Estimates for roots of a polynomial in the field of multiple formal fractional power series in zero characteristic

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Let k be a ground field of zero–characteristic with algebraic closure $\overline{k}$. We assume that k is finitely generated over its primitive subfield. Let $f \in k[X_1, \dots, X_n, Z]$ be a polynomial of degree $\deg_{Z,X_1,\dots,X_n} \leqslant d$ for an integer $d \geqslant 2$.

Consider $f \in k(X_1, \dots, X_n)[Z]$ as a polynomial in one variable Z with coefficients in $k(X_1, \dots, X_n)$. Then the roots $Z = z_\alpha$ of the polynomial f belong to the field of multiple formal fractional power series in $X_1, \dots, X_n$, i.e. to the union by all integers $\nu_1, \dots, \nu_n \geqslant 1$ of the fields of multiple formal fractional power series:

$$\bigcup_{\nu_1,\dots,\nu_n \geqslant 1} \overline{k}((X_1^{1/\nu_1}))((X_2^{1/\nu_2}))\dots((X_n^{1/\nu_2})). \tag{1}$$

This field is algebraically closed. The aim of this talk is to attract the atention to the problem of estimating and constructing the roots z_α in the field (1) (of course one needs to estimate the sizes of coefficients from $\overline{k}$ of z_α in the field (1)). This problem is solved for $n = 1$ in [1]. To our knowledge for the case case $n > 1$ no estimates have been obtained so far.

The problem for an arbitrary n is reduced to the case $\nu_1 = \dots = \nu_n = 1$. Hence now $z_\alpha \in \overline{k}((X_1))((X_2))\dots((X_n))$. Further, for all $1 \leqslant j \leqslant n$ put $X'_j = X_j/(X_1^{\mu_{j,1}} \cdot \ldots \cdot X_{j-1}^{\mu_{j,j-1}})$ for some integers $\mu_{j,i} \geqslant 0$ (so $X'_1 = X_1$). Then one can choose integers $\mu_{j,i}$ such that

$$z_\alpha \in \overline{k}[[X'_1, \dots, X'_n]], \tag{2}$$

i.e., z_α are formal power series in $X'_1, \dots, X'_n$ with coefficients from $\overline{k}$. Good upper bounds for the coefficients of formal power series in (2) can be obtained using the results of [2], [3].

Now it remains to estimate the least possible $\mu_{j,i}$. This can be done applying the results of [1] or [3] recursively. The direct application of [1] or [3] gives double–exponential in n upper bounds for $\mu_{j,i}$. But we hope to improve the estimates from

[3] and obtain upper bounds for $\mu_{j,i}$ which are subexponential in the number of coefficients of the polynomial f, i.e., upper bounds polynomial in $d^{n^{O(1)}}$.

References

[1] A.L. Chistov, *Polynomial complexity of the Newton–Puiseux algorithm*, In: International Symposium on Mathematical Foundations of Computer Science 1986. Lecture Notes in Computer Science Vol. 233 Springer (1986) p. 247–255.

[2] A.L. Chistov *An algorithm for factoring polynomials in the ring of multivariable formal power series in zero–characteristic*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 517 (2022), p. 268–290 (in Russian).

[3] A.L. Chistov *An algorithm for factoring polynomials in the ring of multivariable formal power series in zero–characteristic. II*, Zap. Nauchn. Semin. St-Petersburg. Otdel. Mat. Inst. Steklov (POMI) 529 (2023), p. 261–290 (in Russian).

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