Computing of tropical sequences associated with Somos sequences in Gfan package

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Abstract. The main objective of this work is to study tropical recurrent sequences associated with Somos sequences. For a set of tropical recurrent sequences, D. Grigoriev put forward a hypothesis of stabilization of the maximum dimensions of solutions to systems of tropical equations given by polynomials, which depend on the length of the sequence under consideration. The validity of such a hypothesis would make it possible to calculate the dimensions of these solutions for systems of arbitrary length. The main purpose of this work is to compute tropical sequences associated with Somos sequences using the Gfan package and to test the Grigoriev hypothesis.

Introduction

Tropical mathematics is a young area of modern mathematics related to the study of semirings with idempotent addition. Despite its novelty, it has already found its application in algebra, geometry, mathematical physics, biology, economics, neural network theory, dynamic programming, and other areas.

This work is a continuation of the work [1], which was devoted to tropical linear recurrent sequences. As part of this work, tropical sequences associated with Somos sequences are computed in the Gfan package.

Gfan is a software package for computing universal Gröbner bases, some related geometric objects (Gröbner fans) and tropical varieties, developed in 2005 by A. Jensen, based on the algorithms described and developed in his dissertation [2].

1. Formulation of the problem

One of the main objects of tropical mathematics is the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $x \oplus y := \max\{x, y\}$, $x \otimes y := x + y$. Tropical mathematics has its analogues of polynomial algebra, linear algebra and other areas of mathematics

[3]. Taking the minimum can be considered as tropical addition, then the additional element to the set of real numbers will be plus infinity.

Let $k \geq 2$ be a natural number and

$$\alpha = \{\alpha_i | 1 \leq i \leq [k/2]\}, \quad x = \{x_j | -k/2 < j \leq [k/2]\}$$

- two sets of independent formal variables in the amount of $[k/2]$ in the first case and k in the second. The sequence of rational functions Somos-k of variables from α and x, $S_k(n) = S_k(n; \alpha; x)(n \in \mathbb{Z})$, is defined by the recursive relation

$$S_k\left(n + \left[\frac{k+1}{2}\right]\right) S_k\left(n - \left[\frac{k}{2}\right]\right) = \sum_{1 \leq i \leq k/2} \alpha_i S_k\left(n + \left[\frac{k+1}{2}\right] - i\right) S_k\left(n - \left[\frac{k}{2}\right] + i\right).$$

In this work, we study the tropical sequences $p_k(n)$ associated with $S_k(n)$ that satisfy the recurrent relation

$$p_k\left(n + \left[\frac{k+1}{2}\right]\right) + p_k\left(n - \left[\frac{k}{2}\right]\right) = \min_{1 \leq i \leq k/2} \left\{p_k\left(n + \left[\frac{k+1}{2}\right] - i\right) + p_k\left(n - \left[\frac{k}{2}\right] + i\right)\right\}.$$

An interesting fact is that the tropical analogue of such sequences is related to the classical Somos sequences by some relation. It was proved in [4] that $S_k(n)$ is a Laurent polynomial in the initial variables x_j and an ordinary polynomial in α_i. Therefore, it can be written as

$$S_k(n) = \left(\prod_{-k/2 < j \leq [k/2]} x_j^{p_k^{(j)}(n)}\right) P_k(n),$$

where $P_k(n) = P_k(n; \alpha; x)$ are polynomials with integer coefficients and $p_k^{(j)}(n)$ are integer sequences.

In this work, we will consider all solutions of the finite sequences $p_k(n)$ with $0 \leq n \leq s$ for $k = 4$ and $k = 5$.

