On pictures related to some exponential sums

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Preliminaries

Consider the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of prime order p and a non-trivial character χ of its multiplicative group extended by setting $\chi(0) = 0$. Let e_p be the additive character

$$x \mapsto \exp(2\pi i x/p)$$

of $\mathbb{F}_p$. Given one-variable polynomials f, g over $\mathbb{F}_p$, consider the sum

$$\sum_{x \in \mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big).$$

That is an exponential character sum of mixed type. Under some general assumptions on f, g and χ, one has

$$\Big|\sum_{x \in \mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big)\Big| \leq (m+n-1)\sqrt{p}$$

with $n = \deg(g)$ and $m = \deg(\text{radical of } f)$.

That means, the points

$$\frac{1}{(m+n-1)\sqrt{p}} \sum_{x\in\mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big)$$

belongs to the unit circle $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$.

We refer to J.-P. Serre (Asterisque 41–42, 1977) for review of general theory (involving multi-variable polynomials).

For given character χ and polynomials f and g, we may be interested in these points for all possible prime p and in distribution of these points within D.

We can take $f(x) = g(x) = x$ to get the classical Gauss sums

$$G(\chi) = \sum_{x\in\mathbb{F}_p} \chi(x)\, e_p(x)$$

for which a lot of beautiful formulas are known, say

$$|G(\chi)|^2 = p, \quad G(\chi)\, G(\bar{\chi}) = \chi(-1)\, p \quad \text{and so on.}$$

One say $G(\chi)$ is a quadratic, cubic or sextic sums according to χ is a character of order 2, 3 or 6. The problem of evaluating the Gauss sums is extremely deep. By Gauss, for the quadratic character κ, the sum $G(\kappa)$ is equal to $\sqrt{p}$ or $i\sqrt{p}$ according to $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$.

In the present lecture we consider the exponential sums

$$E_p(\psi) = \frac{1}{2\sqrt{p}} \sum_{x \in \mathbb{F}_p} \psi(x) e_p(x^2)$$

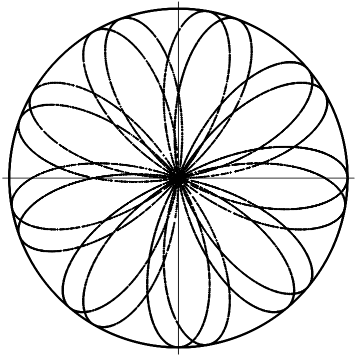
attached to cubic characters ψ. The normalizing factor $2\sqrt{p}$ is involved to ensure that

$$E_p(\psi) \in D = \{z \in \mathbb{C} \mid |z| \leq 1\}.$$

We are interested in distribution of the points $E_p(\psi)$ in the circle D.

Numerical observations

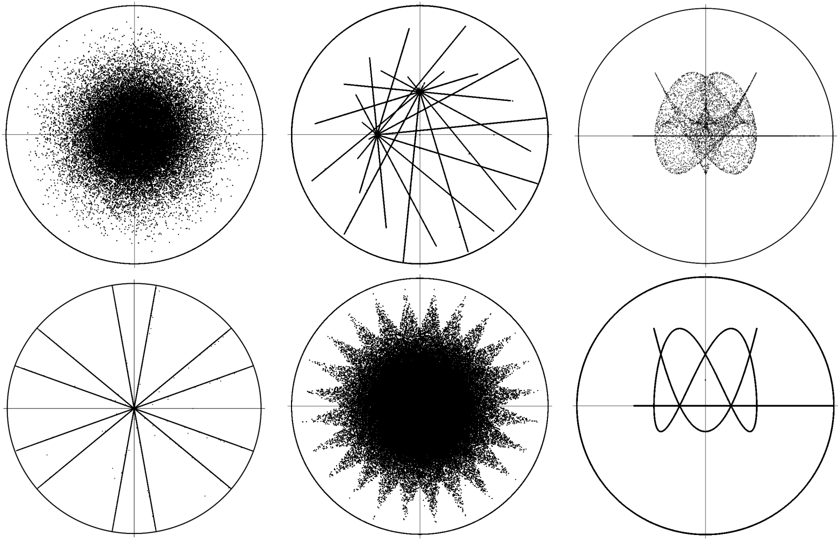
We have evaluated the sums $E_p(\psi)$ for all cubic characters ψ and for all prime $p \equiv 1 \bmod 6$ subject to $p \leq 360000$. The assumption $p \equiv 1 \bmod 6$ is included to ensure the existence of cubic characters.



The real and imaginary axis on $\mathbb{C}$,
the boundary $\overline{D}$ of circle D,
$\overline{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$.

The boundary of 18-petals flower
is formed by the points $E_p(\psi)$.
We will recognise six copies of
the Kepler trifolium here.

