Partially Ordered Derivations in Sequent Calculi for Nonstandard Logics

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Introduction

Nonstandard logics have become the center of modern studies in mathematical logic because these logics have numerous applications especially in the area of artificial intelligence. Characteristic features of such logics are:

- Nonstandard logical connectives or quantifiers requiring peculiar axioms or inference rules.
- Nonlogical axioms specifying properties of concrete many-sorted predicates and functions for a certain set of domains. Mixed axioms applicable to arbitrary logical formulas but also referring to concrete predicates or functions.

Sequent calculi for specifying nonstandard logics:

☑ *Formalization:* Arguably, the sequent notation is the most convenient formalism for logic specification. The simplicity of sequent calculi is due to separate inference rules for every connective and quantifier. Hilbert-style logical axioms can be embedded in sequent calculi.

☑ *Proof search:* Implementation of proof search procedures is straightforward. Sequent calculi support both top-down and bottom-up proof search. It is not realistic to expect highly efficient proof search procedures applicable to a wide variety of logics.

☐ *Normal forms:* Sequent calculi lack normal forms, Numerous derivations exist for any provable formula. Many rule pairs can be permuted. For some standard logics, permutation-free normal forms have been crafted, but these results have not been generalized. Normal forms are important in practice because they impose constraints on derivations and thus reduce proof search space.

Background

Cut elimination and analytic cuts

Local inference rule permutations

Comparison of derivation transformations and term rewriting

Normal forms of derivations for intuitionistic logics

Top-down proof search methods based on Maslov's inverse method
Bottom-up proof search methods based on focusing

Similarity between top-down proof search in sequent calculi with cut and resolution

Sequent Calculi

Sequents are expressions of the form $\{antecedent\} \vdash \{succedent\}$. Antecedents and succedents are multisets of logical formulas. Axioms are sequents. Premises and conclusions of inference rules are sequents:

$$\frac{\{premise_1\} \quad ... \quad \{premise_k\}}{\{conclusion\}}$$

The outcome of inference is sequents of the form $\vdash G$ where formula G is called a goal.
A calculus is called consistent if sequent $\vdash$ is not derivable.

Notation: Metaformulas are built from formula metavariables, substitutions, logical connectives, and quantifiers. Expressions having the following forms are also called metaformulas: $A\theta$ and $A^*\theta$ where A is a formula metavariable and $\theta = \{x_1/t_1, ..., x_k/t_k\}$ is a substitution. The expression $A^*\theta$ means that the formula matching metavariable A is the only formula in its sequent where variables of the substitution θ occur. Multiset metavariables and expressions of the form $\diamond\Pi$, where Π is a multiset metavariable and $\diamond$ is a unary connective, are called metasets. Sequents in logical rules are comprised of metaformulas and metasets.

Inference rules and axioms: Inference rules in sequent calculi are split into structural and logical. The structural rules (contraction, weakening, cut) are essentially universal for all of the calculi whereas logical rules vary. Logical axioms are comprised of metaformulas and possibly formulas. Standard logical axioms contain metavariables only. Nonlogical axioms may contain formulas only. We assume that any axiom has no instances in which there are identical formula in the antecedent or in the succedent.

Definitions

Definition 1. *If all metaformulas/metasets containing the same metavariable are identical in a logical rule, they are called context. All other metaformulas/metasets from the conclusion are called principal. All other metaformulas/metasets from premises are called active. Formulas matching metaformulas/metasets are also called principal, active, context as their respective metaformulas/metasets.*

Definition 2. *A multi-premise logical inference rule is called multiplicative if no context metavariable from one premise occurs in the other premises of the rule. A multi-premise logical inference rule is called additive if every context metavariable occurs in all premises of the rule.*

Definition 3. *A logical rule is called clear if*
- *Every metavariable from any of its premises also occurs in the conclusion.*
- *No multiset metavariable occurs in both antecedents and succedents.*
- *It is single-premise, multiplicative, or additive.*
- *It has one principal metaformula and no principal metasets.*
- *Every premise has one active metaformula if the rule has multiple premises.*
- *Every active formula is a subformula of the principal formula or a result of applying a substitution to such subformula.*
- *The context of any premise antecedent or succedent, if present, is a single multiset metavariable.*
- *There are no constraints on the application of this rule except for those given by metaformulas.*

Definition 4. *A clear rule is called simple if it has a single premise with one active metaformula or it is multiplicative.*

Examples of Rules and Axioms

Standard first-order logics can be specified with using only clear rules, and most of their logical rules can be made simple.

