Binomial Coefficients as Functions of their Denominator; *Another Primality Criteria for Natural Integers*

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Motivation

Binomial coefficients have
surprisingly great expressive power ...
Yu. V. Matiyasevich [1]

In [2] we proved the identity

$$\binom{n}{k} = (-2)^n \sum_{i=0}^{n} \binom{\frac{i-1}{2}}{n} K_i^{(n)}(k), \tag{1}$$

for all integer $k, 0 \leq k \leq n$, where $K_i^{(n)}(k)$ are the Krawtchouk polynomials of order n. [3]

Interpolation

Let $f(x)$ be a real function. The following formula for interpolation polynomial is valid. [4]

$$\mathcal{B}_n(f;x) = \sum_{m=0}^{n} \binom{x}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(k).$$

Let us define $\left\langle \binom{n}{x} \right\rangle$ as a polynomial $\mathcal{B}_n\left(\binom{n}{x}; x\right)$ for fixed integer n.

Main Theorem

Primality Criteria

An odd positive integer n is odd prime iff denominator of the rational number $\langle\binom{n}{n-1}\rangle$ is n^{n-1}, where $\langle\binom{n}{x}\rangle$ is interpolation polynomial on x for the set of binomial coefficients $\{\binom{n}{r}\}_{r=0,\ldots,n}$.

Examples

Primality Criteria

$$\left\langle \binom{3}{1/3} \right\rangle = \frac{17}{9}$$

$$\left\langle \binom{5}{1/5} \right\rangle = \frac{769}{625}$$

"Counter" example

$$\left\langle \binom{2}{1/2} \right\rangle = \frac{7}{4}$$

Examples 2

Primality Criteria

$$\left\langle \binom{7}{1/7} \right\rangle = \frac{233225}{117649} = \frac{491 \times 19 \times 5^2}{7^6}$$

$$\left\langle \binom{11}{1/11} \right\rangle = \frac{115853436093}{25937424601} = \frac{223224347 \times 173 \times 3}{11^{10}}$$

Composite numbers

$$\left\langle \binom{6}{1/6} \right\rangle = \frac{2952251}{1679616} = \frac{967 \times 71 \times 43}{6^8}$$

$$\left\langle \binom{10}{1/10} \right\rangle = \frac{47755338385111}{16000000000000} = \frac{6822191197873 \times 7}{2^{16} \times 5^{12}}$$

Additional examples

It is easy to prove that for any $x \in \mathbb{R}$

$$\left\langle \binom{n}{x} \right\rangle = \left\langle \binom{n}{n-x} \right\rangle .$$

If $x = -1$ then

$$\left\langle \binom{n}{-1} \right\rangle = \left\langle \binom{n}{n+1} \right\rangle = (-1)^{\left[\frac{n}{2}\right] \mod 4} \times \binom{n}{[n/2]}.$$

Sperner's theorem says that $\binom{n}{[n/2]}$ is the maximal number of subsets of an n-set such that no one contains another (A001405 in OEIS).

Additional examples (cellular automaton)

If $x = J_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $\left\langle \binom{2}{J_2^k} \right\rangle = -\left(\left(a_k J_2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right)$,
where $a_k, k > 1$ is a sequence A267816 in OEIS with $a_1 = 1$:
$a_1 = 1 = (1)_2 \quad a_2 = 3 = (11)_2 \quad a_3 = 23 = (10111)_2$
$a_4 = 111 = (1101111)_2 \quad a_5 = 479 = (111011111)_2 \ldots$
These integers a_k are exactly the *decimal representation of the n-th iteration of the "Rule 221" elementary cellular automaton starting with a single ON (black) cell*.

Complexity

At the present moment we consider the Neville's algorithm as the most convenient tool for evaluation of the $\mathrm{Denominator}\left\langle \binom{n}{n-1} \right\rangle$. The complexity of this algorithm can be estimated as $\mathrm{O}(n^2)$ (See [5]).

References I

[1] Y. V. Matiyasevich, “The Riemann hypothesis as the parity of special binomial coefficients”, *Chebyshevskii sbornik.*, vol. 19, no. 3, pp. 46–60, 2018.

[2] N. Gogin and M. Hirvensalo, “On the moments of squared binomial coefficients”,, ser. Polynomial Computer Algebra, Euler International Mathematical Institute, 2020. [Online]. Available: https://pca-pdmi.ru/2020/files/10/GoHi2020ExtAbstract.pdf.

[3] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. North-Holland, 1977.

[4] S. Beresin and N. P. Jhidkov, *Computing Methods*. New York: Pergamon Press, 1973.

References II

[5] E. W. Weisstein, *Neville's algorithm*, [Online]. Available: https://mathworld.wolfram.com/NevillesAlgorithm.html.