This flower is very exceptional and beautiful one. We find a lot of different pictures for other exponential sums. Samples:



Evaluation of $E_p(\psi)$ in terms of Gauss sums

To deal with the sums $E_p(\psi)$ and with cubic characters ψ, assume $p \equiv 1 \bmod 6$. This case we have two cubic characters ψ and $\bar{\psi}$, the quadratic character κ, and the characters $\kappa\psi$ and $\kappa\bar{\psi}$ of order six. Our first observation is that

$$E_p(\psi) = \frac{1}{2\sqrt{p}}\big\{G(\bar{\psi}) + G(\kappa\bar{\psi})\big\}.$$

To prove this formula, notice that $\sharp\{x \in \mathbb{F}_p \mid x^2 = t\} = 1 + \kappa(t)$ for all $t \in \mathbb{F}_p$ and $\psi(x) = \bar{\psi}(x^2)$ for all $x \in \mathbb{F}_p$. We then have

$$2\sqrt{p}\,E_p(\psi) = \sum_{t\in\mathbb{F}_p} \sharp\{x \in \mathbb{F}_p \mid x^2 = t\}\bar{\psi}(t)e_p(t)$$

$$= \sum_{t\in\mathbb{F}_p} (1 + \kappa(t))\,\bar{\psi}(t)e_p(t) = G(\bar{\psi}) + G(\kappa\bar{\psi}), \text{ as claimed.}$$

Our second observation is that the sextic Gauss sums can be evaluated in terms of cubic and quadratic ones by the formula

$$G(\kappa\bar{\psi}) = \bar{\psi}(2)\, G(\kappa)\, G(\psi)^2/p.$$

of B. C. Berndt and R. J. Evans (1979). Also, $G(\bar{\psi}) = p/G(\psi)$. We obtain the following result.

Proposition 1. *For every prime* $p \equiv 1 \bmod 6$, *one has*

$$E_p(\psi) = \frac{1 + QT^3}{2T} \quad \textit{with } T = G(\psi)/\sqrt{p}, \ \ Q = \bar{\psi}(2)\, G(\kappa)/\sqrt{p}.$$

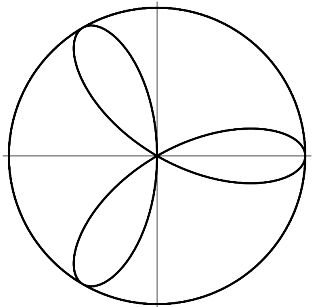
Here ψ *and* κ *are cubic and quadratic characters of* $\mathbb{F}_p$ *and* $|T| = |Q| = 1$.

Kepler trifolium.

Consider the complex plane $\mathbb{C}$ with the coordinates $x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$, $z \in \mathbb{C}$, the unit circle D centred at the origin 0, its boundary $\overline{D}$ and the curve C defined by the equation

$$(x^2 + y^2)^2 + 3xy^2 - x^3 = 0.$$

The C is known as the Kepler trifolium and also as the regular trifolium, the three leaf/petal rose, the three leaf/petal clover.



The curve C remains unchanged when rotated through the angles of $\pm 2\pi/3$. It can be given by the polar equation $r = \cos(3\varphi)$, r being a point on the axis obtained by rotation of the real axis through the angle φ.

Parametrization of trifolium

The Kepler trifolium C can be parametrized by the rational function

$$z \mapsto \frac{1+z^3}{2z} \quad \text{on} \quad \overline{D} = \{z \in \mathbb{C} \mid |z| = 1\}$$

Our observation is as follows.

Proposition 2. *The function above takes any point of* $\overline{D}$ *to some point of* C*. It takes cubic roots of* -1 *to the triple point* 0 *of* C*. It takes cubic roots of* 1 *to the cubic roots of* 1*. Except for the point* 0*, every point of* C *is the image of an unique point of* $\overline{D}$*. The boundary of each petal is the image of someone arc in* $\overline{D}$ *whose endpoints are the cubic roots of* -1*.*

All the statements can be proved by routine computation involving the equations defining $\overline{D}$ and C.

For our purposes, we need to complete the parametrization with one more observation.

Proposition 3 *Let* $v = \exp(it)$, $t \in \mathbb{R}$. *The image of* $\bar{D}$ *under the function*

$$z \mapsto \frac{1 + vz^3}{2z}$$

is $C' = wC$ *with* $w = \exp(it/3)$. *The curve* C' *is obtained by rotation of* C *around* 0 *through the angle* $t/3$.

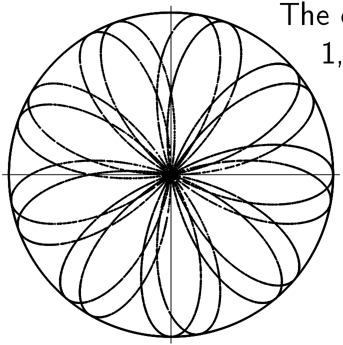
Indeed, as the point z runs over $\bar{D}$, the point wz runs over $\bar{D}$ also, and the point

$$\frac{1 + vz^3}{2z} = w\frac{1 + (wz)^3}{2(wz)}$$

runs over $C' = wC$ (by parametrization above), as claimed.

Distribution of the points $E_p(\psi)$

Now we are ready to give a theoretical explanation to our numerical observation on distribution of the sums $E_p(\psi)$. Compare the function $z \mapsto \dfrac{1+vz^3}{2z}$ in Proposition 3 with the formula $E_p(\psi) = \dfrac{1+QT^3}{2T}$ in Proposition 1. The z and T runs over $\bar{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$.



The only possible values of Q are six numbers $1, \omega, \omega^2, i, i\omega, i\omega^2$ with $\omega^3 = 1$. It follows that any point $E_p(\psi)$ belongs to some of six curves wC obtained by rotation of C around 0 through the angles $w = 0, \pm 2\pi/9, \pi/6, \pi/2 \pm \pi/9$,

$$E_p(\psi) \in \tilde{C} = \bigcup_w wC.$$

One concluding remark.

The points $T = G(\psi)/\sqrt{p}$ in Proposition 1 forms everywhere dense subset of the unit circle $\overline{D}$. That is known from research of the cubic Gauss sums related to the Kummer problem, D. R. Heath-Brown and S. J. Patterson (1979).

It seems likely that

the set of all the points $E_p(\psi)$ is everywhere dense in $\tilde{C}$.

That is an open question.