Khryashchev S. M.

On Multicomponent Continued Fraction Expansions of Hypernumbers of Certain Classes

St. Petersburg State Electrotechnical University

Polynomial Computer Algebra
April 15-20, 2024
Euler International Mathematical Institute, St. Petersburg, Russia

Introduction

A new class of multicomponent continued fractions is investigated. It is assumed that some algebraic structures can be define on a set of approximated elements. This allow consider elements of original sets as hypernumbers. The well-known properties of classical continued fractions can be transfer to the multicomponent class of continued fractions that greatly simplifies research.

Basic concepts of the theory of continued fractions

Let $a_0, a_1, \dots, a_n, \dots$ be some sequence of characters. A finite continued fraction is given in the form

$$a_0 + (a_1 + \dots + (a_{n-1} + (a_n)^{-1})^{-1} \dots)^{-1}, \tag{1}$$

which can also be written as an ordinary fraction.

An infinite continued fraction is given in the form

$$a_0 + (a_1 + \dots + (a_{n-1} + (a_n + \dots)^{-1})^{-1} \dots)^{-1}. \tag{2}$$

Left ordinary fractions for (1):

$$r'_1 = (q'_1)^{-1}p'_1, \quad p'_1 = a_1a_0 + 1,\ q'_1 = a_1,$$
$$r'_2 = (q'_2)^{-1}p'_2, \quad p'_2 = a_2(a_1a_0 + 1) + a_0,\ q'_2 = a_2a_1 + 1,$$

Right ordinary fractions for (1) :

$$r''_1 = p''_1(q''_1)^{-1}, \quad p''_1 = a_0a_1 + 1,\ q''_1 = a_1,$$
$$r''_2 = p''_2(q''_2)^{-1}, \quad p''_2 = (a_0a_1 + 1)a_2 + a_0,\ q''_2 = a_1a_2 + 1, etc.$$

For any $n = 1, 2, \ldots$:

$$r'_n = (q'_n)^{-1} p'_n, \quad r''_n = p''_n (q''_n)^{-1}, \tag{3}$$

which give the same value, i.e. $r'_n = r''_n =: r_n, \quad n = 0, 1, 2, \ldots$. Hence, $q'_n p''_n = p'_n q''_n$.

Continued fraction expansion

Euclidean algorithm and iteration sequences

$\mathbb{T} = \{z\}$ is a state space

$\mathbb{W}$ is a lattice in a state space.

$z = (a_0, a_1, \dots)$ is a continued fraction expansion of z.

Recurrent sequences a_n, α_n, $n = 0, 1, 2, \dots$ are constructed:

Let $\alpha_0 = z$.

For any n, $a_n = [\alpha_n]$ is the whole part of α_n,

$\langle \alpha_n \rangle = \alpha_n - a_n$ is the fractional part. Set $\alpha_{n+1} = \langle \alpha_n \rangle^{-1}$, etc.

The sequence α_n is called an iteration sequence.

$$a_n = \alpha_n - (\alpha_{n+1})^{-1} \in \mathbb{W}, \quad n = 0, 1, 2, \dots. \tag{4}$$

Euler equations

$$p'_{-1} = 1,\ p'_0 = a_0, \quad p'_{n+1} = a_{n+1}p'_n + p'_{n-1}, \tag{5}$$

$$q'_{-1} = 0,\ q'_0 = 1, \quad q'_{n+1} = a_{n+1}q'_n + q'_{n-1}, \tag{6}$$

or

$$p''_{-1} = 1,\ p''_0 = a_0, \quad p''_{n+1} = p''_n a_{n+1} + p''_{n-1}, \tag{7}$$

$$q''_{-1} = 0,\ q''_0 = 1, \quad q''_{n+1} = q''_n a_{n+1} + q''_{n-1}. \tag{8}$$

Let

$$\overline{r}'_n = \mathrm{col}(q'_n, p'_n), \quad \tilde{r}''_n = \mathrm{col}(q''_n, p''_n), \tag{9}$$

where $q'_n \in \mathbb{W}$, $p'_n \in \mathbb{W}$, $q''_n \in \mathbb{W}$, $p''_n \in \mathbb{W}$, $n = -1, 0, 1, 2, \ldots$.

