On pictures related to some exponential sums

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Abstract. By numerical experiments with exponential sums in finite fields, it is discovered relation with flat curve known as Kepler trifolium. A theoretical explanation for this observation is given in the present paper.

0. Preliminaries.

Consider the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of prime order p and a non-trivial character χ of its multiplicative group extended by setting $\chi(0) = 0$. Let e_p be the additive character

$$x \mapsto \exp(2\pi i x/p)$$

of $\mathbb{F}_p$. Given one-variable polynomials f, g over $\mathbb{F}_p$, consider the sum

$$\sum_{x\in\mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big). \tag{1}$$

That is an exponential character sum of mixed type, see [1]. Under some general assumptions on f, g and χ, one has

$$\Big| \sum_{x\in\mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big) \Big| \le (m+n-1)\sqrt{p} \tag{2}$$

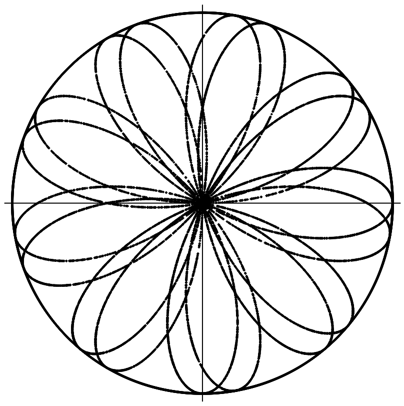
with $n = \deg(g)$ and $m = \deg(\text{radical of}\, f)$, see [1] and [2].
In particular, let ψ be a cubic character and let $f(x) = x$, $g(x) = x^2$. This case (2) with $\chi = \psi$ implies that the sum

$$E_p(\psi) = \frac{1}{2\sqrt{p}} \sum_{x\in\mathbb{F}_p} \psi(x) e_p(x^2) \tag{3}$$

is located in the circle $D = \big\{ z \in \mathbb{C} \;\big|\; |z| \le 1 \big\}$. We are interested in distribution of the points $E_p(\psi)$ in D. In the present paper we provide a theoretical explanation for our numerical experimental observations [4].

1. Numerical observations.

We have evaluated the sums $E_p(\psi)$ for all cubic characters ψ and for all prime $p \equiv 1 \bmod 6$ subject to $p \leq 360000$. The assumption $p \equiv 1 \pmod 6$ is included here just to ensure the existence of cubic characters. The following figure on complex plane $\mathbb{C}$ represents the results originally reported in [4].



On this figure, it is shown the circle D, its boundary $\overline{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$, the real and imaginary axis, and some 18-petals flower. The boundary of the petals are formed by the points $E_p(\psi)$ with p and characters ψ as above. In that follows, we will recognise six copies of Kepler trifolium here.

2. Gauss sums.

The Gauss sums $G(\chi)$ are the ones (1) with $f(x) = g(x) = x$, so that

$$G(\chi) = \sum_{x \in \mathbb{F}_p} \chi(x) e_p(x). \tag{4}$$

For any prime p and non-trivial character χ one has

$$|G(\chi)|^2 = p \quad \text{and} \quad G(\chi)G(\bar{\chi}) = \chi(-1)p, \tag{5}$$

where $\bar{\chi}$ is the complex conjugation of χ. One say $G(\chi)$ is a quadratic, cubic or sextic sums according to χ is a character of order 2, 3 or 6.

By Gauss, for the quadratic character κ, the sum $G(\kappa)$ is equal to $\sqrt{p}$ or $i\sqrt{p}$ according to $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$.

