Analytic solving any equation of polynomial type on variables and derivatives

Alexander D. Bruno
e-mail: abruno@keldysh.ru
http://brunoa.name

Keldysh Institute of Applied Mathematics of RAS

Abstract

A calculus [Bruno, 2023] has been developed which allows one to calculate analytically asymptotic expansions of solutions to equations which are polynomials on variables and their derivatives, as well as to systems of such equations. This calculus is applied to equations of any type: algebraic [Bruno, Azimov, 2023b; 2024], ordinary differential [Bruno, 2004] and partial differential [Bruno, Batkhin, 2023], as well as to their systems. The calculus is based on algorithms of power geometry: (a) selection of truncated equations consisting of all leading terms, as well as (b) power transformations, (c) logarithmic and (d) normalizing coordinate transformations. The required software for this calculus has already been developed.

Talk outline

1. Introduction
2. One algebraic equation
3. One ordinary differential equation
4. One partial differential equation
5. Levels of Power Geometry
6. Bibliography

For a single equation, the sequence of calculations is as follows:

- **A.** First, the truncated equations are selected and the domains where they are first approximations of the original equation are specified.
- **B.** Each truncated equation is then simplified using power transformations and logarithmic coordinate transformations, possibly repeatedly, to an equation that has a simple solution.
- **C.** It is augmented to the solution of the truncated equation.
- **D.** If its perturbation in the full equation has a linear part, then by normalizing transformation we obtain the solution of the original equation.
- **E.** If this perturbation does not have a linear part, we repeat this process for it, i.e., we again select the shortened equations and simplify them until we come to situation D, i.e., to a perturbation with a linear part, for which we find a solution.

The methods of applying this calculus to equations of different types are described below.

The article [Bruno, 2023] outlines the objects and sequences of calculus for:

1. One algebraic equation.
2. One ordinary differential equation (ODE) of order n.
3. An autonomous system of n ODEs.
4. One partial differential equation.

A brief survey of applications is also given there.

Here we give algorithms for nonlinear analysis for cases of a single equation and discuss levels of power geometry.

2.1. One AE. The implicit function theorem (1)

Let $X = (x_1, \ldots, x_n)$, $Q = (q_1, \ldots, q_n)$, then

$$X^Q = x_1^{q_1} \cdots x_n^{q_n}, \quad \|Q\| = |q_1| + \cdots + |q_n|.$$

Theorem 2.1.

Let

$$f(X, \varepsilon, T) = \Sigma a_{Q,r}(T) X^Q \varepsilon^r, \tag{2.1}$$

where $0 \leqslant Q \in \mathbb{Z}^n$, $0 \leqslant r \in \mathbb{Z}$, the sum is finite and $a_{Q,r}(T)$ are some functions of $T = (t_1, \ldots, t_m)$, besides $a_{00}(T) \equiv 0$, $a_{01}(T) \not\equiv 0$. Then the solution to the equation $f(X, \varepsilon, T) = 0$ has the form

$$\varepsilon = \Sigma b_R(T) X^R, \tag{2.2}$$

where $0 \leqslant R \in \mathbb{Z}^n$, $0 < \|R\|$, the coefficients $b_R(T)$ are functions on T that are polynomials from $a_{Q,r}(T)$ with $\|Q\| + r \leqslant \|R\|$ divided by $a_{01}^{2\|R\|-1}$. The expansion (2.2) is unique.

2.1. One AE. The implicit function theorem (2)

After substitution $\varepsilon = \delta + \sum b_R(T)X^R$ the equation $f = 0$ takes the form

$$\delta g(X, \delta, T) = 0. \quad (2.3)$$

So, Theorem 2.1 is on reducing the algebraic equation (2.1) to its normal form (2.3) when the linear part $a_{01}(T) \not\equiv 0$ is nondegenerate. In it, we must exclude the values of T near the zeros of the function $a_{01}(T)$.

Let $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, and $f(X)$ be a polynomial. A point $X = X^0$, $f(X^0) = 0$ is called ***simple*** if the vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ in it is non-zero. Otherwise, the point $X = X^0$ is called ***singular*** or ***critical***. By shifting $X = X^0 + Y$ move the point X^0 to the origin $Y = 0$. If at this point the derivative $\partial f/\partial x_n \neq 0$, then near X^0 all solutions to the equation $f(X) = 0$ have the form $y_n = \Sigma b_{q_1,\ldots q_{n-1}} y_1^{q_1} \cdots y_{n-1}^{q_{n-1}}$, that is, lie in $(n-1)$-dimensional space.

2.2. One AE. Newton's polyhedron (1)

Let the point $X^0 = 0$ be singular. Write the polynomial in the form

$$f(X) = \Sigma a_Q X^Q,$$

where $a_Q = \mathrm{const} \in \mathbb{R}$, or $\mathbb{C}$. Let $\mathbf{S}(f) = \{Q : a_Q \neq 0\}$.

The set $\mathbf{S}$ is called the ***support*** of the polynomial $f(X)$. Let it consist of points $Q_1, \dots, Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(f) = \left\{ Q = \sum_{j=1}^{k} \mu_j Q_j, \mu_j \geqslant 0, \sum_{j=1}^{k} \mu_j = 1 \right\},$$

which is called ***Newton's polyhedron***.

Its boundary $\partial\Gamma(f)$ consists of generalized faces $\Gamma_j^{(d)}$, where d is its dimension, $0 \leqslant d \leqslant n-1$ and j is the number.

2.2. One AE. Newton's polyhedron (2)

Each (generalized) face $\Gamma_j^{(d)}$ corresponds to its:

- ***boundary subset***

$$\mathbf{S}_j^{(d)} = \mathbf{S} \cap \Gamma_j^{(d)},$$

- ***truncated polynomial***

$$\hat{f}_j^{(d)}(X) = \Sigma a_Q X^Q \text{ over } Q \in \mathbf{S}_j^{(d)}, \text{ and}$$

- ***normal cone***

$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle, Q', Q'' \in \mathbf{S}_j^{(d)}, Q''' \in \mathbf{S} \backslash \mathbf{S}_j^{(d)} \right\},$$

where $P = (p_1, \dots, p_n) \in \mathbb{R}_*^n$, the space $\mathbb{R}_*^n$ is conjugate (dual) to the space $\mathbb{R}^n$ and $\langle P, Q \rangle = p_1 q_1 + \dots + p_n q_n$ is the scalar product. Normal cone $\mathbf{U}_j^{(d)}$ consists of all the external to the polyhedron Γ normals to the face $\Gamma_j^{(d)}$.

