Some geometric properties of shifted Young diagrams of maximum dimensions

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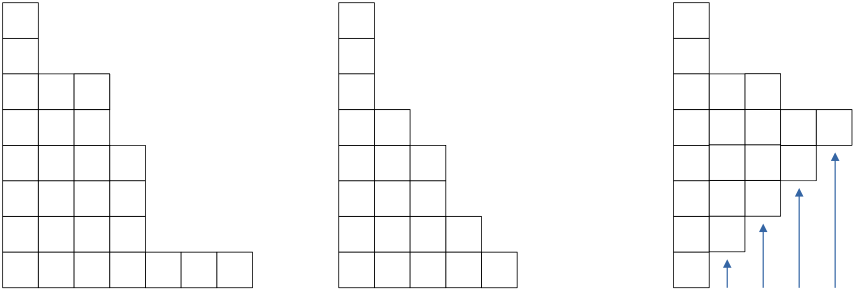
Shifted (strict) Young diagrams

A *strict partition* is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. A strict partition can be represented by its *strict Young diagram* or *shifted Young diagram*.



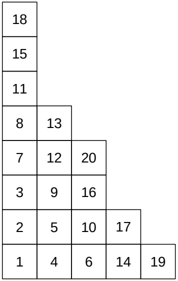
Young diagram (8, 6, 6, 4, 1, 1, 1)

Strict Young diagram (8, 5, 4, 2, 1)

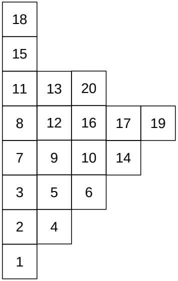
Shifted Young diagram (8, 5, 4, 2, 1)

Shifted (strict) Young tableaux

A *shifted Young tableau* is a shifted Young diagram filled by integers 1..n such that they grow from left to right and from bottom to top.



Strict Young tableau



Shifted Young tableau

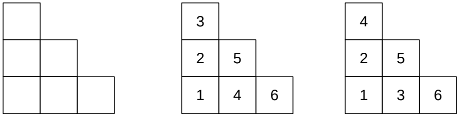
Dimension of a strict Young diagram

Dimension of a strict Young diagram λ is the number of strict Young tableaux of the shape λ.

$$\dim(\lambda) = \prod_{i<j} \frac{x_i - x_j}{x_i + x_j} \cdot \frac{n!}{\prod x_i!},$$

where x_i is the height of i-th column of λ, n is the number of boxes in λ.

Example:



λ

$$\dim(\lambda) = \frac{3-2}{3+2} \cdot \frac{3-1}{3+1} \cdot \frac{2-1}{2+1} \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1 \cdot 1} = 2$$

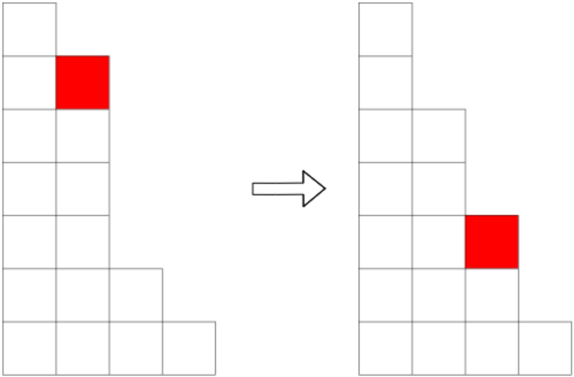
Problem statement

Goal: To find a strict Young diagram with the largest dimension among all strict diagrams of some fixed size n.

Solution method: The idea is to transform λ_n into another strict diagram λ'_n such that $\dim(\lambda'_n) \geq \dim(\lambda_n)$.

Convergence of neighbors

Convergence of neighbors method refers to moving a box from the $(i-1)$ column to the i one.



Neighbor lemma

If the heights of $s-1$ and s columns of strict Young diagrams are x_{s-1} and x_s respectively and

$$x_{s-1} - x_s \geq \frac{3 + \sqrt{9 + 8x_s}}{2}$$

then moving the box from $s-1$ column to s column increase the dimension of diagram.

Proof of neighbor lemma

$$\dim(\lambda) = \prod_{i<j;i,j\neq s,s+1} \frac{x_i - x_j}{x_i + x_j} \cdot \prod_{i<s} \frac{(x_i - (x+k))(x_i - x)}{(x_i + (x+k))(x_i + x)} \cdot$$
$$\prod_{i>s+1} \frac{((x+k) - x_i)(x - x_i)}{(x_i + (x+k))(x_i + x)} \cdot \frac{x+k-x}{x+k+x} \cdot \frac{n!}{\prod\limits_{i\neq s,s+1} x_i! \cdot (x+k)!x!}, \quad (1)$$

where $k = x_{s-1} - x_s$ and $x = x_s$.

