On Multicomponent Continued Fraction Expansions of Hypernumbers of Certain Classes

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Abstract. A new class of multicomponent continued fractions is investigated. It is assumed that some algebraic structures can be define on a set of approximated elements that allow consider elements of original sets as hypernumbers. The well-known properties of classical continued fractions can be transfer to the multicomponent class of continued fractions that greatly simplifies research. In this paper, conditions for approximating elements of the original sets by multicomponent continued fractions and estimating the rate of convergence are obtained.

Introduction

The theory of representing of real numbers by classical continued fractions is given in many works. Results on continued fraction expansions for complex numbers are also known , see for example [1]. There are some results on generalized continued fractions. Many needs of applied sciences lead to the problem of approximation of multidimensional real parameters by a set of rational numbers with the same denominators. To solve such problems, multicomponent continuous fractions can be used. In this paper which continues papers [2], [3] and others, a new class of multicomponent continued fractions is investigated. Multicomponent continued fractions are interpreted as multicomponent scalars. They are elements of certain algebra which is a set of elements with two arithmetic operations. Quaternions are considered as a basic example of such algebra.

1. Basic concepts of the theory of continued fractions

Let $a_0, a_1, \dots, a_n, \dots$ be some sequence of characters. A finite continued fraction is given in the form

$$a_0 + (a_1 + \dots + (a_{n-1} + (a_n)^{-1})^{-1} \dots)^{-1}, \tag{1}$$

which can also be written as an ordinary fraction.

An infinite continued fraction is given in the form

$$a_0 + (a_1 + \cdots + (a_{n-1} + (a_n + \dots)^{-1})^{-1} \dots)^{-1}. \tag{2}$$

In the sequel, we intend to consider continued fraction expansions for a certain set of scalars. Note that commutativity is not assumed for multiplication. Further, we assume that in the set of scalars there is a certain lattice, the elements of which are «integers». Thus, the numbers a_n are elements of the lattice $\mathfrak{G}$. We also assume that the set of scalars is some Euclidean space, i.e. a norm of an element is determined. Below, we give estimates of the rate of convergence of convergent fractions to decomposed elements.

1.1. Euclidean Algorithm and Iteration sequences

The continued fraction expansion of z can be obtained by applying of the Euclidean algorithm. For $n = 0, 1, 2, \dots$, a recurrent sequence α_n is constructed, where $\alpha_0 = z$. For any n, $a_n = [\alpha_n]$ is the whole part of the element α_n. It may depend on the method of rounding and on number n. Next, the fractional part is determined by formula $\langle \alpha_n \rangle = \alpha_n - a_n$.

The previous formula can be rewritten as

$$a_n = \alpha_n - (\alpha_{n+1})^{-1} \in \mathfrak{G}, \quad n = 0, 1, 2, \dots. \tag{3}$$

The sequence α_n is called an iteration sequence.

1.2. Convergent fractions

A finite continued fraction of the form (1) can be written as a ordinary fraction. It is called a convergent fraction for the number $\alpha \in \mathbb{T}$, where $\mathbb{T}$ is some algebra. Since we do not assume that multiplication is commutative, there are two possible representations for a convergent fraction

$$r'_n = (q'_n)^{-1} p'_n, \quad r''_n = p''_n (q''_n)^{-1}, \tag{4}$$

which give the same value, i.e.

$$r'_n = r''_n =: r_n, \quad n = 0, 1, 2, \dots.$$

The (finite or infinite) sequence r_n, $n = 1, 2, \dots$ is called a sequence of convergent fractions.

It can be shown that the quantities p'_n and q'_n, as well as p''_n and q''_n, satisfy the Euler equations where, respectively, the left and right pairs of equations for $n \geq 1$ have the following forms

$$p'_{-1} = 1,\ p'_0 = a_0, \quad p'_{n+1} = a_{n+1} p'_n + p'_{n-1}, \tag{5}$$

$$q'_{-1} = 0,\ q'_0 = 1, \quad q'_{n+1} = a_{n+1} q'_n + q'_{n-1}, \tag{6}$$

and

$$p''_{-1} = 1,\, p''_0 = a_0, \quad p''_{n+1} = p''_n a_{n+1} + p''_{n-1}, \tag{7}$$
$$q''_{-1} = 0,\, q''_0 = 1, \quad q''_{n+1} = q''_n a_{n+1} + q''_{n-1}. \tag{8}$$

1.3. Auxiliary statements

Next, we show that the pairs (p'_n, p''_n) and (q'_n, q''_n) change consistently. Thus, the following statements are true.

