On the parameterizations of the special unitary group SU(4) and related double cosets

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From group to coset

Equivalence relations. Let g and $g' \in G$ be elements of the Lie group, G and $K \subset G \supset H$, then the equivalence relations

$$g \sim g' , \quad \text{if} \quad g' = g_1 g g_2^{-1} , \quad g_1 \in K , g_2 \in H$$

define the double coset $K \backslash G / H$.

Coordinatizing double coset

Starting with the coordinates on the group manifold G and restricting to an appropriate subset, to describe $K \backslash G / H$.
Task: Find a parameterization of G adapted to its subgroups structure resulting in effective description of the corresponding double coset.

Classical results

Cartan decomposition. For the symmetric case, $K = H$, the semisimple Lie group admits factorization

$$G = KP$$

associated to the Cartan involution on Lie algebra with a maximal compact subgroup K of G assuming its center is finite.
Generalized Cartan decomposition For the unitary group $U(n)$ the factorization

$$U = KBH$$

adapted to the double coset of the form

$$U(n_1) \times U(n_2) \times \cdots \times U(n_k) \backslash U(n) / U(m_1) \times U(m_2) \times \cdots \times U(m_r)\,,$$

with double non-commutative involutions.

Motivation from quantum theory

In the Quantum Theory of a binary $N_A \times N_B$ system indwelling in the Hilbert space $\mathcal{H} = \mathbb{C}^{N_A} \otimes \mathbb{C}^{N_B}$, the unitary group $U(N)$ and its subgroup $K = SU(N_A) \times SU(N_B)$ appear as the isometries of a state $\varrho \in \mathfrak{P}_N$, where $\mathfrak{P}_N$ is the quantum state space

$$\mathfrak{P}_N = \{X \in M(N, \mathbb{C}), N = N_A N_B \,|\, X = X^\dagger \,, X \geq 0 \,, \mathrm{Tr}(X) = 1\} \,.$$

$SU(N)$ and K define two kind of orbits in $\mathfrak{P}_N$, respectively:

- the global orbit

$$\mathrm{Orb}_{SU(N)}(\varrho) = g\,\varrho\,g^{-1} \,, \quad \forall\, g \in SU(N)$$

- the local orbit

$$\mathrm{Orb}_{K}(\varrho) = g\,\varrho\,g^{-1} \,, \quad \forall\, g \in SU(N_A) \times SU(N_B)$$

An example: $N = 4 = 2 \times 2$ system, the 2-QUBIT

Global orbits of 2-qubits

The global $\mathrm{Orb}_{SU(4)}(\varrho)$ are determined by the spectrum of ϱ,

$$\varrho = W \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} W^\dagger .$$

The unitary factor W belongs to the coset

$$W \in SU(4)/\mathrm{Iso}(\varrho)\,,$$

where $\mathrm{Iso}(\varrho)$ is the isotropy group of the state ϱ .

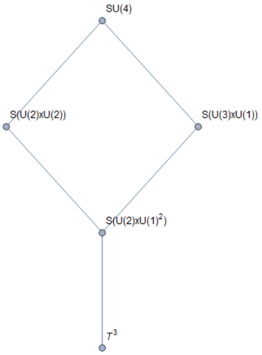
Cosets associated to the global group action

The $SU(4)$ orbits are classified according to the isotropy groups Iso, which form the hierarchy of Lie subgroups in $SU(4)$.
Hence, starting from coordinates on $SU(4)$, we intend to find the parameterization of the following cosets:

$$\frac{SU(4)}{S(U(2)\times U(2))}, \quad \frac{SU(4)}{S(U(3)\times U(1))}$$

and

$$\frac{SU(4)}{S(U(2)\times U(1)^2)}, \quad \frac{SU(4)}{T^3}$$



Local Unitary group action on $\mathrm{Orb}_{SU(4)}$

Considering the local group action $K = SU(2) \times SU(2)$ on the orbits $\mathrm{Orb}_{SU(4)}$, we arrive at the following double cosets:

$$K \backslash SU(4)/S(U(2) \times U(2)), \; K \backslash SU(4)/S(U(3) \times U(1))$$