Vector equations:

$$\overline{r}'_n = a_n \overline{r}'_{n-1} + \overline{r}'_{n-2}, \quad n \geq 1 \tag{10}$$

or

$$\overline{r}''_n = \overline{r}''_{n-1} a_n + \overline{r}''_{n-2}, \quad n \geq 1. \tag{11}$$

Let $\mathbb{T}$ be an algebra over a field $\mathbb{F}$.

We consider $\mathbb{F} = \mathbb{R}$ and $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{C}$, $\mathbb{T} = \mathbb{H}$ and others.

Conjugate element:

$$t \cdot t_r^\star = f_r e_0, \quad t_l^\star \cdot t = f_l e_0,$$

$f_r, f_l \in \mathbb{F}$, e_0 is the unit element in $\mathbb{T}$.

$t_l^\star \cdot t = t \cdot t_r^\star = |t|^2$- pseudonorm;

$(t+s)^\star = t^\star + s^\star$, $(t^\star)^\star = t$, $(t \cdot s)^\star = s^\star \cdot t^\star$, ...

Inverse element:

$t_r^{-1} = \frac{1}{t \cdot t_r^\star} t_r^\star =: \frac{1}{|t|^2} t_r^\star, \quad t_l^{-1} = \frac{1}{t_l^\star \cdot t} t_l^\star =: \frac{1}{|t|^2} t_l^\star.$

The expansion $\mathbb{T}\times\mathbb{T}$ of the state space $\mathbb{T}$.

L) *Left projective space:*

$\mathbb{P}^1_l(\mathbb{T}) := ((\mathbb{T}^2 \setminus \{0\})/\sim)$
$(t,s) \sim (\lambda t, \lambda s) = \lambda(t,s)$ for $(t,s) \in \mathbb{T}^2 \setminus \{0\}$, $\lambda \in \mathbb{T} \setminus \{0\}$.
$\mathrm{Pr}_l : \mathbb{T}^2 \mapsto \mathbb{P}^1_l$ is left projection (partitioning into equivalence classes).

R) *Right projective space:*

$\mathbb{P}^1_r(\mathbb{T}) := ((\mathbb{T}^2 \setminus \{0\})/\sim)$
$(t,s) \sim (t\mu, s\mu) = (t,s)\mu$ for $(t,s) \in \mathbb{T}^2 \setminus \{0\}$, $\lambda \in \mathbb{T} \setminus \{0\}$.
$\mathrm{Pr}_r : \mathbb{T}^2 \mapsto \mathbb{P}^r_l$ is right projection (partitioning into equivalence classes).

Let $(x,y)=(1,yx^{-1})x=x(1,x^{-1}y)$.

$\mathbb{T}\mapsto\mathbb{T}^2:\ z\mapsto(1,z)=:\overset{\circ}{\overline{z}}$ – embedding of set $\mathbb{T}$ into $\mathbb{T}^2$,

$\overset{\circ}{\overline{z}}$ – the basic vector of the corresponding one-dimensional linear subspace,

z – basic direction.

Let $\overset{\circ}{\overline{r}}_n=\mathrm{col}(1,r_n)\in\mathbb{T}^2$. Then

$$\overline{r}'_n=q'_n\overset{\circ}{\overline{r}}_n,\quad \overline{r}''_n=\overset{\circ}{\overline{r}}_n q''_n,\quad n=1,2,\ldots.$$

Bilinear formx on $\mathbb{T}^2 \times \mathbb{T}^2$

Let $\overline{b} \in \mathbb{T}^2, \quad \overline{c} \in \mathbb{T}^2$ where $\overline{b} = \operatorname{col}(b^1, b^2)$, $\overline{c} = \operatorname{col}(c^1, c^2)$.

