Partially Ordered Derivations in Sequent Calculi for Nonstandard Logics

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1. Introduction

Nonstandard logics have become the center of modern studies in mathematical logic because these logics have numerous applications especially in the area of artificial intelligence. These logics have nonstandard logical connectives requiring peculiar axioms or inference rules. These logics may also include nonlogical axioms specifying properties of concrete predicates and functions.

Sequent calculi are perhaps the most common logical formalism [4]. This formalism is simple and versatile. In addition to inference rules for logical connectives and quantifiers, sequent calculi can incorporate arbitrary axioms. Sequent calculi support both top-down and bottom-up proof search. Nonetheless, sequent calculi commonly lack normal forms of derivations. Many rule chains are permutable [2]. Reducing derivation choices arising from rule permutability is a challenging long-standing problem. For some standard logics, permutation-free sequent calculi have been crafted [3, 6, 1]. But these results have not been generalized.

The main result of this work is that inference in a variety of sequent calculi remains complete if it is restricted to derivations in which some pairs of consecutive inference rules are ordered. Additionally, weakening and contraction rules are merged with other rules which reduces choices during inference. These results are applicable to sequent calculi with non-standard inference rules and additional axioms including nonlogical ones. This research is a step forward in the quest for normal forms in various sequent calculi, in particular, in applied calculi with multiple axioms in which cut is heavily used [5].

2. Sequent Calculi

We use standard logical terminology [4]. Upper-case Latin letters are metavariables denoting formulas in inference rules and axioms. Upper-case Greek letters are metavariables denoting formula multisets. Usually, the outcome of inference is sequents of the form $\vdash G$ where formula G is called a goal. A calculus is called consistent if sequent $\vdash$ is not derivable.

Metaformulas are built from formula metavariables, substitutions, logical connectives, and quantifiers. Expressions having the following forms are also called metaformulas: $A\theta$ and $A^*\theta$ where $\theta = \{x_1/t_1, ..., x_k/t_k\}$ is a substitution. The expression $A^*\theta$ means that the formula matching metavariable A is the only formula in its sequent where variables of the substitution θ occur. Multiset metavariables and expressions of the form $\diamond\Pi$, where here Π is a multiset metavariable and $\diamond$ is a unary connective, are called metasets. Sequents in logical rules are comprised of metaformulas and metasets.

Inference rules in sequent calculi are split into structural and logical. The structural rules are essentially universal for all of the calculi whereas logical rules vary. Logical rules contain metaformulas and metasets. Logical and mixed axioms are comprised of metaformulas and possibly formulas. Nonlogical axioms may contain formulas only. We assume that any axiom has no instances in which there are identical formula in the antecedent or in the succedent.

Definition 1. *If all metaformulas/metasets containing the same metavariable are identical, they are called context. All other metaformulas/metasets from the conclusion are called principal. All other metaformulas/metasets from premises are called active. Formulas matching metaformulas/metasets are also called principal, active, context as their respective metaformulas/metasets.*

Definition 2. *A multi-premise logical inference rule is called multiplicative if no context metavariable from one premise occurs in the other premises of the rule. A multi-premise logical inference rule is called additive if every context metavariable occurs in all premises of the rule.*

Definition 3. *A logical rule is called clear if*
- *Every metavariable from any of its premises also occurs in the conclusion.*
- *No multiset metavariable occurs in both antecedents and succedents.*
- *It is single-premise, multiplicative, or additive.*
- *It has one principal metaformula and no principal metasets.*
- *Every premise has one active metaformula if the rule has multiple premises.*
- *Every active formula is a subformula of the principal formula or a result of applying a substitution to such subformula.*
- *The context of any premise antecedent or succedent, if present, is a single multiset metavariable.*
- *There are no constraints on the application of this rule except for those given by metaformulas.*

Definition 4. *A clear rule is called simple if it has a single premise with one active metaformula or it is multiplicative.*

3. Contraction and Weakening Merging

Let $[\Gamma]$ denote the result of applying zero or more possible contractions to multiset Γ. If a calculus does not include contraction, then $[\Gamma] = \Gamma$. If a calculus includes both weakening and contraction, then the [ ] operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic, i.e. none, some, or all contractions are applied.

Let us modify the conclusion of cut and all logical rules by applying [ ] to both the antecedent and the succedent of the conclusion of cut and all logical inference rules. The calculus obtained from calculus L by applying [ ] is denoted L'.

Proposition 1. *For every sequent calculus L, any derivation can be transformed into a L' derivation with the same endsequent and vice versa.*

Theorem 1. *The contraction rules are admissible in any L' sequent calculus.*

Let us modify any calculus L' with weakening. For any single-premise clear rule having more than one metaformula, let us add logical rules to this calculus. Each additional rule is obtained by removing one or more metaformula but not all of them from the premise. Also, additive clear rules are replaced by multiplicative rules if this calculus has both contraction and weakening. The modified calculus will be denoted L". For any calculus L' without weakening, L" is identical to L'.

Proposition 2. *For every sequent sequent calculus L, any L' derivation can be transformed into a L" derivation with the same endsequent and vice versa.*

Theorem 2. *For every consistent sequent calculus L, any L derivation of sequent $\vdash G$ can be transformed into such L" derivation with the same endsequent and without the contraction rules that every weakening rule is either followed by another weakening rule or by a logical rule that is additive or is not clear.*

4. Partially Ordered Derivations

Definition 5. *Strict order relation $\succ$ on formulas and terms is called a simplification order if it satisfies the following conditions:*

- *there is no infinite sequence of formulas $F_0 \succ F_1 \succ \ldots$*
- *if L/l is a particular formula/term occurrence in formula E, formula F is obtained from E by replacing this occurrence with formula/term R/r, and $L \succ R/l \succ r$, then $E \succ F$*
- *if R/r is a subformula/subterm of formula/term L/l, then $L \succ R/l \succ r$*
- *if $L, R/l, r$ are formulas/terms and $L \succ R/l \succ r$, then $L\theta \succ R\eta/l\theta \succ r\theta$ for any substitutions θ and η*

Definition 6. *Formula A is maximal (minimal) with respect to the set of formulas $\mathcal{S}$ if $B \succ A$ ($A \succ B$) does not hold for any other formula $B \in \mathcal{S}$.*

Theorem 3. *For any consistent sequent calculus L and simplification order $\succ$ on its formulas and terms, any L derivation of sequent $\vdash G$ can be transformed into a L" derivation with the same endsequent, without the contraction, with weakening rules satisfying Theorem* 2*, and such that the following holds for any two consecutive inference rules:*
1) If both rules are cut, then the upper cut formula is maximal with respect to the lower cut formula.
2) If the upper rule is simple and the lower rule is cut, then the cut formula is principal in the upper rule.
3) If both rules are simple, then the principal formula of the lower rule is maximal with respect to the principal formula of the upper rule.

Theorem 4. *For every consistent sequent calculus L with both weakening and contraction, Theorem* 3 *holds even if the word 'simple' is changed for the word 'clear'.*

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