and

$$K \backslash SU(4)/S(U(2) \times U(1)^2), \; K \backslash SU(4)/T^3$$

Canonical charts on the Lie group

Theorem on canonical charts

If the Lie algebra $\mathfrak{g}$ of the Lie group G is decomposed in the direct sum of any number of subspaces

$$\mathfrak{g} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots \oplus \mathcal{A}_m \,, \tag{1}$$

then the mapping

$$\Phi : \mathfrak{g} \to G, \;\; \Phi(X) = \exp a_1 \cdot \exp a_2 \cdot \ldots \cdot \exp a_m \,,$$

assuming for any $X \in \mathfrak{g}$ that $a_i \in \mathcal{A}_i \,, i = 1, \ldots, m$ are components of a vector X in decomposition (1), is a diffeomorphism from some neighborhood of $(0, 0, \ldots, 0)$ in $\mathfrak{g}$ to a neighborhood of the identity element in group G.

Canonical charts on the Lie group

The mapping $\Phi(X) = \exp a_1 \cdot \exp a_2 \cdot \ldots \cdot \exp a_m$ defines the *canonical charts* on the group manifold G.
For $m = 1$, - *canonical coordinates of the first kind*.
For $m = \dim T_e(G)$, - *canonical coordinates of the second kind*.

Aiming to describe : $SU(2) \times SU(2) \backslash SU(4)/T^3$.

We extract two subspaces in the Lie algebra $\mathfrak{su}(4)$ – its Cartan subalgebra and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ subalgebra, the corresponding complement of a vector $X \in \mathfrak{su}(4)$ affords for the coordinates of the double coset and its exponential mapping provides the sought-for parameterization of the double coset $SU(2) \times SU(2) \backslash SU(4)/T^3$.

The direct sum decomposition of $\mathfrak{su}(4)$ algebra

Proposition I: The $\mathfrak{su}(4)$ algebra admits decomposition into the following direct sum of subspaces:

$$\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{h}\,, \tag{2}$$

where $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{su}(4)$, $\mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and components $\mathfrak{a}$ and $\mathfrak{a}'$ in (2) are 3-dimensional Abelian subalgebras such that

$$[\mathfrak{a}'\,, \mathfrak{a}] \subseteq \mathfrak{k}\,. \tag{3}$$

Furthermore, the decomposition elements obey:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}\,, \quad [\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{k}\,, \quad [\mathfrak{h}, \mathfrak{a}'] \subseteq \mathfrak{k}\,, \quad [\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{a} \oplus \mathfrak{a}'\,. \tag{4}$$

$K\mathcal{A}T$ -decomposition of $SU(4)$

Proposition II: The decomposition of $\mathfrak{su}(4)$ defined in the Proposition I determines the canonical coordinates in the exponential map $\exp : \mathfrak{su}(4) \to \mathrm{SU}(4)$, of the following form:

$$g := K\,\mathcal{A}\,T^3\,, \qquad g \in SU(4)\,, \tag{5}$$

where K represents subgroup

$$K := \exp(\mathfrak{k}) = \exp(\mathfrak{su}(2)) \times \exp(\mathfrak{su}(2))\,,$$

T^3 is the maximal torus in $SU(4)$ and factor $\mathcal{A}$ associated to the sought-for double coset is product of two conjugated copies of the maximal Abelian subgroup of $SU(4)$:

$$\mathcal{A} := \exp(\mathfrak{a})\exp\big(\mathfrak{a}'\big)$$

Fano basis and adapted direct sum decomposition

The Fano basis of $\mathfrak{su}(4)$ algebra is constructed via tensor products,

$$\sigma_{\mu\nu} = \frac{1}{2\imath}\,\sigma_\mu \otimes \sigma_\nu\,, \qquad \sigma_\mu := (\mathbb{I}_2\,, \boldsymbol{\sigma})\,, \quad \boldsymbol{\sigma} - \text{Pauli matrices.}$$

