Exact solutions to systems of second order ordinary differential equations by Decomposion method

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Abstract. The solvability conditions and exact solutions to the system of linear second order differential equations in terms of the abstract operator equations

$$\mathbb{B}X(t) = X''(t) - S(t)X'(t) - Q(t)X(t) = F(t),$$

with nonlocal multipoint and integral boundary conditions

$$M_0X(0)+\sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}, \quad N_0X'(0)+CX(0)+\sum_{i=1}^{n}[N_iX'(t_i)+V_iX(t_i)] = \vec{0},$$

by decomposition method is proposed in this paper, where $C, M_0, N_0, M_i, N_i, V_i$ are matrices, $\Psi_i(X)$ Fredholm integrals. In the case where the fundamental solution of the first order system is known, the fundamental solution of the corresponding second order system was obtained. The technique is easy to implement to any Computer Algebra System (CAS) and is economic and efficient compared to other existing methods.

Introduction

Everewhere below we denote by $\mathcal{X}$ the space $C[0,1]$ or $L_p(0,1)$ and by $\mathcal{X}_m$ the space of column vectors $X(t) = col(x_1(t),..,x_m(t))$ with elements from $\mathcal{X}$, i.e. $\mathcal{X}_m = C_m[0,1]$ or $\mathcal{X}_m = L_{p_m}(0,1)$. Denote also by $\mathcal{X}^i$ the space $C^i[0,1]$ or the Sobolev space $W_p^i(0,1)$, and by $\mathcal{X}_m^i$ the space $C_m^i[0,1]$ or $W_{p_m}^i(0,1)$, $i = 1,2$. We will also denote by 0_m the zero and by I_m the identity $m \times m$ matrices. By $\vec{0}$ we will denote the zero column vector.

Lemma 1. *Let $P(t), T(t)$ be $m{\times}m$ matrices with element from $\mathcal{X}$, the operators $A_1, A_2 : \mathcal{X}_m \to \mathcal{X}_m$ be defined by*

$$A_1Y(t) = Y'(t) - P(t)Y(t), \quad Y(t) \in D(A_1) = \mathcal{X}_m^1, \tag{1}$$

$$A_2X(t) = X'(t) - T(t)X(t), \quad X(t) \in D(A_2) = \mathcal{X}_m^1, \tag{2}$$

and $Z, \mathcal{Z}$ the fundamental matrices to the homogeneous equations $A_1Y(t)=\vec{0}$, $A_2X(t)=\vec{0}$, respectively, such that $Z(0)=I_m, \det\mathcal{Z}\neq 0$. Then the operators $\widehat{A}_1, \widehat{A}_2$ corresponding to the problems

$$\widehat{A}_1Y(t)=A_1Y(t)=F(t),\ D(\widehat{A}_1)=\{Y(t)\in D(A_1): Y(0)=\vec{0}\}, \tag{3}$$
$$\widehat{A}_2X(t)=A_2X(t)=Y(t),\ D(\widehat{A}_2)=\{X(t)\in D(A_2): X(0)=\vec{0}\}, \tag{4}$$

are correct and their unique solutions are given by

$$Y(t) = \widehat{A}_1^{-1}F(t)=Z(t)\int_0^t Z^{-1}(s)F(s)ds, \tag{5}$$
$$X(t) = \widehat{A}_2^{-1}Y(t)=\mathcal{Z}(t)\int_0^t \mathcal{Z}^{-1}(s)Y(s)ds. \tag{6}$$

Theorem 1. *Let the operators $A_1, A_2, \widehat{A}_1, \widehat{A}_2$, vectors X, Y, F and matrices $Z, \mathcal{Z}$ be defined as in Lemma 1, the vectors $\Psi = col(\Psi_1, ..., \Psi_n)\in\mathcal{X}_n^*$, and $Y(\vec{t}) = col(Y(t_1), ..., Y(t_k))$, $0<t_1<...<t_k\leq 1$, $M=(M_1,...,M_n)$ and $N=(N_1,...,N_k)$ be a $m\times(mn)$ and $m\times(mk)$ constant matrices, respectively, and M_i, N_j the $m\times m$ constant matrices, $i=0,1,...,n$, $j=0,1,...,k$. Then:*
(i) The operator $B_1:\mathcal{X}_m\to\mathcal{X}_m$, corresponding to the problem

$$\begin{aligned} B_1Y(t) &= A_1Y(t)=Y'(t)-P(t)Y(t)=F(t), \\ D(B_1) &= \{Y(t)\in D(A_1)=\mathcal{X}_m^1 : N_0Y(0)+\sum_{j=1}^k N_jY(t_j)=\vec{0}\} \end{aligned} \tag{7}$$