2. Computations of Somos-4 sequences in the Gfan package

To compute the sequences $p_4(n)$, we consider the sequences

$$q_4(n) = \Delta^2 p_4(n) = \Delta p_4(n+1) - \Delta p_4(n) = p_4(n+2) - 2p_4(n+1) + p_4(n).$$

Then the tropical relations will look like

$$q_4(n-1) + q_4(n) + q_4(n+1) + \max\{0, q_4(n)\} = 0$$

For computation in the Gfan package, we reduce this relation to a tropical polynomial. Let $y_n = q_4(n)$. Then we get

$$\max\{y_{n-1} + y_n + y_{n+1}, y_{n-1} + 2y_n + y_{n+1}\} = y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1}$$

To find solutions to this relation, we find tropical prevarieties. Since tropical prevarieties are the set of nonsmoothness of a tropical polynomial, the difficulty for this is that this polynomial is equal to zero. To solve this problem, add 0 as a term to the tropical polynomial

$$y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1} \oplus 0.$$

We can notice that the system of tropical polynomials $\max\{y_{n-1} + y_n + y_{n+1}, y_{n-1} + 2y_n + y_{n+1}\}$ for $1 \le n \le s-1$ reaches a maximum greater than zero only in two cases: $y_0 > 0$ and $y_s > 0$. Because of this, the addition of the term 0 does not affect the dimension of the tropical prevariety. Therefore, to compute the dimensions of the solution space, these cases were excluded. This idea was verified experimentally in the Gfan package for computed finite sequences.

Tropical prevarieties can be computed using the function **gfan_tropicalintersection** of the Gfan package [5]. Denote the dimension of the solution space by d_s. The obtained dimensions of the solution space are presented in Table. 1. The obtained solutions correspond to the calculations carried out in [6].

TABLE 1. Dimensions of the Somos-4 solution space

s	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d_s	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6

3. Computations of Somos-5 sequences in the Gfan package

The tropical relations in this case look like

$$q_5(n-2) + q_5(n-1) + q_5(n) + q_4(n+1) + \max\{0, q_5(n-1) + q_5(n)\} = 0.$$

Let $y_n = q_5(n)$. Then we get

$$\max\{y_{n-2} + y_{n-1} + y_n + y_{n+1}, y_{n-2} + 2y_{n-1} + 2y_n + y_{n+1}\} = 0.$$

Then we consider tropical prevarieties for the following polynomial

$$y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1} \oplus 0.$$

We can notice that the system of tropical polynomials $\max\{y_{n-2} + y_{n-1} + y_n + y_{n+1}, y_{n-2} + 2y_{n-1} + 2y_n + y_{n+1}\}$ for $2 \le n \le s-1$ reaches a maximum greater than zero only in three linear cases: $y_0 > 0$, $y_s > 0$ and $y_n = (-1)^n$. Because of this, the addition of the term 0 does not affect the dimension of the tropical prevariety. Therefore, to compute the dimensions of the solution space, these cases were excluded.

The obtained dimensions of the solution space are presented in Table. 2.

TABLE 2. Dimensions of the Somos-5 solution space

s	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
d_s	3	3	3	4	4	4	4	4	5	5	6	6	6	6	6	7	7	8

Conclusion

Based on the computed tropical prevarieties, we can make the assumption that for Somos-4 sequences $d_s = \left[\frac{s-2}{4}\right] + 2$. Then for such sequences the tropical entropy [1] takes the value $H = 1/4$. For systems of tropical polynomials $y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1}$ for $1 \leq n \leq s-1$ without addition 0, it is obtained that $d_s = 2$ for any s. Then for such sequences the tropical entropy takes the value $H = 0$.

Based on the computed tropical prevarieties, we can make the assumption that for Somos-5 the tropical entropy takes the value $H = 2/7$. For systems of tropical polynomials $y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1}$ for $2 \leq n \leq s-1$ without addition 0, it is obtained that $d_s = 3$ for any s. Then for such sequences the tropical entropy takes the value $H = 0$.

For the Somos-6 and Somos-7 cases, it is more difficult to find the dimension of the solution space using the computation of tropical prevarieties. The problem is that before adding zero as a tropical monomial to tropical polynomials, the solution space of finite sequences increases.

The results obtained are consistent with Grigoriev's hypothesis on the stabilization of the maximum dimensions of solutions to systems of tropical sequences.

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