In a single index notation, $\mathfrak{su}(4)$ algebra basis $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_{15}\}$ is

$$\boldsymbol{\lambda} := \{\sigma_{10}, \sigma_{20}, \sigma_{30}, \sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}, \sigma_{33}\}\,.$$

The sought-for decomposition $\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{h}$ reads:

$$\mathfrak{a} := \mathcal{A}_1\,, \qquad \mathfrak{a}' := \mathcal{A}_2\,, \qquad \mathfrak{k} := \mathcal{A}_3 \oplus \mathcal{A}_4\,, \qquad \mathfrak{h} := \mathcal{A}_5\,,$$

where $\mathcal{A}_i = \text{span}_{\mathbb{R}}(\boldsymbol{\Lambda}_i)$ are constructed by splitting up the basis $\boldsymbol{\lambda} = \bigcup_{i=1}^{5} \boldsymbol{\Lambda}_i$ into five complementary subsets $\boldsymbol{\Lambda}_1 = \{\lambda_1, \lambda_4, \lambda_7\}$, $\boldsymbol{\Lambda}_2 = \{\lambda_9, \lambda_{11}, \lambda_{13}\}$, $\boldsymbol{\Lambda}_3 = \{\lambda_2, \lambda_8, \lambda_{14}\}$, $\boldsymbol{\Lambda}_4 = \{\lambda_5, \lambda_{10}, \lambda_{12}\}$ and $\boldsymbol{\Lambda}_5 = \{\lambda_3, \lambda_6, \lambda_{15}\}\,.$

Explicit coordinate form; $K-$factor

Local subgroup: $\mathrm{K}{=}SU(2)\times SU(2)\,.$

The K factor in KAT-decomposition of $SU(4)$,

$$K=\exp(\mathfrak{k})\,,\qquad \mathfrak{k}=\mathrm{span}_{\mathbb{R}}(\lambda_2,\lambda_8,\lambda_{14},\lambda_5,\lambda_{10},\lambda_{12})\,.$$

Noting that the standard inclusion of $\mathfrak{su}(2)\oplus\mathfrak{su}(2)\hookrightarrow\mathfrak{su}(4)$ is given by $\mathrm{R}^\dagger\mathfrak{k}\mathrm{R}$ with the "magic" 4×4 unitary matrix

$$\mathrm{R}=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&-i\\0&-i&-1&0\\0&-i&1&0\\1&0&0&i\end{pmatrix},$$

then $K=\mathrm{R}^\dagger\left(\exp\left(\imath\sum_i^3\varphi_i\sigma_i\right)\otimes\exp\left(\imath\sum_i^3\psi_i\sigma_i\right)\right)\mathrm{R}\,.$

Hence, two copies of $SU(2)$ coordinates, φ and ψ, parameterize K-factor in KAT-decomposition.

Explicit coordinate form; $\mathcal{A}-$factor

$\mathcal{A}$ factor is identified to $\mathbb{R}^6$ by introducing coordinates α_i and β_i:

$$\mathcal{A} = \mathcal{A}_1\mathcal{A}_2 \quad \text{with} \quad \mathcal{A}_1 = \exp(\boldsymbol{\alpha}, \boldsymbol{\Lambda}_1)\,, \;\; \mathcal{A}_2 = \exp(\boldsymbol{\beta}, \boldsymbol{\Lambda}_2)\,.$$

$\mathcal{A}_1$ and $\mathcal{A}_2$ are diagonalizible via the orthogonal transformations,

$$\mathcal{A}_1 = \mathcal{S}_1\,\mathrm{diag}(\boldsymbol{\alpha})\,\mathcal{S}_1^T\,, \quad \mathcal{A}_2 = \mathcal{S}_2\,\mathrm{diag}(\boldsymbol{\beta})\,\mathcal{S}_2^T\,,$$

$$\mathcal{S}_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \; \mathcal{S}_2 = \frac{1}{2}\begin{pmatrix} -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\mathrm{diag}(\boldsymbol{\alpha}) = ||e^{-\frac{\imath}{2}(\alpha_1+\alpha_2+\alpha_3)}, e^{\frac{\imath}{2}(\alpha_1+\alpha_2-\alpha_3)}, e^{\frac{\imath}{2}(\alpha_1-\alpha_2+\alpha_3)}, e^{\frac{\imath}{2}(-\alpha_1+\alpha_2+\alpha_3)}||$$