2.2. One AE. Newton's polyhedron (3)

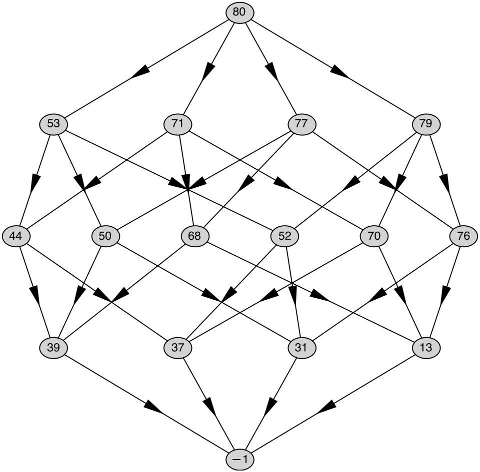
At $X \to 0$ solutions to the full equation $f(X) = 0$ tend to non-trivial solutions of those truncated equations $\hat{f}_j^{(d)}(X) = 0$ whose normal cone $\mathbf{U}_j^{(d)}$ intersects with the negative orthant $P \leqslant 0$ in $\mathbb{R}_*^n$.

If the vector $(\ln x_1, \ldots, \ln x_n)$ tends to infinity along the normal cone $\mathbf{U}_j^{(d)}$, then the truncated polynomial $\hat{f}_j^{(d)}(X)$ is asymptotically the main part of the polynomial $f(X)$.

In fact, we have to work not with a polyhedron Γ, but with its graph.

Example

The polynomial $x_1 + 2x_1x_2 + 3x_1x_2x_3 + 4x_3^3$ has Newton's polyhedron with vertices: $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, $(0,0,3)$. Its graph is as follows:



2.2. One AE. Newton's polyhedron (5)

The top row of the graph corresponds to the entire polyhedron. The next top row contains the hyperfaces, then the faces of dimension $(n-2)$, etc. The last line contains the empty set. The arrows indicate the inclusion of faces.

2.3. One AE. Power transformations (1)

Let $\ln X \stackrel{\text{def}}{=} (\ln x_1, \dots, \ln x_n)$. The linear transformation of the logarithms of the coordinates

$$(\ln y_1, \dots, \ln y_n) \stackrel{\text{def}}{=} \ln Y = (\ln X)\alpha, \tag{2.4}$$

where α is a nondegenerate square n-matrix, is called ***power transformation***.

By the power transformation (2.4), the monomial X^Q tranforms into the monomial Y^R, where $R = Q\alpha^{*-1}$ and the asterisk indicates a transposition.

A matrix α is called ***unimodular*** if all its elements are integers and $\det \alpha = \pm 1$. For an unimodular matrix α, its inverse α^{-1} and transpose α^* are also unimodular.

2.3. One AE. Power transformations (2)

Theorem 2.2.

For the face $\Gamma_j^{(d)}$ *there exists a power transformation* (2.4) *with the unimodular matrix* α *which reduces the truncated sum* $\hat{f}_j^{(d)}(X)$ *to the sum from* d *coordinates, that is,* $\hat{f}_j^{(d)}(X) = Y^S \hat{g}_j^{(d)}(Y)$ *where* $\hat{g}_j^{(d)}(Y) = \hat{g}_j^{(d)}(y_1, \dots, y_d)$ *is a polynomial. Here* $S \in \mathbb{Z}^n$*. The additional coordinates* $y_{d+1}, \dots, y_n$ *are local (small).*

The paper [Bruno, Azimov, 2023a] specifies an algorithm for computing the unimodular matrix α of Theorem 2.2.

2.4. One AE. Parametric expansion of solutions (1)

Let $\Gamma_j^{(d)}$ be a face of the Newton polyhedron $\Gamma(f)$. Let the full equation $f(X) = 0$ is changed into the equation $g(Y) = 0$ after the power transformation of Theorem 2.2. Thus $\hat{g}_j^{(d)}(y_1, \ldots, y_d) = g(y_1, \ldots, y_d, 0, \ldots, 0)$.

Let the polynomial $\hat{g}_j^{(d)}$ be the product of several irreducible polynomials

$$\hat{g}_j^{(d)} = \prod_{k=1}^{m} h_k^{l_k}(y_1, \ldots, y_d), \tag{2.5}$$

where $0 < l_k \in \mathbb{Z}$. Let the polynomial h_k be one of them. Three cases are possible:

2.4. One AE. Parametric expansion of solutions (2)

Case 1.

The equation $h_k = 0$ has a polynomial solution $y_d = \varphi(y_1, \dots, y_{d-1})$. Then in the full polynomial $g(Y)$ let us substitute the coordinates

$$y_d = \varphi + z_d,$$

for the resulting polynomial $\tilde{h}(y_1, \dots, y_{d-1}, z_d, y_{d+1} \dots, y_n)$ again construct the Newton polyhedron, separate the truncated polynomials, etc. Such calculations were made in [Bruno, Batkhin, 2012] and were shown in [Bruno, 2000, Introduction].

Case 2

The equation $h_k = 0$ has no polynomial solution, but has a parametrization of solutions

$$y_j = \varphi_j(T), j = 1, \dots, d, \quad T = (t_1, \dots, t_{d-1}).$$

2.4. One AE. Parametric expansion of solutions (3)

Then in the full polynomial $g(Y)$ we substitute the coordinates

$$y_j = \varphi_j(T) + \beta_j \varepsilon, j = 1, \ldots, d, \tag{2.6}$$

where $\beta_j = \text{const}$, $\Sigma\,|\beta_j| \neq 0$, and from the full polynomial $g(Y)$ we get the polynomial

$$\tilde{h} = \Sigma a_{Q'',r}(T) {Y''}^{Q''} \varepsilon^r, \tag{2.7}$$

where $Y'' = (y_{d+1}, \ldots, y_n)$, $0 \leqslant Q'' = (q_{d+1}, \ldots, q_n) \in \mathbb{Z}^{n-d}$, $0 \leqslant r \in \mathbb{Z}$. Thus $a_{00}(T) \equiv 0$, $a_{01}(T) = \sum\limits_{j=1}^{d} \beta_j \partial \hat{g}_j^{(d)} / \partial y_j(T)$.

