Remarks on Tarski's Elimination

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Tarski's Elimination Theorem

T h e o r e m. Any formula $\varphi(\bar{x})$ of signature $\{=, <, +, \cdot, 0, 1\}$ is equivalent in $\mathbb{R}$ to a quantifier-free formula $\varphi^*(\bar{x})$.

Equivalently, the projection of any semialgebraic set $S \subseteq \mathbb{R}^{n+1}$ along any axis is a semialgebraic subset of $\mathbb{R}^n$.
A subset of $\mathbb{R}^n$ is *semialgebraic*, if it is a finite union of solution sets of systems of polynomial equations $P(\bar{x}) = 0$ and inequalities $Q(\bar{x}) > 0$ with $P, Q \in \mathbb{Z}[\bar{x}]$

There are many extensions and variations on the Tarski theorem. In this talk, we briefly discuss some earlier and some newer variations related to computable model theory, foundations of symbolic computations, and numeric computations.

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Variations on Tarski's Elimination

Tarski's theorem strongly influenced different areas of mathematics including:
1) Axiomatizing and deciding geometry (Tarski, Szmielew, Givant...),
2) Model theory (Robinson, Ax, Macintyre, Ziegler, van den Dries,...),
3) Foundations of PDE-theory (Hörmander, Shilov, Gorin,...),
4) Decidability in fragments of analysis,
5) Computer algebra and computational complexity (Cohen, Collins, Renegar, Grigoriev, Vorobjov, Chistov,...)

Topics of our talk are related to 5) but we mainly consider large complexity classes like general (Turing) computability or primitive recursive (PR) computability. In particular, we are interesting in extending the integer (equivalently, rational) polynomials to larger fields of coefficients which still admit the computability of function $\varphi \mapsto \varphi^*$ in the Tarski theorem and in spectral decomposition.

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Algorithmic problems in field theory

Based on the notion of a computable structure, the computability issues in algebra and model theory were thoroughly investigated. In particular, a rich and useful theory of computable fields was developed.

For instance, Rabin has shown that the algebraic closure of a computable field is computably presentable, and Ershov and Madison have shown that the real algebraic closure of a computable ordered field is computably presentable.

Since the ordered field $\mathbb{Q}$ of rationals is computably presentable, the field $\mathbb{C}_{\mathrm{alg}} = (C_{\mathrm{alg}}; +, \times, 0, 1)$ of complex algebraic numbers and the ordered field $\mathbb{R}_{\mathrm{alg}} = (R_{\mathrm{alg}}; \leq, +, \times, 0, 1)$ of algebraic reals are computably presentable.

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Constructive structures

D e f i n i t i o n. A structure $\mathbb{B} = (B; \sigma)$ of a finite signature σ is called constructivizable iff there is a numbering β of B such that all signature predicates and functions, and also the equality predicate, are β-computable. Such a numbering β is called a constructivization of $\mathbb{B}$, and the pair $(\mathbb{B}, \beta)$ is called a constructive structure.

Obviously, $(\mathbb{B}, \beta)$ is a constructive structure iff given a quantifier-free σ-formula $\phi(v_1, \ldots, v_k)$ with free variables among $v_1, \ldots, v_k$ and given $n_1, \ldots, n_k \in \mathbb{N}$, one can compute the truth-value $\phi^{\mathbb{B}}(\beta(n_1), \ldots, \beta(n_k))$ of ϕ in $\mathbb{B}$ on the elements $\beta(n_1), \ldots, \beta(n_k) \in B$.

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Strongly constructive structures

D e f i n i t i o n. A structure $\mathbb{B} = (B;\sigma)$ of a finite signature σ is called strongly constructivizable iff there is a numbering β of B such that, given a first-order σ-formula $\phi(v_1,\ldots,v_k)$ with free variables among $v_1,\ldots,v_k$ and given $n_1,\ldots,n_k \in \mathbb{N}$, one can compute the truth-value $\phi^{\mathbb{B}}(\beta(n_1),\ldots,\beta(n_k))$ of ϕ in $\mathbb{B}$ on the elements $\beta(n_1),\ldots,\beta(n_k) \in B$. Such a numbering β is called a strong constructivization of $\mathbb{B}$, and the pair $(\mathbb{B},\beta)$ is called a strongly constructive structure.

