Computing of tropical sequences associated with Somos sequences in Gfan package

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1. **Basic objects of tropical math**
   - Tropical semiring
   - Tropical polynomial
   - Tropical prevariety

2. **Somos sequences**
   - Classical Somos sequences
   - Computing of tropical prevarieties, associated with Somos sequences

Introduction

The main goals of this work are

- computing finite tropical sequences associated with Somos sequences using the Gfan package;
- testing D.Yu. Grigoriev hypothesis.

For a set of tropical recurrent sequences, D.Yu. Grigoriev put forward a **hypothesis** of stabilization of the maximum dimensions of solutions to systems of tropical equations given by polynomials, which depend on the length of the sequence under consideration.

Gfan is a software package for computing universal Gröbner bases, some related geometric objects (Gröbner fans) and tropical varieties, developed in 2005 by A. Jensen, based on the algorithms described and developed in his dissertation.

Tropical semiring

Tropical semiring

Tropical semiring is a semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ with tropical operations:

$$x \oplus y := \max(x, y) \quad \text{and} \quad x \otimes y := x + y.$$

Examples:

$$5 \oplus 2 = 5, \quad 5 \otimes 2 = 7.$$

One of the main differences between tropical and classical mathematics is that tropical addition is **idempotent** $x \oplus x = x$.
It is also important that there is no subtraction operation in a tropical semiring. The equation $a \oplus x = -\infty$ has no solution for any a except minus infinity itself.

Tropical monomial

Tropical monomial

Let $x_1, \dots, x_n$ be variables which represent elements in the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$. By commutativity, we can sort the product and write *tropical monomial* in the usual notation:

$$q(x_1, \dots, x_n) = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n} = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n.$$

Tropical polynomial

Tropical polynomial

A *tropical polynomial* is a finite linear combination of tropical monomials:

$$p(x_1,\ldots,x_n) = \bigoplus_j \left(a_j \otimes x_1^{\otimes i_{j_1}} \otimes \cdots \otimes x_n^{\otimes i_{j_n}}\right) =$$

$$= \max_j \left(a_j + i_{j_1}x_1 + \cdots + i_{j_n}x_n\right).$$

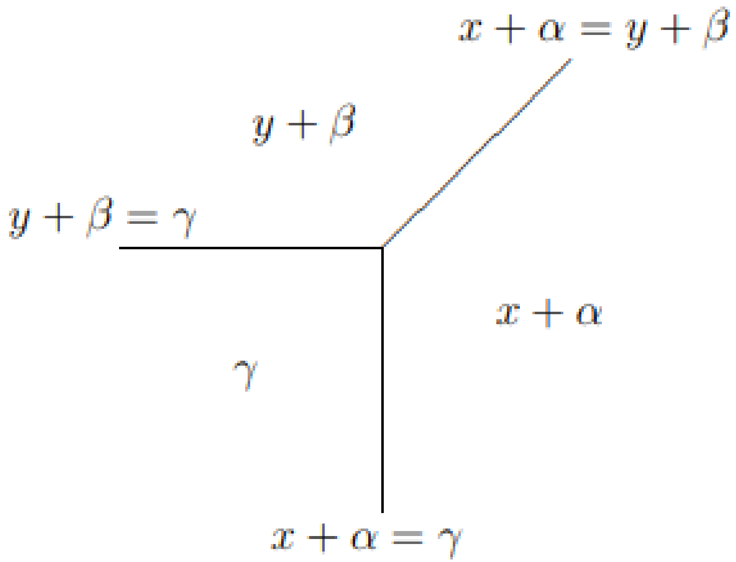
Tropical root

$x = (x_1,\ldots,x_n)$ is a *tropical root* of th polynomial p, if maximum of tropical monomials q_j is reached at least two different values of j.

Tropical line

Let's consider a tropical line as an example

$$f(x, y) = \alpha \otimes x \oplus \beta \otimes y \oplus \gamma = \max(\alpha + x, \beta + y, \gamma)$$

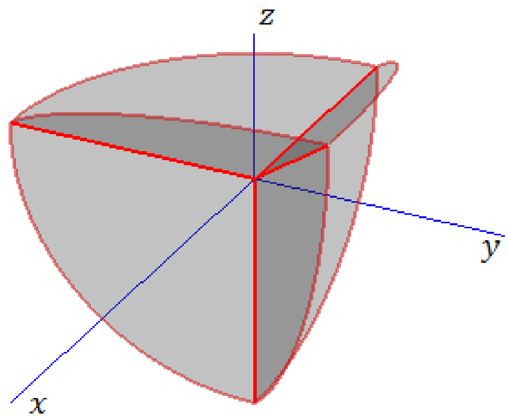


Tropical hypersurface

Tropical hypersurface

Tropical hypersurface $\mathcal{T}(f)$ of the polynomial f is the set of tropical roots of tropical polynomial $Trop(f)$. *Tropicalization* $Trop(f)$ is the transition from objects of classical mathematics to objects of tropical, and the coefficients at monomials are assumed to be equal to zero.