2.4. One AE. Parametric expansion of solutions (4)

If in the expansion (2.5) $l_k = 1$, then $a_{01} \not\equiv 0$. By Theorem 2.1, solutions to the equation $\tilde{h} = 0$ have the form

$$\varepsilon = \Sigma b_{Q''}(T) Y''^{Q''},$$

i.e., according to (2.6) the solutions to the equation $g = 0$ have the form

$$y_j = \varphi_j(T) + \beta_j \Sigma b_{Q''}(T) Y''^{Q''}, j = 1, \dots, d.$$

Such calculations were proposed in [Bruno, 2018].

If in (2.5) $l_k > 1$, then in (2.7) $a_{01}(T) \equiv 0$ and for the polynomial (2.7) from Y'', ε we construct a Newton polyhedron by support $\mathbf{S}(h) = \{Q'', r : a_{Q'',r}(T) \not\equiv 0\}$, separate the truncations and so on.

2.4. One AE. Parametric expansion of solutions (5)

Case 3

The equation $h_k = 0$ has neither a polynomial solution nor a parametric one. Then, using Hadamard's polyhedron [Bruno, 2018; 2019b], one can compute a piecewise approximate parametric solution to the equation $h_k = 0$ and look for an approximate parametric expansion.

Similarly, one can study the position of an algebraic manifold in infinity.

2.4. One AE. Parametric expansion of solutions (6)

Most applications of Nonlinear Analysis are related to ordinary differential equations, such as:

- a single non-autonomous ODE of any finite order and a system of such equations;
- a system of autonomous ODEs;
- a Hamiltonian system.

Each of them has its own specific features.

2.5. Software for convex hull and normal cones computation (1)

Various software is available for working with convex sets. Here we briefly describe only those programs that can be used both to compute convex hulls and to compute their normal cones. The references to described software see in [Bruno, Batkhin, 2021].

Let's consider the following software

- ***Qhull*** [Barber, Dobkin, Huhdanpaa, 1996];
- ***The PolyhedralSets*** for CAS `Maple` [Thompson, 2016];
- packages for Sage [The Sage Developers, 2022].

2.5. Software for convex hull and normal cones computation (2)

Qhull

The ***Qhull*** package is used in many application software packages, both commercial and free, the ***Qhull*** package has a software interface with the `Matlab` scientific calculation system, `GNU Octave`, computer algebra systems `Mathematica` and `Maple`, libraries `SciPy` and `geometry` for programming languages `Python` and `R` respectively.

The main feature of the package is that the calculations are performed using real numbers rather than in the field of rational numbers, which is convenient when working with the Hadamard polyhedron. When calculating the Newton polyhedron, additional steps are required to bring the results of the calculations to rational values.

2.5. Software for convex hull and normal cones computation (3)

PolyhedralSets

Since the 2015 version, the `Maple` computer algebra system includes the ***PolyhedralSets*** package. It allows, in particular, to compute the convex hull of a set, to give its H- or V-representations, i.e., either as equations of hyperplanes of the boundary, or as a set of extreme points and rays, the linear combination of which gives an unbounded convex hull.

In this package all calculations are performed in the field of rational numbers, which somewhat simplifies its use for the study of the Newton polyhedron, but makes it useless when working with the Hadamard polyhedron. Note that ***PolyhedralSets*** has extremely low performance compared to ***Qhull***.

A large selection of libraries for research in computational geometry is provided by the freeware computer algebra system `Sage`. It allows you to work with libraries: `cdd`, `PPL` (Parma Polyhedral Library), `polymake`, `Qhull` and others.

2.6. Software for plane curve investigation (1)

The CAS `Maple` system has a package ***algcurves*** that allows to study planar algebraic curves: build their sketches with high precision, calculate their genus, find singular points, for curves of genus 0 find rational parameterization and construct expansion of the curve in the form of Puiseux series. The package allows to construct a sketch of the curve $f(x, y) = 0$ by numerical integration of the differential equation $f'_x + f'_y y' = 0$ for some set of initial conditions defined by points in which at least one of the partial derivatives of the function $f(x, y)$ is equal to zero. Using this package to investigate a set of curves with different orders of peculiarities has shown that in the case of high order peculiarities, the quality of the sketch is not very high.

Another Maple package named ***MultiSeries*** performs asymptotic and series expansions in general asymptotic scales.

2.6. Software for plane curve investigation (2)

Since version 12, `Wolfram Mathematica` has included an ***AsymptoticSolve*** procedure that implements an asymptotic representation of solutions to equations or systems of equations (not necessarily algebraic) in the form of either Taylor, Laurent, or Puiseux series near finite or infinite points. If the point is singular, the procedure tries to calculate the asymptotic expansions of all branches. In this case we can specify that we should restrict ourselves to real expansions only.

No one of these packages can provide asymptotic expansion in the case when a singular point is defined by algebraic numbers.

Remark

For investigation algebraic variety in three variables near its singular points and infinity a `Maple` library ***PGlib*** was implemented [Bruno, Batkhin, 2012]. This library essentially uses ***Qhull*** package for convex hull computation and ***Groebner*** package of CAS `Maple`.

3. One ODE. Problem statement (1)

. Here we consider an ordinary differential equation of the form

$$f\left(x, y, y', \dots, y^{(n)}\right) = 0, \tag{3.1}$$

where x is the independent variable, y is the dependent variable, $y' = dy/dx$, and f is the polynomial of the arguments.

Near $x^0 = 0$ or ∞, we look for solutions to equation (3.1) in the form of an asymptotic series

$$y = \sum_{k=1}^{\infty} b_k x^{s_k}, \tag{3.2}$$

where b_k are functions of $\log x$ and $\omega s_k > \omega s_{k+1}$ with

$$\omega = \begin{cases} -1, & \text{if } x^0 = 0, \\ 1, & \text{if } x^0 = \infty. \end{cases} \tag{3.3}$$

3. One ODE. Problem statement (2)

Let $X = (x, y)$. By the differential monomial $a(x, y)$ we mean the product of the ordinary monomial

$$cx^{r_1}y^{r_2} \stackrel{\text{def}}{=} cX^R, \tag{3.4}$$

and a finite number of derivatives $d^l y/dx^l$. Their sum is called the ***differential sum***. In the equation (3.1), the polynomial f is a differential sum.