Note that we used above "Russian" terminology; the equivalent "American" notions for "constructivizable" and "constructive" are "computably presentable" and "computable", resp.
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Presentaion complexity of structures

In applications one of course has to pay attention to the complexity of implemented algorithms and of structure presentations. The complexity of structure presentations was first studied by Nerode, Cenzer and Remmel. In particular, the notion of a polynomial-time (p-time) structure was introduced. To our knowledge, the complexity issues for presentations of fields were not studied in computability theory so far.

At the same time, there exists a well-developed theory of symbolic computations (closely related to computer algebra) which investigates the complexity of algorithms in fields, of concrete presentations of fields and rings, and aims to implement these in computer systems.

Although the mentioned theories are clearly related, they developed independently and there are essentially no references between them. We promote the theory of feasible presentations of structures as a foundation for symbolic computations.

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Computable reals

A real is *computable* if it is the limit of a sequence of rationals $\{q_i\}$ such that $|q_i - q_{i+1}| < 2^{-i}$. The field $\mathbb{R}_c$ of computable reals is countable, real closed, and not computably presentable. But, in a sense, it is "partially computably presentable".

Let $\varkappa$ — be a constructivisation of $\mathbb{Q}$ and $\{\varphi_n\}$be the standard computable numbering of all computable partial functions on $\mathbb{N}$. Define a partial function ρ from $\mathbb{N}$ onto $\mathbb{R}_c$: $\rho(n) = x$ iff φ_n is total and $\{\varkappa\varphi_n(i)\}_i$ is a fast Cauchy sequence converging to x.

A numbering μ is *reducible* to a (partial) numbering ν ($\mu \leq \nu$), if $\mu = \nu \circ f$ for some computable function f on $\mathbb{N}$.

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Constructive fields of reals

P r o p o s i t i o n 1. Let $\mathbb{B}$ be a computable ordered subfield of $\mathbb{R}$, and β be a constructivisation of $\mathbb{B}$. Then $\beta \leq \rho$, in particular $\mathbb{B} \subseteq \mathbb{R}_c$.

P r o p o s i t i o n 2. Let $\mathbb{B}$ be a subfield of $(\mathbb{R}; +, \cdot, 0, 1)$ and β be a constructivisation of $\mathbb{B}$ such that $\beta \leq \rho$. Then β is a constructivisation of the ordered field $(\mathbb{B}; <)$.

P r o p o s i t i o n 3. Let $\mathbb{B}$ be a real closed subfield of $(\mathbb{R}; +, \cdot, 0, 1)$ and β be a constructivisation of $\mathbb{B}$. Then β is a strong constructivisation of the ordered field $(\mathbb{B}; <)$.

Adjoining computable reals

We add the following theorem to the results of the previous slide. The theorem relates constructive fields of reals to the field $\mathbb{R}_c$ of computable reals.

T h e o r e m (with S. Selivanova). For any finite set $F \subseteq \mathbb{R}_c$ there is a strongly constructive real closed subfield $(\mathbb{B}, \beta)$ of the ordered field $\mathbb{R}_c$ such that $F \subseteq B$.
Cf. an independent result by R. Miller and V. Ocasio Gonzalez.

Thus, the union of all computably presentable real closed fields of reals is $\mathbb{R}_c$.

Example: For any fixed computable real matrix there is a strongly constructive real closed subfield $(\mathbb{B}, \beta)$ of $\mathbb{R}_c$ containing all the matrix coefficients.

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Computability in linear algebra

Let $(\mathbb{B}, \beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Then one can compute, given a polynomial $p(x) = a_0 + a_1 x^1 \cdots + a_k x^k$ with coefficients in $\mathbb{B}$ (i.e., given a string $n_0, \ldots, n_k$ of naturals with $\beta(n_0) = a_0, \ldots, \beta(n_k) = a_k$) the string $r_1 < \cdots < r_m$, $m \geq 0$, of all distinct real roots of $p(x)$ (i.e., a string $l_1, \ldots, l_m$ of naturals with $\beta(l_1) = r_1, \ldots, \beta(l_m) = r_m$), as well as the multiplicity of any root r_j.

Spectral decomposition of a symmetric real matrix $A \in M_n(\mathbb{R})$ is a pair $((\lambda_1, \ldots, \lambda_n), (\mathbf{v}_1, \ldots, \mathbf{v}_n))$ where $\lambda_1 \leq \cdots \leq \lambda_n$ is the sequence of all eigenvalues of A (each eigenvalue occurs in the sequence several times, according to its multiplicity) and $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a corresponding orthonormal basis of eigenvectors.

P r o p o s i t i o n. Let $(\mathbb{B}, \beta)$ be a strongly constructive real closed ordered subfield of $\mathbb{R}_c$. Given a symmetric $n \times n$-matrix A with coefficients in $\mathbb{B}$, one can compute its spectral decomposition.

