Some geometric properties of shifted Young diagrams of maximum dimensions

Vasilii Duzhin and Egor Smirnov-Maltsev

The problem of finding Young diagrams of straight shape with large dimensions, i. e. those with a large number of Young tableaux, was previously studied in [1, 2]. This research is a continuation of works [3, 4] in which a similar problem was investigated for strict Young diagrams. The approaches used in the above works have in common that the original diagram λ_n is transformed into a new one λ'_n of the same size n such that the dimension of λ'_n is greater than the dimension of λ_n. Here we present two new methods for finding strict Young diagrams of larger dimensions which are based on the same idea.

Consider a strict Young diagram $\lambda = (x_1, x_2, ..., x_s)$ of size n. The first method is to move a box from the $(i-1)$th column to the ith one. Thus, the resulting diagram of size n will be $\lambda' = (x_1, ..., x_{i-2}, x_{i-1} - 1, x_i + 1, x_{i+1}, ..., x_s)$. We prove that the dimension of λ' is greater than the dimension of λ if the following condition is met:

$$x_{i-1} - x_i \geq \frac{3 + \sqrt{9 + 8x_i}}{2}. \quad (1)$$

Note that in the case when inequality (1) is not satisfied, the above transformation can lead to both a decrease in dimension and an increase in it.

Consider a strict Young diagram $\lambda = (x_1, x_2, ..., x_s)$. The *tail* of length t is the last t columns of λ. Tail size $\tilde{n}$ is the number of boxes in these t columns. The second method is to change a tail of length t of size $\tilde{n}$ to a tail of length $t+1$ of the same size. The resulting Young diagram is $\lambda' = (x_1, ..., x_{s-t}, y_1, ..., y_{t+1})$.

During the research, it was hypothesized that the dimension of the diagram λ is less than the dimension of the diagram λ' if 2 conditions are met:

1. The dimension of a diagram $\lambda_t = (x_{s-t+1}, x_{s-t+2}, ..., x_s)$ is not greater than the dimension of a diagram $\lambda'_t = (y_1, y_2, ..., y_{t+1})$.
2. There is no $t_1 < t$ such that there is a diagram $\lambda'_{t_1} = (\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_{t_1+1})$ whose dimension is not less than the dimension of the diagram

$$\lambda_{t_1} = (x_{s-t_1+1}, x_{s-t_1+2}, ..., x_s).$$

This conjecture was proven for t equal to 1 and 2. However, already for t equal to 7 a counterexample to it was found. This work was supported by grant RSF 22-21-00669.

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Vasilii Duzhin
Saint Petersburg Electrotechnical University
ul. Professora Popova 5, 197376 St. Petersburg, Russian Federation
e-mail: vsduzhin@etu.ru

Egor Smirnov-Maltsev
Department of Applied Informatics and Probability Theory
Peoples' Friendship University of Russia
Moscow, Russia
e-mail: 1032212272@rudn.ru