To each differential monomial $a(X)$ corresponds its (vector) ***power exponent*** $Q(a) = (q_1, q_2) \in \mathbb{R}^2$ by the following rules:

- for a monomial of the form (3.4) $Q\left(cX^R\right) = R$, that is, $Q\left(cx^{r_1}y^{r_2}\right) = (r_1, r_2)$;
- for the derivative $Q\left(d^l y/dx^l\right) = (-l, 1)$;
- when differential monomials are multiplied, their exponents are summed as vectors: $Q\left(a_1 a_2\right) = Q\left(a_1\right) + Q\left(a_2\right)$.

- The set $\mathbf{S}(f)$ of indices $Q\left(a_i\right)$ of all differential monomials $a_i(X)$ in a differential sum is called the ***bearer*** of the sum $f\left(X\right)$. Clearly, $\mathbf{S}(f) \in \mathbb{R}^2$.

3. One ODE. Problem statement (3)

- The convex hull of the $\Gamma(f)$ carrier $\mathbf{S}(f)$ is called the ***polygon*** of the sum $f(X)$.
- The boundary of a $\partial\Gamma(f)$ polygon $\Gamma(f)$ consists of the vertices $\Gamma_j^{(0)}$ and the edges $\Gamma_j^{(1)}$. These objects are called (generalized) ***faces*** $\Gamma_j^{(d)}$, where the upper index indicates the dimension of the face and the lower index indicates its number.
- Each face of $\Gamma_j^{(d)}$ has a ***boundary subset*** of $\mathbf{S}_j^{(d)} = \mathbf{S}(f) \cap \Gamma_j^{(d)}$ of the set $\mathbf{S}$ and ***truncated sum***

$$\hat{f}_j^{(d)}(X) = \sum a_i(X) \quad \text{by} \quad Q\left(a_i\right) \in \mathbf{S}_j^{(d)}. \tag{3.5}$$

3. One ODE. Problem statement (4)

Let $\mathbb{R}^2_*$ be a plane conjugate to the plane $\mathbb{R}^2$ such that the inner (scalar) product of $\langle P, Q\rangle \stackrel{\text{def}}{=} p_1 q_1 + p_2 q_2$ is defined for any $P = (p_1, p_2) \in \mathbb{R}^2_*$ and $Q = (q_1, q_2) \in \mathbb{R}^2$. Any face of $\Gamma_j^{(d)}$ corresponds to its ***normal cone***

$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q\rangle = \left\langle P, Q'\right\rangle, \langle P, Q\rangle > \left\langle P, Q'\right\rangle, Q, Q'. \in \mathbf{S}_j^{(d)}, Q'' \in \mathbf{S}(f) \backslash \mathbf{S}_j^{(d)} \right\}$$

and the shortened sum (3.5). All these constructions apply to the equation (3.1), where f is the differential sum.

Let $x \to 0$ or $x \to \infty$ and assume that the solution to equation (3.1) has the form

$$y = c_r x^r + o\left(|x|^{r+\varepsilon}\right), \tag{3.6}$$

where c_r is a coefficient, $c_r = \text{const} \in \mathbb{C}$, $c_r \neq 0$, the exponents r and ε are in $\mathbb{R}$, and $\varepsilon\omega < 0$.

3. One ODE. Problem statement (5)

Then we say that the expression

$$y = c_r x^r, \quad c_r \neq 0. \tag{3.7}$$

gives the ***power asymptotics*** of the solution (3.6). Thus, any edge $\Gamma_j^{(d)}$ corresponds to a normal cone $\mathbf{U}_j^{(d)}$ in $\mathbb{R}_*^2$ and the truncated equation

$$\hat{f}_j^{(d)}(X) = 0. \tag{3.8}$$

Theorem 3.1.

If the equation $f\left(x, y, y', \dots, y^{(n)}\right) = 0$ has a solution of the form $y = c_r x^r + o\left(|x|^{r+\varepsilon}\right)$ and if $\omega(1, r) \in \mathbf{U}_j^{(d)}$, then the truncation $y = c_r x^r, \quad c_r \neq 0$ of this solution is a solution to the truncated equation $\hat{f}_j^{(d)}(X) = 0$.

3. One ODE. Solution of the truncated equation (1)

Here we consider two cases separately:

- vertex $\Gamma_j^{(0)}$ and
- edge $\Gamma_j^{(1)}$.

The vertex $\Gamma_j^{(0)} = \{Q\}$ corresponds to the truncated equation (3.8) with a one-point support Q and with $d = 0$. Let $g(X) \stackrel{\mathrm{def}}{=} X^{-Q} \hat{f}_j^{(0)}(X)$. Then the solution (3.5), (3.8) satisfies the equation $g(X) = 0$.

3. One ODE. Solution of the truncated equation (2)

Substituting $y = cx^r$ into $g(X)$, we see that $g\left(x, cx^r\right)$ is independent of x, c and is a polynomial of r, that is, $g\left(x, cx^r\right) \equiv \chi(r)$, where $\chi(r)$ is the ***characteristic polynomial*** of the differential sum $\hat{f}_j^{(0)}(X)$. Hence, in the solution (3.7) of equation (3.8), the exponent r is the root of the characteristic equation

$$\chi(r) \stackrel{\text{def}}{=} g\left(x, x^r\right) = 0, \tag{3.9}$$

and the coefficient c_r is arbitrary.

Among the roots r_i of the equation (3.9), we need to select only those for which one of the vectors $\omega(1, r)$, where $\omega = \pm 1$, belongs to the normal cone $\mathbf{U}_j^{(0)}$ of the vertex $\Gamma_j^{(0)}$. In this case, the value of ω is uniquely determined. The corresponding sum expressions with arbitrary constant c_r are candidates for truncated solutions to the equation (3.1).

3. One ODE. Solution of the truncated equation (3)

Moreover, by (3.3), if $\omega = -1$, then $x \to 0$, and if $\omega = 1$, then $x \to \infty$. Complex roots r of the characteristic equation (3.9) can lead to ***exotic expansions*** of solutions (3.2) where the coefficients b_k are power series on $x^{\alpha i}$ with real $\alpha \in \mathbb{R}$ and $i^2 = -1$.