Proof of neighbor lemma

$$\dim(\lambda') = \prod_{i<j;i,j\neq s,s+1} \frac{x_i - x_j}{x_i + x_j} \cdot \prod_{i<s} \frac{(x_i - (x+k-1))(x_i - (x+1))}{(x_i + (x+k-1))(x_i + (x+1))} \cdot$$
$$\cdot \prod_{i>s+1} \frac{((x+k-1) - x_i)((x+1) - x_i)}{(x_i + (x+k-1))(x_i + (x+1))} \cdot \frac{x+k-1-(x+1)}{x+k-1+x+1} \cdot$$
$$\cdot \frac{n!}{\prod_{i\neq s,s+1} x_i! \cdot (x+k-1)!(x+1)!}. \quad (2)$$

Proof of neighbor lemma

Let us compare $\dim(\lambda)$ and $\dim(\lambda')$.

Proof of neighbor lemma

Let us compare $\dim(\lambda)$ and $\dim(\lambda')$.

$$\frac{(x_i-(x+k))(x_i-x)}{(x_i+(x+k))(x_i+x)} < \frac{(x_i-(x+k-1))(x_i-(x+1))}{(x_i+(x+k-1))(x_i+(x+1))}$$

for all $x_i > x+k$ and

$$\frac{((x+k)-x_i)(x-x_i)}{(x_i+(x+k))(x_i+x)} < \frac{((x+k-1)-x_i)((x+1)-x_i)}{(x_i+(x+k-1))(x_i+(x+1))}$$

for all $x_i < x$ are satisfied when $k > 1$.

Proof of neighbor lemma

$$\frac{x+k-x}{x+k+x} \cdot \frac{1}{(x+k)!x!} < \frac{x+k-1-(x+1)}{x+k-1+x+1} \cdot \frac{1}{(x+k-1)!(x+1)!}$$

is equivalent to

$$k^2 - 3k - 2x > 0.$$

This inequality is satisfied when

$$k \geq \frac{3+\sqrt{9+8x}}{2}.$$

Regular tails

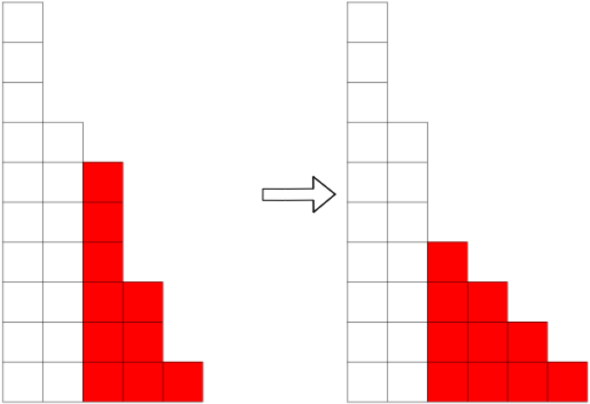
Let us call the rightmost part of a partition *a regular tail* if it contains only consecutive odd (..., 7, 5, 3, 1) or consecutive even (..., 8, 6, 4, 2) numbers.

It was discovered [Duzhin, Vasilyev '16] that most of partitions corresponding to diagrams of maximum dimensions have a regular tail of a relatively large size. Two examples of such partitions are listed in the table below:

Size	Partition
200	34, 30, 26, 23, 20, 17, 14, **11, 9, 7, 5, 3, 1**
250	38, 34, 30, 27, 24, 21, 18, 15, 13, **10, 8, 6, 4, 2**

Tail transforming

The *tail* of length t is the last t columns of strict Young diagram. The second method is to transform a tail of length t of size $\tilde{n}$ to a tail of length $t+1$ of the same size.



Tail transforming hypothesis

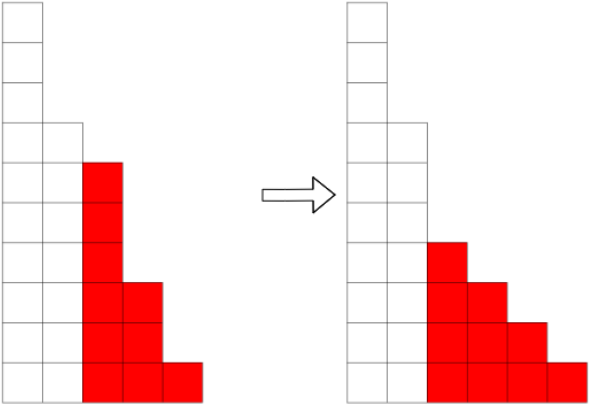
During the research, it was hypothesized that the dimension of the diagram λ is less than the dimension of the diagram λ' if 2 conditions are met:

1. The dimension of a diagram $\lambda_t = (x_{s-t+1}, x_{s-t+2}, ..., x_s)$ is not greater than the dimension of a diagram $\lambda'_t = (y_1, y_2, ..., y_{t+1})$.
2. There is no $t_1 < t$ such that there is a diagram $\lambda'_{t_1} = (\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_{t_1+1})$ whose dimension is not less than the dimension of the diagram

$$\lambda_{t_1} = (x_{s-t_1+1}, x_{s-t_1+2}, ..., x_s).$$

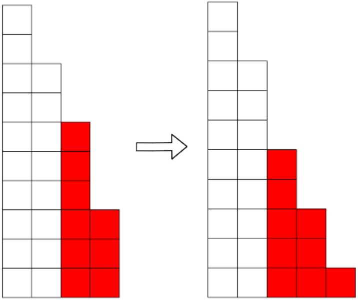
First condition of the hypothesis

We make this replacement only if dimension of left tail is bigger than or equal to dimension of right tail.

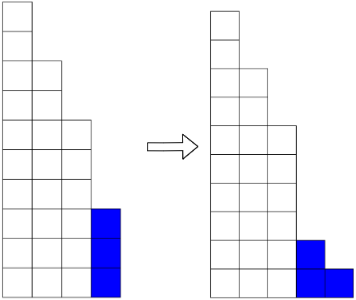


Second condition of the hypothesis

We do not make red replacement if we can make blue replacement.



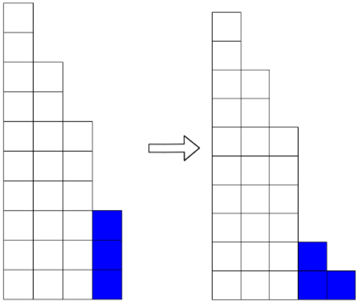
length of tail is 2



length of tail is 1

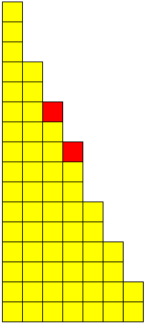
Tail transforming hypothesis

We proved this hypothesis for tails with length equal to 1 and 2. In particular, we proved that height of last columns is equal to 1 or 2.

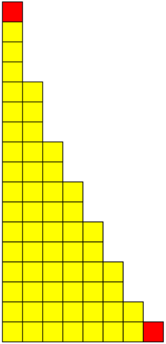


Counterexample to tail transforming hypothesis

However for tails with length equal to 7 this hypothesis is false.
Dimension of λ_t is less than dimension of λ'_t:



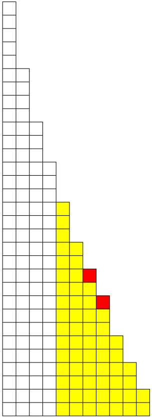
λ_t



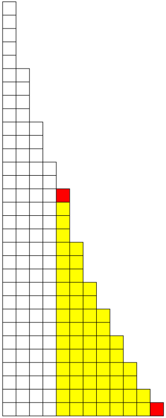
λ'_t

Counterexample to tail transforming hypothesis

but dimension of λ is bigger than dimension of λ':



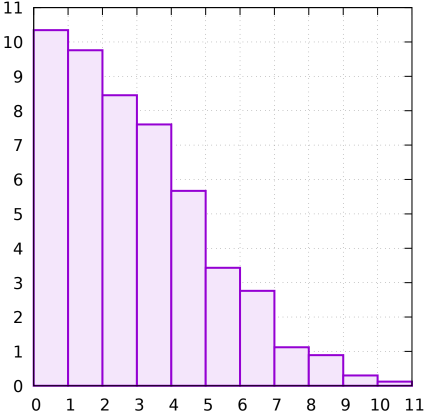
λ



λ'

Young diagrams with columns of real height

Future plans: to consider strict Young diagrams with columns of real height.



Young diagrams with columns of real height

Let us define dimension of such diagrams using the formula:

$$\dim(\lambda) = \prod_{i<j} \frac{x_i - x_j}{x_i + x_j} \cdot \frac{\Gamma(n+1)}{\prod \Gamma(x_i + 1)},$$

where x_i is the height of i-th column of λ, n is the area of λ.
Such Young diagrams of size n with maximum dimension could help to find discrete Young diagrams of size n with maximum dimension.

Further research

1. To correct the tail transforming hypothesis.
2. To make a search for strict Young diagrams of maximum dimension limited by proven statements.
3. To study strict Young diagrams with columns of real height.

Thanks for your attention!