Define a function Δ of $\overline{b}, \overline{c}$ «determinant rule»:

$$\Delta(\overline{b}, \overline{c}) := \det(\overline{b}, \overline{c}) := \begin{vmatrix} b^1 & c^1 \\ b^2 & c^2 \end{vmatrix} = b^1 c^2 - b^2 c^1,$$

1. $\Delta(\overline{b}' + \overline{b}'', \overline{c}) = \Delta(\overline{b}', \overline{c}) + \Delta(\overline{b}'', \overline{c}); \Delta(\overline{b}, \overline{c}' + \overline{c}'') = \Delta(\overline{b}, \overline{c}') + \Delta(\overline{b}, \overline{c}'')$,
2. $\Delta(\delta\overline{b}, \overline{c}) = \delta\Delta(\overline{b}, \overline{c}), \quad \Delta(\overline{b}, \overline{c}\gamma) = \Delta(\overline{b}, \overline{c})\gamma, \quad \gamma, \delta \in \mathbb{T}$.
3. $(\Delta(\overline{b}, \overline{c}))^\star = -\Delta(\overline{c}^\star, \overline{b}^\star)$.

The function Δ can be interpreted as a symplectic scalar product. Let $\overline{b} = (b^1, b^2), \overline{c} = (c^1, c^2)$, $b^1 \neq 0, c^1 \neq 0$.

$$\Delta(\overline{b}, \overline{c}) = 0 \quad \Leftrightarrow \quad (b^1)^{-1} b^2 = c^2 (c^1)^{-1} =: r \quad \Leftrightarrow \quad \overline{b} = b^1 \overset{\circ}{\overline{r}},\ \overline{c} = \overset{\circ}{\overline{r}} c^1,$$

where $\overset{\circ}{\overline{r}} = \mathrm{col}(1, r)$.

Thus, the parallelogram has zero volume if and only if the vectors $\overline{b}, \overline{c}$ have same base direction r, i.e., they are parallel.
In particular, the condition $r'_n = r''_n$ can be rewritten as

$$\Delta(\overline{r}'_n, \overline{r}''_n) = 0, \quad n = 0, 1, \ldots .$$

This means that p'_n, q'_n and p''_n, q''_n change consistently.

Orientation of ordered pairs of $\mathbb{T}$-vectors

The value of $\Delta(\overline{b}, \overline{c})$ can also be interpreted as the value of the $\mathbb{T}$-valued volume of the parallelogram spanned by the vectors $\overline{b}, \overline{c}$.

For some ordered pairs $(\overline{b}, \overline{c})$ the value Δ can be a real.
For such pairs, we can introduce a relation of a orientation.

l) $\Delta(\overline{b}, \overline{c}) > 0$: the pair $(\overline{b}, \overline{c})$ is called *positively oriented*,

r) $\Delta(\overline{b}, \overline{c}) < 0$: the pair $(\overline{b}, \overline{c})$ is called *negatively oriented*.

For any $\overline{b}, \overline{c}$ there exist $\delta \in \mathbb{T}$ and $\gamma \in \mathbb{T}$ such that

$$\Delta(\delta\overline{b}, \overline{c}) \in \mathbb{R}, \quad \Delta(\overline{b}, \overline{c}\gamma) \in \mathbb{R}.$$

For example, $\delta = (\Delta(\overline{b}, \overline{c}))^\star$, $\gamma = (\Delta(\overline{b}, \overline{c}))^\star$.

Thus, it is defined an orientation of 1D linear subspaces.

Definition. The vector $\overline{z}$ *lies between* the vectors $\overline{r}'$ and $\overline{r}''$ if the pairs $(\overline{r}', \overline{r}'')$ and $(\overline{z}, \overline{r}'')$ have the same orientation, i.e. $\Delta(\overline{r}', \overline{r}'') \cdot \Delta(\overline{z}, \overline{r}'') > 0$.

See fig. 2.

The bilinear form Δ is a certain metric characteristic of the state space $\mathbb{T}$, in terms of which many of the properties of associated elements for continued fractions given below can be expressed.