Lemma 1. *The relations are valid:*

$$V_{n-1,n} := p'_{n-1}q''{}_n - q'_{n-1}p''{}_n = (-1)^n, \tag{9}$$
$$V_{n.n-1} := p'{}_n q''_{n-1} - q'_n p''_{n-1} = (-1)^n. \tag{10}$$

Lemma 2. *The relations are valid:*

$$P'_n = P''_n, \quad Q'_n = Q''_n, \tag{11}$$

where $P'_n = p'_n(p'_{n-1})^{-1}$ *and* $Q'_n = q'_n(q'_{n-1})^{-1}$.

Let $q_n^2 := |q'_n|^2 = |q''_n|^2$. Then $|q_n| := \sqrt{q_n^2}$.

2. The main theorems on continued fractions

By formulas (9) and (10), we have the following relations:

$$zq''_n - p''_n = (-1)^n(\alpha_{n+1}q''_n + q''_{n-1})^{-1} = (-1)^n(q''_n)^{-1}(\alpha_{n+1} + (Q''_n)^{-1}), \tag{12}$$
$$q'_n z - p'_n = (-1)^n(q'_n\alpha_{n+1} + q'_{n-1})^{-1} = (-1)^n(\alpha_{n+1} + (Q'_n)^{-1})(q'_n)^{-1}. \tag{13}$$

Theorem 1. *For residuals, the following relations are valid*

$$|zq''_n - p''_n| = |q'_n z - p'_n| = \frac{1}{|\alpha_{n+1} + (Q_n)^{-1}|}\frac{1}{|q_n|}, \tag{14}$$

$$|z - r_n| = \frac{1}{|\alpha_{n+1} + (Q_n)^{-1}|}\frac{1}{|q_n|^2}. \tag{15}$$

2.1. Some special conditions for the convergence of continued fractions

In formulas (14) and (15), the inequalities $|\alpha_{n+1}| > 1$ and $|(Q_n)^{-1}| < 1$ are true for all n. However, this does not guarantee that the denominators of the fractions are separated from zero. To ensure this condition, we introduce additional enhanced restrictions:

$$|\alpha_{n+1}| \geq \alpha > 1 \,\&\, |(Q_n)^{-1}| < 1, \tag{16}$$

or

$$|(Q_n)^{-1}| \leq 1 - c^{-1} < 1 \,\&\, |\alpha_{n+1}| > 1. \tag{17}$$

Remark 1. The inequalities $\alpha_n \geq \underline{\alpha} > 1$ from formula (17) can be interpreted as conditions of strong non-degeneracy of a iteration sequence. In some cases, these may be provided by selecting of a rounding function. Inequalities $|Q_n^{-1}| \leq c^{-1} < 1$ from formula (17) is equivalent to the inequalities $|q_n| > c|q_{n-1}|$. These relations are strong conditions for exponential growth of denominators of a convergent fraction.

By the triangle inequality, for $C = \min\{\underline{\alpha} - 1, 1 - c^{-1}\}$, $c > 1$, the following condition is satisfied:

$$0 < C \leq |\alpha_{n+1} + (Q_n)^{-1}|, \quad n \geq 0. \tag{18}$$

Theorem 2. Let condition (17) be satisfied. Then the following estimates for the residuals are valid:

$$|q'_n z - p'_n| = |zq''_n - p''_n| \leq \frac{1}{C|q_n|^2}, \tag{19}$$

$$|z - r_n| \leq \frac{1}{C|q_n|^2}.$$

Conclusion

The results obtained in this work were used to solve some control theory problems related to determining switching instants at discrete times, [3].

References

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