Nonstandard inference rules:

$$\frac{\mathcal{G}\Gamma, \Gamma \vdash A, \mathcal{H}\Delta, \mathcal{H}\Sigma}{\mathcal{G}\Gamma \vdash \mathcal{G}A, \Delta, \mathcal{H}\Sigma}\, R\mathcal{G} \qquad\qquad \frac{\Gamma \vdash A}{\mathcal{O}\Gamma \vdash \mathcal{O}A}\, D\mathcal{O}$$

Nonstandard logical axiom:

$$\vdash (A \supset B) \vee (B \supset A)$$

Logical axioms with concrete predicates or functions:

$$y = z, A\{x/y\} \vdash A\{x/z\}$$

$$A\{x/0\}, A\{x/k\} \supset A\{x/k+1\} \vdash A\{x/n\} \qquad (k, n \text{ do not occur in } A)$$

Nonlogical axioms:

$$\vdash t(x, x) \qquad\qquad e(x, y), t(y, z) \vdash t(x, z)$$

Nonstandard inference rules are one obstacle for cut elimination. All kinds of additional axioms are another obstacle. Cut is an essential rule for calculi incorporating domain knowledge.

Contraction and Weakening Merging

Let $[\Gamma]$ denote the result of applying zero or more possible contractions to multiset Γ. If a calculus does not include contraction, then $[\Gamma] = \Gamma$. If a calculus includes both weakening and contraction, then the [ ] operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic, i.e. none, some, or all contractions are applied.

Let us modify the conclusion of cut and all logical rules by applying [ ] to both the antecedent and the succedent of the conclusion of these rules. The calculus obtained from calculus L by applying [ ] is denoted L'.

Proposition 1. *For every sequent calculus L, any derivation can be transformed into a L' derivation with the same endsequent and vice versa.*

Theorem 1. *The contraction rules are admissible in any L' sequent calculus.*

Let us modify any calculus L' with weakening. For any single-premise clear rule having more than one metaformula, let us add logical rules to this calculus. Each additional rule is obtained by removing one or more metaformula but not all of them from the premise. Also, additive clear rules are replaced by multiplicative rules if this calculus has both contraction and weakening. The modified calculus will be denoted L". For any calculus L' without weakening, L" is identical to L'.

Proposition 2. *For every sequent calculus L, any L' derivation can be transformed into a L" derivation with the same endsequent and vice versa.*

Theorem 2. *For every consistent sequent calculus L, any L derivation of sequent $\vdash G$ can be transformed into such L" derivation with the same endsequent and without the contraction rules that every weakening rule is either followed by another weakening rule or by a logical rule that is additive or is not clear.*

Partially Ordered Derivations

Definition 5. *Strict order relation $\succ$ on formulas and terms is called a simplification order if it satisfies the following conditions:*
- *there is no infinite sequence of formulas $F_0 \succ F_1 \succ ...$*
- *if L/l is a particular formula/term occurrence in formula E, formula F is obtained from E by replacing this occurrence with formula/term R/r, and $L \succ R/l \succ r$, then $E \succ F$*
- *if R/r is a subformula/subterm of formula/term L/l, then $L \succ R/l \succ r$*
- *if $L, R/l, r$ are formulas/terms and $L \succ R/l \succ r$, then $L\theta \succ R\eta/l\theta \succ r\theta$ for any substitutions θ and η*

Definition 6. *Formula A is maximal (minimal) with respect to the set of formulas $\mathcal{S}$ if $B \succ A$ ($A \succ B$) does not hold for any other formula $B \in \mathcal{S}$.*

Theorem 3. *For any consistent sequent calculus L and simplification order $\succ$ on its formulas and terms, any L derivation of sequent $\vdash G$ can be transformed into a L" derivation with the same endsequent, without the contraction, with weakening rules satisfying Theorem 2, and such that the following holds for any two consecutive inference rules:*
1) If both rules are cut, then the upper cut formula is maximal with respect to the lower cut formula.
2) If the upper rule is simple and the lower rule is cut, then the cut formula is principal in the upper rule.
3) If both rules are simple, then the principal formula of the lower rule is maximal with respect to the principal formula of the upper rule.

Theorem 4. *For every consistent sequent calculus L with both weakening and contraction, Theorem 3 holds even if the word 'simple' is changed for the word 'clear'.*

Conclusion

This research is a step forward in finding normal forms of derivations in sequent calculi for nonstandard logics. Normal forms of sequent derivations are currently known for variants of intuitionistic logic only.

Finding normal forms of derivations is somewhat orthogonal to the research of proof search methods. The constraints imposed by normal forms can be potentially used in arbitrary proof search methods.

Our ordered form is more suitable for top-down proof search because it established for multiplicative rules. Bottom-up proof search works better with additive rules. Bottom-up proof search does not work well in the presence of cut.

Our ordered form may be beneficial for calculi mixing Gentzen's and Hilbert's styles. If standard connectives and quantifiers are specified by logical rules and nonstandard connectives are specified by logical axioms, then entire derivation trees are ordered.