Continued fractions (1) (2) can be written uniformly:

$$r_n = (a_0; a_1, \dots, a_{n-1}, a_n, \overline{0}), \quad n = 0, 1, 2, \dots,$$

$$z = (a_0; a_1, \dots, a_{n-1}, a_n, \alpha_{n+1}), \quad n = 0, 1, 2, \dots.$$

Hence, for $n \geq 1$:

$$\overline{r}'_{n+1} = a_{n+1}\overline{r}'_n + \overline{r}'_{n-1}, \quad \overline{r}''_{n+1} = \overline{r}''_n a_{n+1} + \overline{r}''_{n-1},$$

$$\overline{z}'_{n+1} := \alpha_{n+1}\overline{r}'_n + \overline{r}'_{n-1}, \quad \overline{z}''_{n+1} := \overline{r}''_n \alpha_{n+1} + \overline{r}''_{n-1},$$

where

$$\overline{r}'_n := \mathrm{col}(q'_n, p'_n), \quad \overline{r}''_n := \mathrm{col}(q''_n, p''_n),$$
$$\overline{z}'_n := \mathrm{col}(x'_n, y'_n), \quad \overline{z}''_n := \mathrm{col}(x''_n, y''_n),$$
$$r'_n = (q'_n)^{-1} p'_n, \quad r''_n = p''_n (q''_n)^{-1},$$
$$z = z' = z'_n = (x'_n)^{-1} y'_n, \quad z = z'' = z''_n = y''_n (x''_n)^{-1},$$
$$\overset{\circ}{\overline{z}} = (1, z), \quad \overline{z}'_n = x'_n \overset{\circ}{\overline{z}}, \quad \overline{z}''_n = \overset{\circ}{\overline{z}} x''_n.$$

The illustration of relation «lies between».

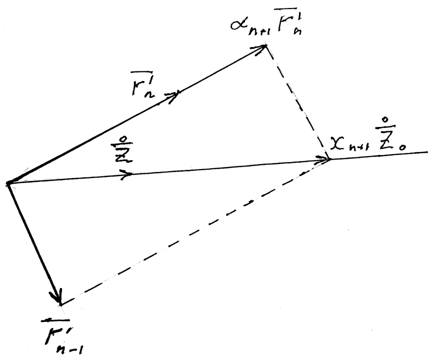

Figure 1: x''_{n+1} is denominator of the fraction $z := z_{n+1} = y''_{n+1}(x''_{n+1})^{-1}$, $\overset{\circ}{\bar{z}} = (1, z)$.

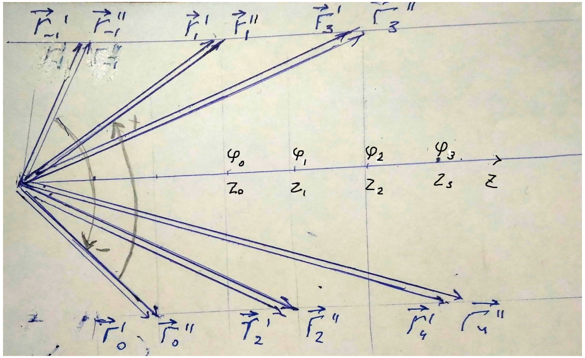

Figure 2: Bounding sequences $\overline{r}'_n$ and $\overline{r}''_n$, $n = 0, 1, \ldots$ for direction z.

Auxiliary statements

Lemma 1 The relations are valid:

$$V_{n-1,n} := \Delta(\overline{r}'_{n-1}, \overline{r}''_{n}) = p'_{n-1}q''{}_{n} - q'_{n-1}p''{}_{n} = (-1)^{n}, \tag{-6}$$

$$V_{n.n-1} := \Delta(\overline{r}'{}_{n}, \overline{r}'{}_{n-1}) = p'{}_{n}q''_{n-1} - q'_{n}p''_{n-1} = (-1)^{n}. \tag{-5}$$

Lemma 2 The relations are valid:

$$P'_n = P''_n =: P_n, \quad Q'_n = Q''_n =: Q_n,$$

where $P'_n = p'_n(p'_{n-1})^{-1}$ *and* $Q'_n = q'_n(q'_{n-1})^{-1}$.