The edge $\Gamma_j^{(1)}$ corresponds to the truncated equation (3.8) with $d = 1$, whose normal cone $\mathbf{U}_j^{(1)}$ is the ray $\{\lambda N_j, \lambda > 0\}$. If $\omega(1, r) \in \mathbf{U}_j^{(1)}$, then this condition uniquely determines the exponent r of the truncated solution (3.7) and the value of $\omega = \pm 1$ in (3.3).

To find the coefficient of c_r, we need to substitute the expression (3.7) into the truncated equation (3.8). After reduction by degree x, we get the algebraic ***defining equation*** for the coefficient c_r : $\tilde{\tilde{f}}(c_r) \stackrel{\text{def}}{=} x^{-s} \hat{f}_j^{(1)}(x, c_r x^r) = 0$.

3. One ODE. Solution of the truncated equation (4)

Each root $c_r = c_r^{(i)} \neq 0$ of this equation corresponds to an expression of the form (3.7), which is a candidate for being a truncated solution of the equation (3.1). Moreover, by (3.3), if $p_1 < 0$ in the normal cone $\mathbf{U}_j^{(1)}$, then $x \to 0$, and if $p_1 > 0$, then $x \to \infty$.

If in the truncated equation (3.8), we make a ***power transformation*** $y = x^p z$ and a ***logarithmic transformation*** $\xi = \log x$, then we get an ODE

$$\varphi(\xi, z) = 0, \tag{3.10}$$

where φ is the differential sum, i.e., has the form (3.1).

3. One ODE. Solution of the truncated equation (5)

If Equation (3.10) has a solution in the form $z = \sum_{j=1}^{\infty} c_j \xi^{r_j}, \quad r_j > r_{j+1}$, then in the expansion (3.2) the coefficients b_k are functions of $\log x$. If $b_1 = \mathrm{const}$, then it is a ***power-logarithmic expansion*** where the remaining b_k are polynomials of $\log x$. If b_1 depends on $\log x$, then all b_k are power series on $\log x$ and the expansion (3.2) is ***complicated***.

3. One ODE. Solving the equation (3.1) as an expansion of (3.2) (1)

From the polygon Γ of the original equation (3.1), we take the vertex or edge $\Gamma_j^{(d)}$. Then we find the degree solution $y = b_1 x^{p_1}$ of the truncated equation $\hat{f}_j^{(d)}(X) = 0$ as described above, put $y = b_1 x^{p_1} + z$ and get the new equation $g(x, z) = 0$.

Let's construct a polygon ${}_1\Gamma$ for the new equation, take a vertex or edge ${}_1\Gamma_k^{(\mathrm{e})}$, solve the truncated equation $\hat{g}_k^{(\mathrm{e})}(x, z) = 0$, and get the second term $b_2 x^{p_2}$ of the expansion (3.2), and so on. In [Bruno, 2004] some properties that simplify the calculations are given.

3. One ODE. Solving the equation (3.1) as an expansion of (3.2) (2)

After obtaining the solution $y = Cx^\alpha$ of the truncated equation, we substitute $y = Cx^\alpha + z$ into the full equation $f(x, y) = 0$, obtaining the equation $g(x, z) \stackrel{\text{def}}{=} f(x, Cx^\alpha + z) = 0$. If it does not have a linear term on y, we continue the calculations until we obtain an equation with a linear part on y. We reduce it to normal form by Theorem 3.2 (below).

Condition 1

The point $(v, 1)$ is a vertex of the polygon $\Gamma(f)$. In the sum $f(x, y)$, it corresponds to the summand $\mathcal{L}(x)y$ and only to it. Here $\mathcal{L}(x)$ is a linear operator without logarithms.

3. One ODE. Solving the equation (3.1) as an expansion of (3.2) (3)

Theorem 3.2.

If equation (3.1) *satisfies Condition 1, then it has a formal solution*

$$y = \sum_{k=1}^{\infty} \beta_k(\ln x) x^{s_k},$$

where $\beta_k(\ln x)$ *are polynomials from* $\ln x$, $\omega s_k > \omega s_{k+1}$, $|s_k| \to \infty$ *at* $k \to \infty$.

3. One ODE. Solving the equation (3.1) as an expansion of (3.2) (4)

Thus, we get 2 kinds of expansions (3.2) of solutions to the equation (3.1):

1. ***Power***: when all $b_k = \text{const}$ [Bruno, 2004];
2. ***Power-logarithmic***: when $b_1 = \text{const}$ and the remaining b_k is polynomial in $\log x$ [Bruno, 2004].

Other types of expansion of solutions:

3. ***Complicated***: when all b_k are power series over $\log x$ [Bruno, 2006];
4. ***Exotic***: when all b_k are power series on $x^{i\alpha}$ [Bruno, 2007].

In addition to these, there are ***exponential expansions***

$$y = \sum_{k=1}^{\infty} b_k(x) \exp\left[k\varphi(x)\right],$$

where $b_k(x)$ and $\varphi(x)$ are power series on x [Bruno, 2012c] and many others. There are also solutions in the form of transseries [Bruno, 2019a].

Let $X = (x_1, \ldots, x_n) \in \mathbb{C}^n$ or $\mathbb{R}^n$ be independent variables and $y \in \mathbb{C}$ or $\mathbb{R}$ be a dependent one. Consider $Z = (z_1, \ldots, z_n, z_{n+1}) = (x_1, \ldots, x_n, y)$.

Differential monomial $a(Z)$ is called a product of an ordinary monomial $cZ^R = cz_1^{r_1} \cdots z_{n+1}^{r_{n+1}}$, where $c = \mathrm{const}$, and a finite number of derivatives of the following form

$$\frac{\partial^l y}{\partial x_1^{l_1} \cdots \partial^{l_n} x_n} \stackrel{\mathrm{def}}{=} \frac{\partial^l y}{\partial X^L},\ 0 \leqslant l_j \in \mathbb{Z}, \sum_{j=1}^{n} l_j = l,\ L = (l_1, \ldots, l_n)\,.$$

Vector ***power exponent*** $Q(a) \in \mathbb{R}^{n+1}$ corresponds to the differential monomial $a(Z)$, it is constructed according to the following rules:

$$Q(c) = 0, \text{ if } c \neq 0, \quad Q\left(Z^R\right) = R, \quad Q\left(\partial^l y / \partial X^L\right) = (-L, 1).$$

The product of monomials corresponds to the sum of their vector power exponents:

$$Q(ab) = Q(a) + Q(b).$$

Differential sum is the sum of differential monomials

$$f(Z) = \sum a_k(Z). \tag{4.1}$$

If $f(Z)$ has no similar terms, then the set $\mathbf{S}(f) = \{Q(a_k)\}$ is called ***support*** of the sum (4.1).