is injective if and only if

$$\det L_1=\det[N_0+\sum_{j=1}^k N_jZ(t_j)]\neq 0. \tag{8}$$

(ii) If the operator B_1 is injective, then it is correct and a unique solution to Problem (7) is

$$Y(t)=B_1^{-1}F(t)=\widehat{A}_1^{-1}F(t)-Z(t)L_1^{-1}\sum_{j=1}^k N_j(\widehat{A}_1^{-1}F)(t_j), \tag{9}$$

where $\widehat{A}_1^{-1}F(t)$ is given by (5).
(iii) The operator $B_2:\mathcal{X}_m\to\mathcal{X}_m$, corresponding to the problem

$$\begin{aligned} B_2X(t) &= A_2X(t)=X'(t)-T(t)X(t)=Y(t), \\ D(B_2) &= \{X(t)\in D(A_2)=\mathcal{X}_m^1 : M_0X(0)+\sum_{i=1}^n M_i\Psi_i(X)=\vec{0}\} \end{aligned} \tag{10}$$

is injective if and only if

$$\det L_2=\det[M_0\mathcal{Z}(0)+M\Psi(\mathcal{Z})]\neq 0. \tag{11}$$

(iv) If the operator B_2 is injective, then it is correct and a unique solution to Problem (10) is

$$X(t) = B_2^{-1}Y(t) = \widehat{A}_2^{-1}Y(t) - \mathcal{Z}(t)L_2^{-1}M\Psi(\widehat{A}_2^{-1}Y), \tag{12}$$

where $\widehat{A}_2^{-1}Y(t)$ is given by (6).

Theorem 2. *Let a vector Ψ be defined as in Theorem 1, vectors $U = U(t) = col\,(u_1(t), ..., u_{2m}(t)) \in \mathcal{X}_{2m}^1$, $\mathcal{F} = \mathcal{F}(t) = col\,(f_1(t), ..., f_{2m}(t)) \in \mathcal{X}_{2m}$, $\mathcal{M}_0$ and $\mathcal{M}$ be the $2m \times 2m$ and $2m \times 2mn$ constant matrices, respectively, S, Q constant $m \times m$ matrices, $\det Q \neq 0$ and the operator $\mathbf{B}$ be defined by*

$$\begin{gathered}\mathbf{B}U(t) = \mathbf{A}U(t) = U'(t) - DU(t) = \mathcal{F}, \\ D(\mathbf{B}) = \{U(t) \in D(\mathbf{A}) = \mathcal{X}_{2m}^1 : \mathcal{M}_0U(0) + \mathcal{M}\Psi(U) = \vec{0}\},\end{gathered} \tag{13}$$

where $D = \begin{pmatrix} S & Q \\ I_m & 0_m \end{pmatrix}$. Suppose also that there exist a constant matrix T satisfying the matrix equation $T^2 - ST = Q$, $\det T \neq 0$ and the fundamental matrices $Z = Z(t), \mathcal{Z} = \mathcal{Z}(t)$ to the systems $Y'(t) - PY(t) = \vec{0}$, $A_2X(t) = X'(t) - TX(t) = \vec{0}$, respectively, where $P = S - T$, $Z(0) = I_m$, $\det \mathcal{Z}(0) \neq 0$. Then:

(i) The $2m \times 2m$ matrix

$$\mathbf{Z}(t) = \begin{pmatrix} \mathcal{Z}(t) & \widehat{A}_2^{-1}Z(t) \\ \int_0^t \mathcal{Z}(s)ds + T^{-1}\mathcal{Z}(0) & \int_0^t \widehat{A}_2^{-1}Z(s)ds - (PT)^{-1} \end{pmatrix} \tag{14}$$

is a fundamental matrix to $U'(t) - DU(t) = \vec{0}$, where $S = P + T$, $Q = -PT$, $\widehat{A}_2$ as in Lemma 1.

(ii) Problem (13) is uniquely solvable if and only if

$$det\mathbf{L} = det[\mathcal{M}_0\mathbf{Z}(0) + \mathcal{M}\Psi(\mathbf{Z})] \neq 0, \tag{15}$$

and the unique solution to Problem (13) is given by

$$U(t) = \widehat{\mathbf{A}}^{-1}\mathcal{F}(t) - \mathbf{Z}\mathbf{L}^{-1}\mathcal{M}\Psi\left(\widehat{\mathbf{A}}^{-1}\mathcal{F}(t)\right), \tag{16}$$

where $\widehat{\mathbf{A}}^{-1}\mathcal{F}(t) = \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathcal{F}(s)ds$, $\quad \mathbf{Z}(0) = \begin{pmatrix} \mathcal{Z}(0) & 0_m \\ T^{-1}\mathcal{Z}(0) & -(PT)^{-1} \end{pmatrix}$.