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The status of spectral decomposition in computable analysis is quite different. Martin Ziegler and Vasco Brattka (based on old results by F. Rellich) have shown that the spectral decomposition of symmetric real matrices is not computable in the Turing sense (because it is not continuous). But this problem becomes computable if the number of distinct eigenvalues of the matrix is given as input.

This result was a motivation for our work because the second author was interested in the symmetric hyperbolic systems of PDEs, and difference schemes used in numeric methods for solving such systems require to compute spectral decompositions of symmetric matrices and matrix pencils.

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Bit complexity version

Based on deep facts of Computer Algebra, Alaev and S. have shown PTIME-presentability of $\mathbb{R}_{\mathrm{alg}}, \mathbb{C}_{\mathrm{alg}}$, and PTIME-computability of some versions of root-finding in these fields. We used this to establish upper bounds of bit complexity of some problems in linear algebra and PDEs. Examples:

T h e o r e m (with S. Selivanova). 1) For any fixed $n \geq 1$, there is a polynomial time algorithm which, given a symmetric matrix $A \in M_n(\mathbb{R}_{\mathrm{alg}})$, computes a spectral decomposition of A.
2) There is a polynomial time algorithm which, given a symmetric matrix $A \in M_n(\mathbb{Q})$, computes a spectral decomposition of A uniformly on n. The same holds if we replace $\mathbb{Q}$ by $\mathbb{Q}(\alpha)$ where α is any fixed algebraic real.

Similar results hold for matrix pencils.

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Primitive recursive version

More recently, we (with S. Selivanova) developed a PR-version of our approach, in particular of the Ershov-Madison theorem. A numbering $\alpha : \mathbb{N} \to \mathbb{R}$ is a *PR-archimedean field*, if $A = rng(\alpha)$ is an ordered subfield of $\mathbb{R}$, all $+, \cdot, -, ^{-1}, <, =$ are α-PR, and $\alpha(n) < f(n)$ for a PR-function f. Examples of typical results:

T h e o r e m. Given a PR-archimedean field α, one can find a PR-archimedean field $\hat{\alpha}$ s.t. $\alpha \leq \hat{\alpha}$ and $\hat{A}$ is the real closure of A.

T h e o r e m. If α is a PR-archimedean field with PR-splitting then $\hat{\alpha}$ and the algebraic closure $\overline{\alpha}$ have PR-root-finding.

Also, the spectral decomposition and Tarski's elimination are uniformly PR-computable when restricted to coefficients in a PRAS-field.

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The union of PR-archimedean fields coincides with $\mathbb{R}(\varkappa)$ — the set of PR reals b such that the sign of polynomials in $\mathbb{Q}[x]$ at b is checked primitive recursively. There is a PR real which is not in $\mathbb{R}(\varkappa)$.

Many transcendental reals, in particular e and π, may be (separately) included in a PRAS-field.

BUT, many interesting questions remain open, in particular:
1) Is $\mathbb{Q}(e, \pi)$ a PRAS-field?
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PR functions

The PR functions are generated from the distinguished functions $o = \lambda n.0$, $s = \lambda n.n+1$, and $I_i^n = \lambda x_1, \ldots, x_n.x_i$ by repeated applications of the operators of superposition S and primitive recursion R. Thus, any PR function is represented by a "correct" term in the partial algebra of functions over $\mathbb{N}$. Intuitively, any total function defined by an explicit definition using (not too complicated) recursion is PR; the unbounded μ-operator is of course forbidden but the bounded one is possible.

Consider the structure $(\mathcal{N}; +, \circ, J, s, q)$ where $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is the set of unary functions on $\mathbb{N}$, $+$ and $\circ$ are binary operations on $\mathcal{N}$ defined by $(p+q)(n) = p(n) + q(n)$ and $(p \circ q)(n) = p(q(n))$, J is a unary operation on $\mathcal{N}$ defined by $J(p)(n) = p^n(0)$ where $p^0 = id_{\mathbb{N}}$ and $p^{n+1} = p \circ p^n$, s and q are distinguished elements defined by $s(n) = n+1$ and $q(n) = n - [\sqrt{n}]^2$ where, for $x \in \mathbb{R}$, $[x]$ is the unique integer m with $m \leq x < m+1$. The PR unary functions coincide with the subalgebra generated by s, q (R. Robinson).

PR functions

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THANK YOU FOR YOUR ATTENTION!!

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