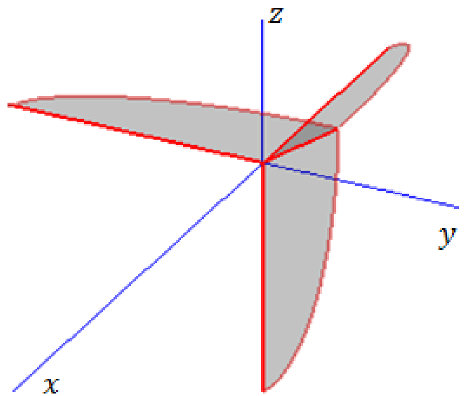
For example, consider the tropical hypersurface of polynomial $g = x + 2y + z + 1$. $Trop(g) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 0 = \max(x, y, z, 0)$.



Tropical prevariety

Tropical prevariety of a system of polynomials $f_1, \dots, f_n$ is the finite intersection of tropical hypersurfaces $\mathcal{T}(f_1) \cap \dots \cap \mathcal{T}(f_n)$.

For example, consider the tropical hypersurface of polynomial $g = x + 2y + z + 1$ and $h = x + y + 2z$.
$Trop(g) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z \oplus 0 = \max(x, y, z, 0)$;
$Trop(h) = 0 \otimes x \oplus 0 \otimes y \oplus 0 \otimes z = \max(x, y, z)$.



Classical Somos sequences

Somos-k sequence

Let $k \geq 2$ be natural and

$$\alpha = \{\alpha_i | 1 \leq i \leq [k/2]\}, \quad x = \{x_j | -k/2 < j \leq [k/2]\}$$

are two sets of independent formal variables in the amount of $[k/2]$ in the first case and k in the second. **The sequence of rational Somos-k functions** of variables from α and x, $S_k(n) = S_k(n; \alpha; x)(n \in \mathbb{Z})$, is determined by the recursive relation

$$S_k\left(n + \left[\frac{k+1}{2}\right]\right) S_k\left(n - \left[\frac{k}{2}\right]\right) = \sum_{1 \leq i \leq k/2} \alpha_i S_k\left(n + \left[\frac{k+1}{2}\right] - i\right) S_k\left(n - \left[\frac{k}{2}\right] + i\right)$$

For example, for $k = 4, \alpha_1 = \alpha_2 = 1$ the recurrence relation has the form

$$S_4(n+2)S_4(n-2) = S_4(n+1)S_4(n-1) + S_4^2(n)$$

If $S_4(0) = S_4(1) = S_4(2) = S_4(3) = 1$, then $S_4(4) = 2$, $S_4(5) = 3$, $S_4(6) = 7$, $S_4(7) = 23$, $S_4(8) = 59$, $S_4(9) = 314$, $S_4(10) = 1529$, $S_4(11) = 8209$, $S_4(12) = 83313$.

Classical Somos sequences

For $k = 2, \alpha = \{\alpha_1\}, x = \{x_0, x_1\}$

$$S_2(n+1)S_2(n-1) = \alpha_1 S_2^2(n)$$

using induction on n we can obtain the equality

$$S_2(n) = \alpha_1^{n(n-1)/2} x_0^{1-n} x_1^n.$$

For $k = 3, \alpha = \{\alpha_1\}, x = \{x_{-1}, x_0, x_1\}$

$$S_3(n+2)S_3(n-1) = \alpha_1 S_3(n+1)S_3(n)$$

using induction on n we can obtain the equality

$$S_3(n) = \begin{cases} \alpha_1^{n^2/4} x_{-1}^{-n/2} x_0 x_1^{n/2}, \text{if } n \text{ is even,} \\ \alpha_1^{(n^2-1)/4} x_{-1}^{(1-n)/2} x_1^{(1+n)/2}, \text{if } n \text{ is odd.} \end{cases}$$

Tropical Somos sequences

Tropical Somos-k sequences

Tropical sequence $p_k(n)$, associated with Somos-k sequence $S_k(n)$ satisfies the recurrence relation

$$p_k\left(n+\left[\frac{k+1}{2}\right]\right)+p_k\left(n-\left[\frac{k}{2}\right]\right)=\min_{1\le i\le k/2}\left\{p_k\left(n+\left[\frac{k+1}{2}\right]-i\right)+p_k\left(n-\left[\frac{k}{2}\right]+i\right)\right\}.$$

An interesting fact is that the tropical analogue of such sequences is related to the classical Somos sequences by the following relation

$$S_k(n)=\left(\prod_{-k/2<j\le[k/2]} x_j^{p_k^{(j)}(n)}\right)P_k(n),$$

where $P_k(n)=P_k(n;\alpha;x)$ is polynomial with integer coefficients and $p_k^{(j)}(n)$ are tropical sequences.