To deal with cubic characters, assume $p \equiv 1 \bmod 6$. This case we have two cubic characters, say ψ and $\bar{\psi}$, the quadratic character κ, and sextic characters $\kappa\psi$ and $\kappa\bar{\psi}$. The sextic sums can be evaluated (see theorem 3.1 in [3]) in terms of cubic and quadratic ones by the formula

$$G(\kappa\bar{\psi}) = \bar{\psi}(2)G(\kappa)G(\psi)^2/p. \tag{6}$$

For the cubic characters ψ one has $\psi(-1) = 1$, so that (5) implies

$$G(\bar{\psi}) = p/G(\psi). \tag{7}$$

3. Evaluation of sums (3) in terms of Gauss sums.

Consider the sum $E_p(\psi)$ in (3) with a cubic character ψ, $p \equiv 1 \bmod 6$. One has $\psi(x) = \bar{\psi}(x^2)$ for all $x \in \mathbb{F}_p$ and

$$\sharp\{x \in \mathbb{F}_p \mid x^2 = t\} = 1 + \kappa(t) \quad \text{for all} \quad t \in \mathbb{F}_p.$$

Recall, it is assumed $\kappa(0) = \psi(0) = 0$. It follows,

$$\begin{aligned} 2\sqrt{p}\, E_p(\psi) &= \sum_{t\in\mathbb{F}_p} \sharp\{x \in \mathbb{F}_p \mid x^2 = t\}\bar{\psi}(t) e_p(t) \\ &= \sum_{t\in\mathbb{F}_p} \big(1 + \kappa(t)\big)\bar{\psi}(t) e_p(t) = G(\bar{\psi}) + G(\kappa\bar{\psi}). \end{aligned}$$

This can be rewritten as $2\sqrt{p}\, E_p(\psi) = p/G(\psi) + \bar{\psi}(2)G(\kappa)G(\psi)^2/p$, see (6) and (7), and finally as follows.

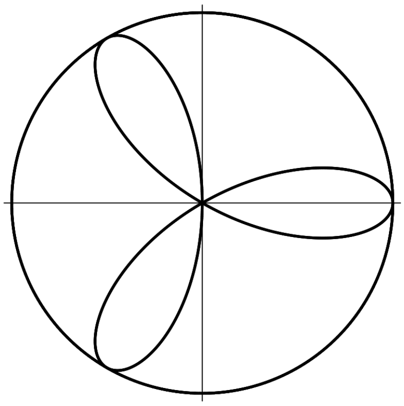
Proposition 1. *For every prime* $p \equiv 1 \bmod 6$, *one has*

$$E_p(\psi) = \frac{1 + QT^3}{2T} \qquad \textit{with } T = G(\psi)/\sqrt{p}, \;\; Q = \bar{\psi}(2)G(\kappa)/\sqrt{p}. \tag{8}$$

Here ψ *and* κ *are cubic and quadratic characters of* $\mathbb{F}_p$ *and* $|T| = |Q| = 1$. □

4. Kepler trifolium.

Consider the complex plane $\mathbb{C}$ with the Cartesian coordinates $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $z \in \mathbb{C}$, the unit circle D centred at the origin 0, its boundary $\overline{D}$ and the curve C



defined by the equation

$$(x^2 + y^2)^2 + 3xy^2 - x^3 = 0. \tag{9}$$

This curve is known as Kepler trifolium and also as regular trifolium, three leaf/petal rose, three leaf/petal clover. It remains unchanged when rotated through the angles of $\pm 2\pi/3$ and it can be given by the polar equation $r = \cos(3\varphi)$, r being a point on the axis obtained by rotation of the real axis through the angle φ.

5. Parametrization.

Consider again the Kepler trifolium C. We intend to show that C can be parametrized by a rational function on $\overline{D}$. To be precise, we mean the complex function

$$z \mapsto \frac{1+z^3}{2z} \tag{10}$$

and its restriction to $\overline{D}$.

Proposition 2. *The function* (10) *takes any point of* $\overline{D}$ *to some point of* C. *In particular, it takes cubic roots of* -1 *to the triple point* 0 *of* C. *It takes cubic roots of* 1 *to the cubic roots of* 1. *Except for the point* 0, *every point of* C *is the image of an unique point of* $\overline{D}$. *The boundary of each petal is the image of someone arc in* $\overline{D}$ *whose endpoints are the cubic roots of* -1.