Recurrent sequences:

$$P_n = a_n + (P_{n-1})^{-1}, \quad P_0 = a_0, \quad n \geq 1,$$

$$Q_n = a_n + (Q_{n-1})^{-1}, \quad Q_1 = a_1, \quad n \geq 2,$$

$$Q_n = (a_n, \ldots, a_1).$$

Let $q_n^2 := |q'_n|^2 = |q''_n|^2$. Then $|q_n| := \sqrt{q_n^2}$.

By formulas (-6) and (-5):

$$\Delta(\vec{z}, \vec{r}''_n) := zq''_n - p''_n = (-1)^n (x''_{n+1})^{-1} = (-1)^n (q''_n)^{-1} c_n^{-1},$$

and

$$\Delta(\vec{r}'_n, \vec{z}) := q'_n z - p'_n = (-1)^n (x'_{n+1})^{-1} = (-1)^n c_n^{-1} (q'_n)^{-1},$$

where

$$c_n = [\alpha_{n+1} + (Q_n)^{-1}]. \qquad \text{(-10)}$$

The value $\theta_n = c_n^{-1}$ is the approximation coefficient.

Theorem 1 For residuals, the following relations are valid

$$|zq''_n - p''_n| = |q'_n z - p'_n| = \frac{1}{|c_n|}\frac{1}{|q_n|}, \tag{-9}$$

$$|z - r_n| = \frac{1}{|c_n|}\frac{1}{|q_n|^2}. \tag{-8}$$

Condition for the convergence of convergents as $n \to \infty$:
$(x_n)^{-1} \to 0 \quad \Leftrightarrow \quad \theta_n = o(|q_n|)$,
where $|q_n| \to \infty$ as $n \to \infty$.

1.1 Some special conditions for the convergence of continued fractions

In formula (-10), for all $n \geq 0$:

$$|\alpha_{n+1}| > 1 \,\&\, |(Q_n)^{-1}| < 1.$$

However, this does not guarantee that c_n is separated from zero.

To ensure this condition, we introduce additional enhanced restrictions:

$$|\alpha_{n+1}| \geq \alpha > 1 \,\&\, |(Q_n)^{-1}| < 1,$$

or

$$|\alpha_{n+1}| > 1 \,\& \quad |(Q_n)^{-1}| \leq 1 - c^{-1} < 1.$$

Remark 1 The inequalities $\alpha_n \geq \underline{\alpha} > 1$ *can be interpreted as conditions of strong non-degeneracy of a iteration sequence.*

Inequalities $|Q_n^{-1}| \leq c^{-1} < 1$ *from formula is equivalent to the inequalities* $|q_n| > c|q_{n-1}|$.

By the triangle inequality, for $C = \min\{\underline{\alpha} - 1, 1 - c^{-1}\}$, $c > 1$, the following condition is satisfied:

$$0 < C \leq |c_n| = |\alpha_{n+1} + (Q_n)^{-1}|, \quad n \geq 0.$$

Theorem 2 The following estimates for the residuals are valid:

$$|q'_n z - p'_n| = |z q''_n - p''_n| \leq \frac{1}{C|q_n|^2},$$

$$|z - r_n| \leq \frac{1}{C|q_n|^2}.$$

References

[1] S. G. Dani, *Continued fraction expansions for complex numbers – a general approach.* Acta Aritm., 171 [2015] , 355-369.

[2] S. M. Khryashchev, *A Finding of Switching Instants for Dynamical Polysystems by Applying Continued Fractions.* Nonlinear Phenomena in Complex Systems. 2021, Vol.24. No.2, 175 - 183

[3] S. M. Khryashchev, *A Method of Determining of Switching Instants for Discrete-time Control Systems.* SCP 2020. Lecture Notes in Control and Information Sciences - Proceedings. Springer, Cham. https: // doi.org/10.1007/978 -3-030- 87966-$2_1$8.