Analytic solving any equation of polynomial type on variables and derivatives

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Abstract. A calculus [1] has been developed which allows one to calculate analytically asymptotic expansions of solutions to equations which are polynomials on variables and their derivatives, as well as to systems of such equations. This calculus is applied to equations of any type: algebraic [2, 3], ordinary differential [4] and partial differential [5], as well as to their systems. The calculus is based on algorithms of power geometry: (a) selection of truncated equations consisting of all leading terms, as well as (b) power transformations, (c) logarithmic and (d) normalizing coordinate transformations. The required software for this calculus has already been developed.

1. Introduction

For a single equation, the sequence of calculations is as follows:

I. First, the truncated equations are selected and the regions where they are first approximations of the original equation are specified.

II. Each truncated equation is then simplified using power and logarithmic coordinate transformations, possibly repeatedly, to an equation that has a simple solution.

III. This is augmented to the solution of the truncated equation.

IV. If its perturbation in the full equation has a linear part, we obtain the solution of the original equation by the normalizing transformation.

V. If this perturbation does not have a linear part, we repeat this process for it, i.e., we again separate the truncated equations and simplify them until we come to situation IV, i.e., to a perturbation with a linear part, for which we find a solution.

The methods of applying this calculus to equations of different types are described below.

The article [1] outlines the objects and sequences of calculus for:

1. One algebraic equation.
2. One ordinary differential equation (ODE) of order n.
3. An autonomous system of n ODEs.
4. One partial differential equation.

A brief review of applications is also given there.

Here we give algorithms of nonlinear analysis for cases of one equation and discuss levels of power geometry.

2. Levels of Power Geometry

Everything that has been told in [1] refers to the zero level of power geometry, for there it has been «sealed» that

$$\operatorname{ord} y' = \operatorname{ord} y - 1. \tag{1}$$

But this is not always the situation. By rejecting this property, we get a wider set of solutions. Let's discuss this in more detail.

In the algebraic equation

$$f(X) \stackrel{\text{def}}{=} \sum a_Q X^Q = 0,$$

with $X \in \mathbb{R}^n$ or $\mathbb{C}^n$ to each monomial $a_Q X^Q$ can be assigned a point $\check{Q} = \{Q, \ln|a_Q|\}$ in $\mathbb{R}^{n+1}$. Their set forms the *supersupport* $\check{S}(f)$, and its convex hull $\check{H}(f)$ is the *Adamar polyhedron* [6].

We build truncated equations on its faces. They are simpler than the truncated equations corresponding to the faces of Newton's polyhedron, and allow us to study cases where Newton's polyhedron fails.

For a single ODE, one can search for solutions that have $\operatorname{ord} y - \operatorname{ord} y' \neq 1$ by introducing a new coordinate for the order of the derivative y'. This was done in [7] and allowed us to obtain solutions in the form of power expansions whose coefficients are trigonometric or elliptic functions.

We can consider solutions where $\operatorname{ord} y^{(k)} - \operatorname{ord} y^{(k+1)}$ is arbitrary, or several such differences are arbitrary, and obtain new types of solutions. For details see [8, 9, 10].

A similar thing could be done with partial differential equations, but it has not been done yet.

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