Proof. Let $z = x + iy$ with real x and y. If $z \in \overline{D}$ then $x^2 + y^2 = 1$ and

$$\frac{1+z^3}{2z} = \frac{\bar{z}+z^2}{2} = X + iY$$

$$\text{with} \quad X = (1+x)S, \quad Y = yS, \quad S = x - \frac{1}{2}.$$

It follows, $X^2 = (1+x)^2S^2$, $Y^2 = (1-x^2)S^2$, $X^2 + Y^2 = 2(1+x)S^2$, and then

$$(X^2+Y^2)^2 + 3XY^2 - X^3 = 0.$$

According to (9), that means $Z = X + iY$ is a point of the curve C, as required. In particular, if z is a cubic root of -1 then $1 + z^3 = 0$ and $Z = 0$. If z is a cubic root of 1 then $Z = \bar{z}$ and that is a cubic root of 1 as well.

Now let X and Y be the real and the imaginary parts of some point $Z \in C$. Assume, this Z is a point of the right petal on the figure and $Z \neq 1$. We have $X > 0$ and $X \neq 1$. For every such X, there is a unique $x \geq -1$ satisfying

$$(1+x)S = X \quad \text{with} \quad S = x - \frac{1}{2}.$$

This x satisfies $1/2 < x < 1$. Then we should take y satisfying $Y = yS$ and to check that $x^2 + y^2 = 1$. The point $x + iy$ is the only one of $\overline{D}$ whose image is equal to Z. This point belongs to the arc of $\overline{D}$ that passes through 1 and whose endpoints are $\exp(\pm\pi i/3)$. The points Z of another two petals can be treated similarly. □

Proposition 3. *Let* $v = \exp(it)$ *and* $w = \exp(it/3)$ *with some* $t \in \mathbb{R}$. *The image of* $\overline{D}$ *under the function*

$$z \mapsto \frac{1+vz^3}{2z} \tag{11}$$

is the curve $C' = wC$ *obtained by rotation of* C *around* 0 *through the angle* $t/3$.

Proof. As the point z runs over $\overline{D}$, the point wz runs over $\overline{D}$ and the point

$$\frac{1+vz^3}{2z} = w\frac{1+(wz)^3}{2(wz)}$$

runs over $C' = wC$ by Proposition 2, as required. □

6. Distribution of the sums (3).

Now we are ready to give a theoretical explanation to the shown above distribution of the sums $E_p(\psi)$, see Section 1. Compare the formula (8) in Proposition 1 with the formula (11) in Proposition 3. It follows from the Gauss formulas for the quadratic sums that the only possible values of the coefficient Q in (8) are either 1, ω, ω^2 or i, $i\omega$, $i\omega^2$ according to $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$. Here $\omega = \exp(2\pi i/3)$, so that 1, ω, ω^2 are all possible values of $\psi(2)$. We find easily that any point $E_p(\psi)$ in (3) belongs to some of six curves wC obtained by rotation of C around 0 through the angles $w = 0$, $\pm 2\pi/9$, $\pi/6$, $\pi/2 \pm \pi/9$, so that

$$E_p(\psi) \in \tilde{C} = \bigcup_w wC. \tag{12}$$

This is consistent with the figure presented in Section 1.
It remains an open question whether the countable set of all the points $E_p(\psi)$ is everywhere dense in the curve $\tilde{C}$ in (12). It seems likely that *the set of all the points $E_p(\psi)$ is everywhere dense in $\tilde{C}$* (with the topology induced by the canonical topology in $\mathbb{C}$). In the meantime, the points T in Proposition 1 forms everywhere dense subset in $\overline{D}$. That is known from research of the cubic Gauss sums related to the Kummer problem [5].

References

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