Theorem 3. *Let the operator A, the matrices $S(t), Q(t), M_0, M = (M_1, ..., M_n)$, the vectors X, F, Ψ be defined as in Theorem 1 and $N = (N_1, ..., N_n)$, $V = (V_1, ..., V_n)$ be the $m \times mn$ matrices with $m \times m$ constant matrices N_i, V_i. Suppose also that N_0, C are $m \times m$ constant matrices, the vectors $X(\vec{t}) = col(X(t_1), ..., X(t_n))$, $X'(\vec{t}) = col(X'(t_1), ..., X'(t_n))$, where $0 < t_1 < ... < t_n \leq 1$, and the operator $\mathbf{B} : \mathcal{X}_m \to \mathcal{X}_m$*

is defined by

$$\begin{aligned} \mathbf{B}X(t) &= AX(t) = X''(t) - S(t)X'(t) - Q(t)X(t) = F(t), \qquad (17)\\ D(\mathbf{B}) &= \{X(t) \in D(A) = \mathcal{X}_m^2 : M_0X(0) + \sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}, \\ & N_0X'(0) + CX(0) + \sum_{i=1}^{n}[N_iX'(t_i) + V_iX(t_i)] = \vec{0}\}. \end{aligned}$$

If there exists a differentiable $m \times m$ matrix $T = T(t)$, such that

$$T'(t) - S(t)T(t) + T^2(t) = Q(t),\ C = -N_0T(0),\ V_i = -N_iT(t_i),\ i = 1, ..., n, \quad (18)$$

then there exist the matrix $P(t) = S(t) - T(t)$ and the operators $A_1, A_2, \widehat{A}_1, \widehat{A}_2$ defined by (1), (2), (3), (4), respectively, such that:
(i) The operator $\mathbf{B}$ is decomposed in $\mathbf{B} = B_1B_2$, where the operators $B_1, B_2 : \mathcal{X}_m \to \mathcal{X}_m$ are given by

$$\begin{aligned} B_1Y(t) = A_1Y(t) = Y'(t) - P(t)Y(t) = F(t), \qquad (19)\\ D(B_1) = \{Y(t) \in D(A_1) : N_0Y(0) + \textstyle\sum_{i=1}^{n} N_iY(t_i) = \vec{0}\},\\ B_2X(t) = A_2X(t) = X'(t) - T(t)X(t) = Y(t), \qquad (20)\\ D(B_2) = \{X(t) \in D(A_2) : M_0X(0) + \textstyle\sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}\}. \end{aligned}$$

(ii) The operator $\mathbf{B}$ is injective if and only if

$$\det L_1 = \det[N_0 + \sum_{i=1}^{n} N_iZ(t_i)] \neq 0,\ \det L_2 = \det[M_0\mathcal{Z}(0) + M\Psi(\mathcal{Z})] \neq 0, \quad (21)$$

where $Z, \mathcal{Z}$ are the fundamental matrices of the equations $A_1Y(t) = \vec{0}$, $A_2X(t) = \vec{0}$, respectively.
(iii) If the operator $\mathbf{B}$ is injective, then it is correct and a unique solution to Problem (17) is given by

$$\begin{aligned} X(t) = \mathbf{B}^{-1}F(t) = \widehat{A}_2^{-1}Y(t) - \mathcal{Z}(t)L_2^{-1}M\Psi(\widehat{A}_2^{-1}Y), \quad \text{where} \qquad (22)\\ Y(t) = \widehat{A}_1^{-1}F(t) - Z(t)L_1^{-1}\textstyle\sum_{i=1}^{n} N_i(\widehat{A}_1^{-1}F)(t_i), \qquad (23) \end{aligned}$$

and $\widehat{A}_1^{-1}F(t)$, $\widehat{A}_2^{-1}Y(t)$ are given by (5), (6), respectively.

Conclusion

The solvability conditions and exact solutions to the nonlocal boundary value problems (BVPs) for the systems of first and second order ordinary differential equations were obtained in the terms of abstract operators. BVP for the system of second order were solved by Decomposition method. Some examples in [1], [2], [3] can be solved more easily by the proposed method for the systems of first order.

References

[1] I.T. Kiguradze, *Boundary value problems for systems of ordinary differential equations*, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Nov. Dostizh., 30, p.3–103, 1987, (in Russian)

[2] V.A. Churikov, *Solution of a Multipoint Boundary-Value Problems for a System of Linear Ordinary Differential Equations with holomorphic coefficients*, Ukrainian Mathematical Journal. 42, p. 113-115, 1990.

[3] M.M. Baiburin, *On Multipoint boundary value problems for first order linear differential equations, system*, Applied Mathematics and Control Sciense. 6, p. 16-30, 2018, (in Russian)

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