4.2. Resonant monomials (1)

Let the support $\mathbf{S}(a)$ of the differential sum $a(Z)$ consists of one point $E_{n+1} = (0, \dots, 0, 1)$. Then the substitution

$$y = cX^P, \quad P = (p_1, \dots, p_n) \in \mathbb{R}^n \tag{4.2}$$

in $a(Z)$ gives the monomial

$$c\omega_P(P)X^P$$

where $\omega_P\,(P)$ is a ***polynomial*** of P which coefficients depend on P.

Monomial (4.2) will be called ***resonant*** for $a(Z)$ if for it

$$\omega_P(P) = 0.$$

4.2. Resonant monomials (2)

Let μ_k be the maximal order of the derivative over x_k in $a(Z)$, $k = 1, \ldots, n$. If in $P = (p_1, \ldots, p_n)$

$$p_k \geqslant \mu_k, \quad k = 1, \ldots, n, \tag{4.3}$$

then

$$a(Z) = c\chi(P)X^P,$$

where $\chi(P)$ is the ***characteristic polynomial*** of the sum of $a(Z)$ and its coefficients do not depend on P. But if the inequalities (4.3) are not satisfied, then $\omega_P(P) \neq \chi(P)$.

4.3. Normal form (1)

For a differential sum $f(Z)$ we denote by $f_k(Z)$ the sum of all differential monomials of $f(Z)$ which have $n+1$ coordinate q_{n+1} of vector power exponents $Q = (q_1, \dots, q_n, q_{n+1})$ equal to k: $q_{n+1} = k$. Denote $\mathbb{Z}^n_+ = \{P : 0 \leqslant P \in \mathbb{Z}^n\}$.

Consider the PDE

$$f(Z) = 0. \tag{4.4}$$

Theorem 4.1.

Let

$$f(Z) = \sum_{k=0}^{\infty} f_k(Z),$$

where all $\mathbf{S}(f_k) \subset \mathbb{Z}_+^n \times \{q_{n+1} = k\}$*. Suppose*

1. $f_0(Z) = \varphi(X)$ *is a power series from* X *without a free term,*
2. $f_1(Z) = a(Z) + b(Z)$*, where* $\mathbf{S}(a) = E_{n+1} = (0, \ldots, 0, 1)$*,* $\mathbf{S}(b) \subset \left(\mathbb{Z}_+^{n+1} \backslash 0\right) \times \{q_{n+1} = 1\}$*.*

Then there exists a substitution $y = \zeta + \psi(X)$*, where* $\psi(X)$ *is a power series from* X *without a free term, which transforms the equation* (4.4) *to the* ***normal form***

$$g(X, \zeta) = 0, \tag{4.5}$$

where $g_0(X) = \sum c_P X^P$ *is a power series without a free term* $\left(P \in \mathbb{Z}_+^n\right)$*, containing only resonant monomials* $c_P X^P$ *for sum* $a(Z)$*.*

Corollary 4.2.

If the sum $a\,(Z)$ *has no resonance monomials* cX^P *with* $P \in \mathbb{Z}_+^n$, $P \neq 0$, *then* $g_0(X) \equiv 0$ *and*

$$y = \psi(X)$$

is the formal solution to the equation (4.4).

If in equation (4.4) differential sum $a(Z)$ does not contain derivatives, then $a(Z) = \mathrm{const} \cdot z_{n+1} = \mathrm{const} \cdot y$. Hence $a(Z)$ has no resonant monomials and in the normal form (4.5) the series $g_0(X) \equiv 0$. So Theorem 4.1 give Implicit Function Theorem 2.1 without T.

4.4 Polyhedron and truncated equations (1)

Let $\mathbf{S}(f)$ be a support of a differential sum (4.1). The convex hull

$$\mathbf{\Gamma}(f) = \left\{ Q = \sum \lambda_j Q_j, Q_j \in \mathbf{S}, \lambda_j \geqslant 0, \sum \lambda_j = 1 \right\}$$

of the support $\mathbf{S}(f)$ is called the ***polyhedron of sum*** $f(Z)$.

The boundary $\partial \mathbf{\Gamma}$ of the polyhedron $\mathbf{\Gamma}(f)$ consists of generalized faces $\mathbf{\Gamma}_j^{(d)}$, where $d = \dim \mathbf{\Gamma}_j^{(d)}$, $0 \leqslant d \leqslant n$.

4.4 Polyhedron and truncated equations (2)

Each face $\mathbf{\Gamma}_j^{(d)}$ corresponds to:
normal cone

$$\mathbf{U}_j^{(d)} = \left\{ P \in \mathbb{R}_*^{n+1} : \langle P, Q \rangle = \left\langle P, Q' \right\rangle > \left\langle P, Q'' \right\rangle, \text{ where } Q, Q' \in \mathbf{\Gamma}_j^{(d)}, Q'' \in \mathbf{\Gamma} \setminus \mathbf{\Gamma}_j^{(d)} \right\},$$

where the space $\mathbb{R}_*^{n+1}$ is conjugate to the space $\mathbb{R}^{n+1}$, $\langle \cdot, \cdot \rangle$ is the scalar product, and ***truncated sum***

$$\hat{f}_j^{(d)}(Z) = \sum a_k(Z) \text{ over } Q(a_k) \in \mathbf{\Gamma}_j^{(d)} \bigcap \mathbf{S}.$$

4.4 Polyhedron and truncated equations (3)

Consider the equation

$$f(Z) = 0, \tag{4.6}$$

where f is the differential sum. In the solution of equation (4.6)

$$y = \varphi(X),$$

where φ is a series on the powers of x_k and their logarithms, the series φ corresponds to its support, polyhedron, normal cones $\mathbf{u}_k$ and truncations.