Somos-4

For $k = 4$ tropical recurrent relation sequence $p_4(n)$ looks like

$$p_4(n+2) + p_4(n-2) = \min\left\{p_4(n+1) + p_4(n-1), 2p_4(n)\right\}.$$

To compute sequences change the variable

$$q_4(n) = \Delta^2 p_4(n) = \Delta p_4(n+1) - \Delta p_4(n) = p_4(n+2) - 2p_4(n+1) + p_4(n)$$

Then the tropical relations will look like

$$q_4(n) + 2q_4(n-1) + q_4(n-2) = \min\{0, q_4(n-1)\}.$$

or

$$q_4(n) + q_4(n-1) + q_4(n-2) + \max\{0, q_4(n-1)\} = 0.$$

Let $y_n = q_4(n)$, using distributivity we get

$$\max\{y_{n-2} + y_{n-1} + y_n, y_{n-2} + 2y_{n-1} + y_n\} = 0$$

or in tropical form

$$f_4 = y_{n-2} \otimes y_{n-1} \otimes y_n \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n = 0.$$

Somos-4

$$f_4 = y_{n-2} \otimes y_{n-1} \otimes y_n \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n = 0.$$

Computing the tropical prevariety of a system of polynomials f_4 with different n will allow us to find out the points at which the monomials $y_{n-2} \otimes y_{n-1} \otimes y_n$ and $y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n$ are equal. To find equality to zero, we tropically add 0 to the polynomial f_4.

$$\hat{f}_4 = f_4 \oplus 0 = y_{n-2} \otimes y_{n-1} \otimes y_n \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n \oplus 0.$$

Let compute the tropical prevariety of the system of polynomials f_4, compute the tropical prevariety of the system of polynomials $\hat{f}_4$, and subtract from the second prevariety those rays and cones that do not satisfy the equation.

Table 1: Dimensions of tropical prevarieties at $k = 4$

s	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim(T(f_4))$	2	2	2	2	2	2	2	2	2	2	2	2	2	2
$\dim(T(\hat{f}_4))$	2	2	2	2	3	3	3	3	4	4	4	4	5	5
$d_s(f_4)$	2	2	2	2	2	3	3	3	3	4	4	4	4	5

Somos-5

For $k = 5$ tropical recurrent relation sequence $p_4(n)$ looks like

$$p_5(n+3) + p_5(n-2) = \min\left\{p_5(n+2) + p_5(n-1), p_5(n+1) + p_5(n)\right\}.$$

To compute sequences change the variable

$$q_5(n) = \Delta^2 p_5(n)$$

Then the tropical relations will look like

$$q_5(n-2) + q_5(n-1) + q_5(n) + q_5(n+1) + \max\{0, q_5(n-1) + q_5(n)\} = 0.$$

Let $y_n = q_5(n)$, using distributivity we get

$$f_5 = y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1} = 0.$$

Somos-5

$$f_5 = y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1} = 0.$$

By analogy with the previous case, in this case we tropically add 0, obtaining a tropical polynomial

$$\hat{f}_5 = f_5 \oplus 0 = y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1} \oplus 0.$$

Table 2: Dimensions of tropical prevarieties at $k = 5$

s	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\dim(T(f_5))$	3	3	3	3	3	3	3	3	3	3	3	3	3	3
$\dim(T(\hat{f}_5))$	3	4	4	4	4	4	5	5	6	6	6	6	6	7
$d_s(f_5)$	3	3	3	4	4	4	4	4	5	5	6	6	6	6

Somos-6 and Somos-7

Table 3: Dimensions of tropical prevarieties at f_6 and $\hat{f}_6$

s	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\dim(T(f_6))$	4	4	4	4	5	5	5	5	6	6	6	6	7	7
$\dim(T(\hat{f}_6))$	4	4	5	5	6	6	7	7	8	8	9	9	10	10

Table 4: Dimensions of tropical prevarieties at f_7 and $\hat{f}_7$

s	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\dim(T(f_7))$	5	5	6	6	6	6	6	7	7	8	8	8	8	8
$\dim(T(\hat{f}_7))$	5	5	6	6	7	7	7	8	8	8	9	9	10	10

Conclusion

The results obtained are consistent with Grigoriev's hypothesis on the stabilization of the maximum dimensions of solutions to systems of tropical sequences. A complete proof of Grigoriev's hypothesis would allow us to construct a tropical analogue of the classical Hilbert polynomial.
As a continuation of the current work, the following tasks can be set:

- find solutions to tropical recurrent sequences associated with the Somos-6 and Somos-7 sequences;
- consider Somos sequences with non-zero α_i;
- consider nonlinear tropical recurrent sequences, in particular, tropical analogues of special functions.