On computer experiments with reversible difference schemes in Sage

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Abstract. The results of computer experiments with reversible difference schemes approximating differential equations with a quadratic right side are presented, they made in the routine called fdm for sage. The generalization of the theory of reversible difference schemes for dynamical systems with a polynomial right part was discussed. It is proposed to use for this the quadratization of the system of differential equations with polynomial part according to Appelroth. The results of computer experiments with such schemes are presented.

1. Reversible schemes

Consider an autonomous system of di erential equations

$$\frac{dx_1}{dt} = f_1(x_1, \dots, x_n), \quad \frac{dx_n}{dt} = f_n(x_1, \dots, x_n). \tag{1}$$

Cremona transformations are a very interesting algebraic object. They were discovered relatively recently, in the middle of the 19th century. After the first successes in studying their properties, undertaken by Cremona, Netter, Rosanes and Mlodsijewski, there was a long pause due to the unexpected breadth of the question. Now this vastness is clear — any dynamical system with a quadratic right-hand side is described using the Cremona transformation.

The algorithm for constructing a reversible scheme, presented at PCA'2021 and described in [1], is implemented in our fdm for sage system in the Sage computer algebra system [2]. In this system, the initial problem is specified separately from the method for solving it. Its solution, say, according to the explicit Euler scheme is specified as **`erk(pr)`**, and according to the reversible scheme as **`cremona_scheme(pr)`**. Both functions return an element belonging to the **`Numsol`** class, so you can work with the new scheme in this system in the same way as with Runge-Kutta schemes. Of course, the externally obtained solutions using these schemes di er significantly (see Fig. 1). Using this implementation, several computer experiments were carried out, during which very unexpected properties of these schemes were found. Later, it turned out to be possible to substantiate some of them.

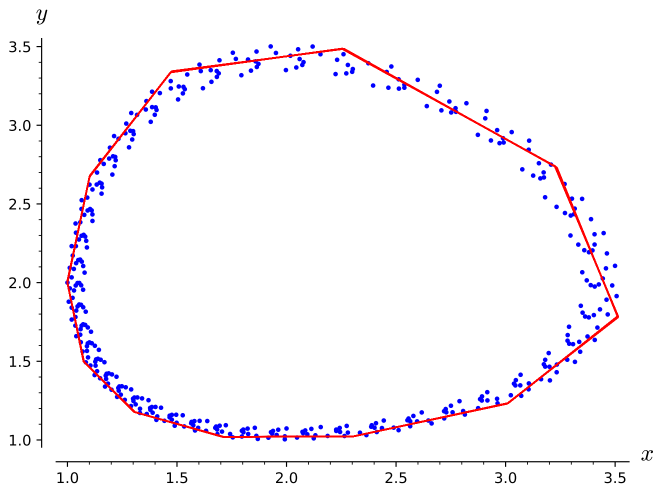

FIGURE 1. Solution of the initial value problem for the Volterra–Lotka system of equations: dots — the solution found by the Runge–Kutta scheme, polygon — the solution found by the reversible scheme.

2. Quadratization of dynamical systems

Transferring the developed technique to the case of equations with a polynomial right-hand side does not cause significant di culties, since back at the beginning of the 20th century G.G. Appelroth proposed a method that allows, by increasing the number of unknowns, to reduce a system with a polynomial right-hand side to a system with a quadratic right-hand side. This procedure was later called quadratization; Ref. [6] describes the software that allows performing such quadratization of any system with a quadratic right-hand side. This makes it possible to construct a reversible di erence scheme for any dynamical system with a polynomial right-hand side.

Computer experiments have shown that the relationships between new and old variables, which are valid for the exact solution, are no longer valid for the approximate solution, which is especially noticeable near moving singular points of the solution. The appearance of moving algebraic singular points is typical for nonlinear systems. In the case of poles, the solution found using a reversible scheme passes through the pole without distortion and, after the pole, fits perfectly on the integral curve [1]. However, in the case of an algebraic singular point, the situation inevitably changes: there may not be a real integral curve behind the singular point. However, the approximate solution continues beyond such singular points while remaining real. In the future, we plan to combine software for squared di erential equations and our system for numerical integration of di erential equations [2].

3. Properties of reversible circuits

Classical nonlinear oscillators integrable in elliptic functions, are dynamic systems with quadratic right-hand side; a top fixed in its center of gravity is an example. In this case, the new discrete theory completely repeats the continuous theory: i)the points of the approximate solution lie on a certain elliptic curve, which at $\Delta t \to 0$ transforms into an integral curve [3]; ii) the di erence scheme allows a quadrature representation [4]; iii) the approximate solution can be presented by means of an elliptic function of discrete argument [4]. All the di erence reduces to the fact that the place of birational transformations on an integral curve is occupied by the Cremona transformations of the entire three-dimensional space of velocities.

In the case of nonlinear oscillators nonintegrable in elliptic functions, e.g., in the case of the Volterra–Lotka system, the points are arranged along some closed curve, which, however, is not algebraic. In contrast to the case of elliptic oscillators, here it is impossible to choose the step Δt such that the points of the trajectory would form a periodic sequence. In particular, the polygon shown in Fig. 1, actually changes with time, but very slowly.

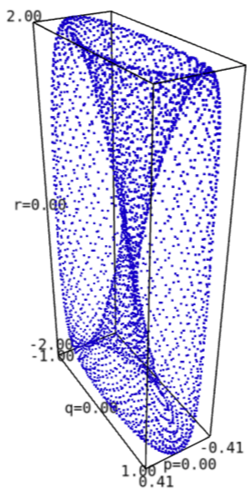

FIGURE 2. Typical solution of the initial-value problem for an asymmetric top.

More complex conservative systems, e.g., the system describing the motion of an asymmetric top, are also described by a system of the form (1) with the quadratic right-hand side. Computer experiments show that the points of one trajectory in this case fill everywhere densely some surfaces, see Fig. 2.

In the case of dissipative systems, the trajectory already cannot fill densely the entire surface. However, here it is of interest that by choosing Δt it turns out to be possible to transform complex limit structures into multiple loops.

We believe that the development of methods for studying trajectories obtained using reversible schemes will make it possible to look at non-integrable systems from a new angle. The advantage of this point of view is that it is always possible to calculate arbitrarily many solution points using a reversible scheme, and therefore to see the structures into which the solution points are arranged in phase space.

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