The logarithm $\ln x_i$ has a zero power exponent on x_i. The truncated solution $y = \hat{\varphi}_k$ corresponds to the normal cone

$$\mathbf{u}_k \subset \mathbb{R}_*^{n+1}.$$

4.4 Polyhedron and truncated equations (4)

Theorem 4.3.

If the normal cone $\mathbf{u}$ *intersects with the normal cone* (4.2), *then the truncation* $y = \hat{\varphi}(X)$ *of the solution* (4.3) *satisfies the truncated equation*

$$\hat{f}_j^{(d)}(Z) = 0. \qquad (4.7)$$

4.5 Power transformations (1)

To simplify the truncated equation (4.7), it is convenient to use a power transformation. Let α be a square real nondegenerate block matrix of dimension $n+1$ of the form

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{pmatrix},$$

where α_{11} and α_{22} are square matrices of dimensions n and 1, respectively. We denote $\ln Z = (\ln z_1, \dots, \ln z_{n+1})$, and by the asterisk * we denote transposition.

Variable change.

$$\ln W = (\ln Z)\,\alpha \tag{4.8}$$

is called ***the power transformation***.

Theorem 4.4 ([Bruno, 2000]).

The power transformation (4.8) *reduces a differential monomial* $a(Z)$ *with a power exponent* $Q(a)$ *into a differential sum* $b(W)$ *with a power exponent* $Q(b)$*:*

$$R = Q(b) = Q(a)\alpha^{-1*}.$$

Power transformation preserves the dimension of the polyhedron.

Corollary 4.5.

The power transformation (4.8) *reduces the differential sum* (4.1) *with support* $\mathbf{S}(f)$ *to the differential sum* $g(W)$ *with support* $\mathbf{S}(g) = \mathbf{S}(f)\alpha^{-1*}$*, i.e.*

$$\mathbf{S}(f) = \mathbf{S}(g)\alpha^{*}$$

4.5 Power transformations (3)

Theorem 4.6.

For the truncated equation

$$\hat{f}_j^{(d)}(Z) = 0$$

there is a power transformation (4.8) *and monomial Z^T that translates the equation above into the equation*

$$g(W) = Z^T \hat{f}_j^{(d)}(Z) = 0,$$

where $g(W)$ is a differential sum whose support has $n + 1 - d$ zero coordinates.

4.6 Logarithmic transformation

Let z_j be one of the coordinates x_k or y. ***Transformation***

$$\zeta_j = \ln z_j \tag{4.9}$$

is called ***logarithmic***.

Theorem 4.7.

Let $f(Z)$ be a differential sum such that all its monomials have a jth component q_j of the vector exponent of degree $Q = (q_1, \dots, q_{n+1})$ equal to zero, then the logarithmic transformation (4.9) *reduces the differential sum $f(Z)$ into a differential sum from $z_1, \dots, \zeta_j, \dots, z_{n+1}$.*

Logarithmic transformation is used only for differential equations. It blows up the polyhedron and increases its dimension up to $n+1$.

4.7 Calculating asymptotic forms of solutions

Usually boundary conditions allow to find an order $P = (p_1, \ldots, p_{n+1})$ of all variables. Let $P \in \mathbf{U}_j^{(d)}$. Then we take truncated equation $\hat{f}_j^{(d)}(Z) = 0$. If it cannot be solved, then a power transformation of the Theorem 4.6 and then a logarithmic transformation of the Theorem 4.7 should be performed. It gives a polyhedron of dimension $n+1$. Its truncated equations are simpler then its own equation $\hat{f}_j^{(d)}(Z) = 0$. In case that this new truncated equation is not solvable again, the above procedure is repeated until we get a solvable equation. Having its solutions, we can return to the original coordinates by doing inverse coordinate transformations. So the solutions, written in original coordinates, are the asymptotic forms of solutions to the original equation (4.4).

In [Bruno, Shadrina, 2007; Bruno, Batkhin, 2023] method of selecting truncated equations was applied to systems of PDE.

Software for investigation ODE and PDE

For investigation systems of non-linear PDE by Power Geometry algorithm a Maple library was implemented in [Bruno, Batkhin, 2023, Subsect. 4.5]. Support and Newton's polygon associated with each equation in the system are computed with the help of special parsing algorithm. Other procedures of the library allows to computed corresponding normal cones, truncated system and produce power and logarithmic transformations of the set of independent and dependent variables.

Another approach called the ***method of power substitution*** for computation the objects of Power Geometry for systems of ODE was proposed in [Aranson, 2023a,b]. The power substitution allows to compute vector power exponents by builtin functions of CAS and allows substitute power exponent of substituted variable in form of expression in several variables.

5. Levels of Power Geometry (1)

Everything that has been told in [Bruno, 2023] refers to the zero level of power geometry, for there it has been «sealed» that

$$\operatorname{ord} y' = \operatorname{ord} y - 1.$$

But this is not always the situation. By rejecting this property, we get a wider set of solutions. Let's discuss this in more detail.

In the algebraic equation

$$f(X) \stackrel{\text{def}}{=} \sum a_Q X^Q = 0,$$

with $X \in \mathbb{R}^n$ or $\mathbb{C}^n$ to each monomial $a_Q X^Q$ can be assigned a point $\check{Q} = \{Q, \ln |a_Q||\}$ in $\mathbb{R}^{n+1}$. Their set forms the ***supersupport*** $\check{S}(f)$, and its convex hull $\check{H}(f)$ is the ***Adamar polyhedron*** [Bruno, 2018].

5. Levels of Power Geometry (2)

We build truncated equations on its faces. They are simpler than the truncated equations corresponding to the faces of Newton's polyhedron, and allow us to study cases where Newton's polyhedron fails.

For a single ODE, one can search for solutions that have $\operatorname{ord} y - \operatorname{ord} y' \neq 1$ by introducing a new coordinate for the order of the derivative y'. This was done in [Bruno, 2015] and allowed us to obtain solutions in the form of power expansions whose coefficients are trigonometric or elliptic functions.

We can consider solutions where $\operatorname{ord} y^{(k)} - \operatorname{ord} y^{(k+1)}$ is arbitrary, or several such differences are arbitrary, and obtain new types of solutions. For details see [Bruno, 2012a,b; Bruno, Parusnikova, 2012].

A similar thing could be done with partial differential equations, but it has not been done yet.

