) dt =$$

$$= \theta\left(-\frac{r_1}{r_2}\right) \frac{u_2^{-\frac{c}{r_2}}}{|r_2|} \mathrm{H}_{1,4}^{2,1}\left[u_1 u_2^{-\frac{r_1}{r_2}} \middle| \begin{matrix} (0,1) \\ \left(\frac{c}{2r_2}+\frac{\nu}{2}, -\frac{r_1}{2r_2}\right), (0,1), (1-a,\alpha), \left(\frac{c}{2r_2}-\frac{\nu}{2}, -\frac{r_1}{2r_2}\right) \end{matrix}\right] +$$
$$+\theta\left(\frac{r_1}{r_2}\right) \frac{u_2^{-\frac{c}{r_2}}}{|r_2|} \mathrm{H}_{3,2}^{1,2}\left[u_1 u_2^{-\frac{r_1}{r_2}} \middle| \begin{matrix} (0,1), \left(1-\frac{c}{2r_2}-\frac{\nu}{2}, \frac{r_1}{2r_2}\right), \left(1-\frac{c}{2r_2}+\frac{\nu}{2}, \frac{r_1}{2r_2}\right) \\ (0,1), (1-a,\alpha) \end{matrix}\right].$$

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Oleg Marichev
Wolfram Research Inc
Champaign, USA
e-mail: `oleg@wolfram.com`

Elina Shishkina
Mathematical and Applied Analysis Department
Voronezh State University
Voronezh, Russia
e-mail: `ilina_dico@mail.ru`