$$\mathrm{diag}(\boldsymbol{\beta}) = ||\, e^{-\frac{\imath}{2}(\beta_1+\beta_2+\beta_3)}, e^{\frac{\imath}{2}(\beta_1+\beta_2-\beta_3)}, e^{\frac{\imath}{2}(\beta_1-\beta_2+\beta_3)}, e^{\frac{\imath}{2}(-\beta_1+\beta_2+\beta_3)}||$$

Analyticity domain of KAT- parameters

Theorem on exponential mapping

Let G be a compact and connected matrix Lie group. The function $\exp_A$ is analytic on a bounded open neighborhood of the origin:

$$\mathcal{U} = \{A \in \mathfrak{g} \,|\, |\mathrm{Im}[\mathrm{Eigenvalue}(A)]| < \pi\}\,. \tag{6}$$

Eq.(6) determines the chart with KAT-parameters covering almost the whole SU(4). Particularly, it holds for 3-tuples $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ each from the regular octahedron:

$$\begin{aligned}
|\alpha_1 + \alpha_2 + \alpha_3| &< 2\pi\,, \qquad & |\beta_1 + \beta_2 + \beta_3| &< 2\pi\,,\\
|\alpha_1 + \alpha_2 - \alpha_3| &< 2\pi\,, \qquad & |\beta_1 + \beta_2 - \beta_3| &< 2\pi\,,\\
|\alpha_1 - \alpha_2 + \alpha_3| &< 2\pi\,, \qquad & |\beta_1 - \beta_2 + \beta_3| &< 2\pi\,,\\
|\alpha_1 - \alpha_2 - \alpha_3| &< 2\pi\,, \qquad & |\beta_1 - \beta_2 - \beta_3| &< 2\pi\,.
\end{aligned}$$

Analyticity domain of $\mathcal{A}$-factor

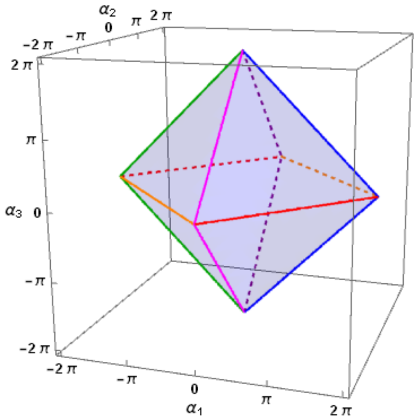


Figure 1: The regular octahedron as the domain of the analiticity of $\mathcal{A}_1$-factor in KAT-decomposition

Fundamental domain of KAT- parameters

The symmetry of KAT -decomposition.

Problem: Find a subgroup $S \subset SU(4)$ such that

$$s\,\mathcal{A}_i\,s^{-1} \in \mathcal{A}_i\,, \quad \forall\, s \in S, \quad \text{and} \quad \forall\, i = 1, 2, 3, 4, 5\,.$$

Answer: The decomposition symmetry group is the Klein four-group V_4, the order 4 non-cyclic subgroup of the symmetric group $\mathfrak{S}_4$.

Hence, reduction of the analiticity domain to the fundamental one is in order. As a result, the fundamental domain of the Abelian factors $\mathcal{A}_1$ and $\mathcal{A}_2$ geometrically is given by the pair of factor spaces:

$$\frac{\text{Regular octahedron}}{\text{Klein four-group } V_4}$$

Discussion

Future plans

- Generalization to the case of the non-generic global orbits $\mathrm{Orb}_{SU(4)}$;
- Description of the entanglement space of 2-qubits $\mathfrak{P}_4/SU(2) \times SU(2)$ in terms of the two octahedron coordinates and simplex of the density matrix eigenvalues;
- Construction of the Wigner function of 2-qubits in terms of the introduced parameterization of $SU(4)$.