Bibliography (1)

- *Barber C. B., Dobkin D. P., Huhdanpaa H. T.* The Quickhull algorithm for convex hulls. // ACM Trans. on Mathematical Software. 1996. Vol. 22, no. 4. P. 469–483. DOI: 10.1145/235815.235821.
- *Bruno A. D.* Power Geometry in Algebraic and Differential Equations. Amsterdam: Elsevier Science, 2000.
- *Bruno A. D.* Asymptotics and expansions of solutions to an ordinary differential equation. // Russian Mathem. Surveys. 2004. Vol. 59, no. 3. P. 429–480.
- *Bruno A. D.* Complicated expansions of solutions to an ordinary differential equation. // Doklady Mathematics. 2006. Янв. Т. 73, № 1. С. 117—120. ISSN 1531-8362. DOI: 10.1134/s1064562406010327. URL: http://dx.doi.org/10.1134/S1064562406010327.
- *Bruno A. D.* Exotic expansions of solutions to an ordinary differential equation. // Doklady Mathematics. 2007. Окт. Т. 76, № 2. С. 729—733. ISSN 1531-8362. DOI: 10.1134/s1064562407050237. URL: http://dx.doi.org/10.1134/S1064562407050237.

Bibliography (2)

- *Bruno A. D., Shadrina T. V.* Axisymmetric boundary layer on a needle. // Transactions of Moscow Math. Soc. 2007. Vol. 68. P. 201–259.
- *Bruno A. D.* Chapter 6. Space Power Geometry for one ODE and P1–P4, P6. // Proceedings of the International Conference, Saint Petersburg, Russia, June 17-23, 2011 / ed. by A. D. Bruno, A. B. Batkhin. Berlin, Boston: De Gruyter, 2012a. P. 41–52. DOI: doi:10.1515/9783110275667.41.
- *Bruno A. D.* Power-elliptic expansions of solutions to an ordinary differential equation. // Computational Mathematics and Mathematical Physics. 2012b. Vol. 51, no. 12. P. 1650–1661.
- *Bruno A. D.* Power-exponential expansions of solutions to an ordinary differential equation. // Doklady Mathematics. 2012c. Май. Т. 85, № 3. С. 336—340. ISSN 1531-8362. DOI: 10.1134/s106456241203009x. URL: http://dx.doi.org/10.1134/S106456241203009X.
- *Bruno A. D., Batkhin A. B.* Resolution of an algebraic singularity by power geometry algorithms. // Programming and Computer Software. 2012. Vol. 38, no. 2. P. 57–72. DOI: 10.1134/S036176881202003X. URL: https://disk.yandex.ru/i/dmylwZQZxQ_wTw.

Bibliography (3)

- *Bruno A. D.*, *Parusnikova A. V.* Chapter 7. Elliptic and Periodic Asymptotic Forms of Solutions to P5. // Proceedings of the International Conference, Saint Petersburg, Russia, June 17-23, 2011 / ed. by A. D. Bruno, A. B. Batkhin. Berlin, Boston: De Gruyter, 2012. P. 53–66. DOI: doi:10.1515/9783110275667.53.
- *Bruno A. D.* Power geometry and elliptic expansions of solutions to the Painlevé equations. // International Journal of Differential Equations. 2015. Vol. 2015. P. 340715. DOI: 10.1155/2015/340715.
- *Thompson I.* Understanding Maple. Cambridge University Press, 2016. 228 p.
- *Bruno A. D.* Algorithms for solving an algebraic equation. // Programming and Computer Software. 2018. Vol. 44, no. 6. P. 533–545. DOI: 10.1134/S0361768819100013.
- *Bruno A. D.* Expansion of Solutions to an Ordinary Differential Equation into Transseries. // Doklady Mathematics. 2019a. Янв. Т. 99, № 1. С. 36—39. ISSN 1531-8362. DOI: 10.1134/s1064562419010113. URL: http://dx.doi.org/10.1134/S1064562419010113.

Bibliography (4)

- *Bruno A. D.* On the parametrization of an algebraic curve. // Mathematical Notes. 2019b. Vol. 106, no. 6. P. 885–893.
- *Bruno A. D.*, *Batkhin A. B.* Algorithms and software for computation of roots of a polynomial in one or two variables. // Programming and Computer Software. 2021. Vol. 46, no. 5. (to be published).
- *Aranson A. A.* // Computer algebra: 5th International Conference Materials. Moscow, 26–28 June, 2023 / ed. by S. A. Abramov, A. B. Batkhin, L. A. Sevastyanov. Moscow: KIAM, 2023a. Chap. Power Algebra for Power Geometry. P. 33–36. DOI: 10.20948/ca-2023.
- *Aranson A. A.* Painleve Test and Integrability of Polynomial ODEs. // International Conference Polynomial Computer Algebra '2023; St. Petersburg, April 17-22, 2023 / ed. by N. N. Vassiliev. St. Petersburg, Russia: VVM Pubishing, 2023b. P. 10–11.
- *Bruno A. D.* Nonlinear Analysis as a Calculus. // London Journal of Research in Science: Natural and Formal. 2023. Vol. 23, no. 5. P. 1–31. URL: https://journalspress.com/LJRS_Volume23/Nonlinear-Analysis-as-a-Calculus.pdf ; (open access).

Bibliography (5)

- *Bruno A. D., Azimov A. A.* Computation of unimodular matrices of power transformations. // Programming and Computer Software. 2023a. Vol. 49, no. 1. P. 32–41. DOI: 10.1134/S0361768823010036. URL: https://disk.yandex.ru/i/zXlMlHu91CcQiw.
- *Bruno A. D., Azimov A. A.* Parametric expansions of an algebraic variety near its singularities. // Axioms. 2023b. Vol. 12, no. 5. P. 469. URL: https://doi.org/10.3390/axioms12050469 ; (open access).
- *Bruno A. D., Batkhin A. B.* Asymptotic forms of solutions to system of nonlinear partial differential equations. // Universe. 2023. Vol. 9, no. 1. P. 35. DOI: 10.3390/universe9010035. (open access).
- *Bruno A. D., Azimov A. A.* Parametric expansions of an algebraic variety near its singularities II. // Axioms. 2024. Vol. 13, no. 2. P. 106. URL: https://doi.org/10.3390/axioms13020106 ; (open access).
- *The Sage Developers*. SageMath, the Sage Mathematics Software System (Version 9.7). 2022. https://www.sagemath.org.