PERIODIC ORBITS in the SHAPE SPACE

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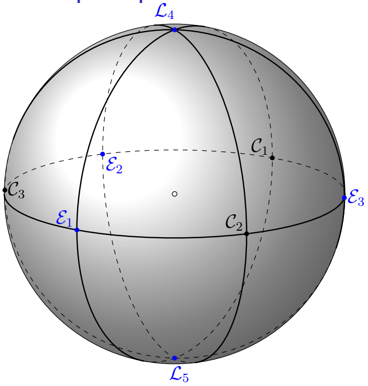
Shape space



Jacobie coordinates:

$$\mathbf{Q}_1 = \mathbf{r}_2 - \mathbf{r}_1,$$
$$\mathbf{Q}_2 = \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2},$$

Hopf transformation

$$\xi_1 = \tfrac{1}{2}\mu_1|\mathbf{Q}_1|^2 - \tfrac{1}{2}\mu_2|\mathbf{Q}_2|^2,$$
$$\xi_2 + i\,\xi_3 = \sqrt{\mu_1\mu_2}\mathbf{Q}_1\bar{\mathbf{Q}}_2$$

Each element of the space $\mathbb{X}$ is a class of oriented congruent triangles, it is called *form space*. In this space, the length of the vector (ξ_1, ξ_2, ξ_3) is equal to the moment of inertia:

$$I = m_1|\mathbf{r}_1|^2 + m_2|\mathbf{r}_2|^2 + m_3|\mathbf{r}_3|^2 = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}.$$

$$r_{12}^2 = \frac{m_1+m_2}{2m_1m_2}\left(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2} + \xi_1\right)$$
$$r_{13}^2 = \frac{m_1+m_3}{2m_1m_3}\sqrt{\xi_1^2+\xi_2^2+\xi_3^2} + \frac{m_2m_3 - m_1(m_1+m_2+m_3)}{2m_1m_3(m_1+m_2)}\xi_1 + \frac{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}}{m_1m_3(m_1+m_2)}\xi_2$$
$$r_{23}^2 = \frac{m_2+m_3}{2m_2m_3}\sqrt{\xi_1^2+\xi_2^2+\xi_3^2} + \frac{m_1m_3 - m_2(m_1+m_2+m_3)}{2m_2m_3(m_1+m_2)}\xi_1 - \frac{\sqrt{m_1m_2m_3(m_1+m_2+m_3)}}{m_2m_3(m_1+m_2)}\xi_2$$

Shape space, spherical coordinates

$$
\begin{aligned}
\xi_1 &= \rho\cos\varphi\cos\theta, \\
\xi_2 &= \rho\sin\varphi\cos\theta, \\
\xi_3 &= \rho\sin\theta,
\end{aligned}
\qquad
\begin{aligned}
r_{12}^2 &= \frac{m_1+m_2}{2m_1m_2}\,\rho(1+\cos\varphi\cos\theta), \\
r_{13}^2 &= \frac{m_1+m_3}{2m_1m_3}\,\rho(1-\cos(\varphi-\varphi_{13})\cos\theta), \\
r_{23}^2 &= \frac{m_2+m_3}{2m_2m_3}\,\rho(1-\cos(\varphi-\varphi_{23})\cos\theta).
\end{aligned}
$$

$$
V(\rho,\theta,\varphi) = \frac{1}{\sqrt{\rho}}\left(\frac{\nu_{12}}{\sqrt{1+\cos\theta\cos\varphi}} + \frac{\nu_{13}}{\sqrt{1-\cos\theta\cos(\varphi-\varphi_{13})}} + \frac{\nu_{23}}{\sqrt{1-\cos\theta\cos(\varphi-\varphi_{23})}}\right) = \frac{1}{\sqrt{\rho}}\,D(\theta,\varphi),
$$

Invariable configurations ($\theta = \mathrm{const}, \varphi = \mathrm{const}$):

$$
\begin{aligned}
\frac{\partial D(\varphi,\theta)}{\partial\theta} &= 0, \\
\frac{\partial D(\varphi,\theta)}{\partial\varphi} &= 0
\end{aligned}
$$

Invariable configurations, $m_1 - m_3 - m_2$

$\theta = 0$

$$
\begin{aligned}
r_{12} &= \sqrt{\rho}\sqrt{\frac{m_1+m_2}{2m_1m_2}}\cos\varphi/2,\\
r_{13} &= \sqrt{\rho}\sqrt{\frac{m_1+m_3}{2m_1m_3}}\sin(\varphi-\varphi_{13})/2,\\
r_{23} &= -\sqrt{\rho}\sqrt{\frac{m_2+m_3}{2m_2m_3}}\sin(\varphi-\varphi_{23})/2,
\end{aligned}
$$

we have

$$D(0,\varphi) = \frac{1}{\sqrt{2}}\left(\frac{\nu_{12}}{\cos\varphi/2} + \frac{\nu_{13}}{\sin(\varphi-\varphi_{13})/2} - \frac{\nu_{23}}{\sin(\varphi-\varphi_{23})/2}\right),$$

and

$$\frac{\partial D(0,\varphi)}{\partial\varphi} = \frac{1}{\sqrt{8}}\left(\nu_{12}\frac{\sin\varphi/2}{\cos^2\varphi/2} - \nu_{13}\frac{\cos(\varphi-\varphi_{13})/2}{\sin^2(\varphi-\varphi_{13})/2} + \nu_{23}\frac{\cos(\varphi-\varphi_{23})/2}{2\sin^2(\varphi-\varphi_{23})/2}\right).$$

$$z = r_{13}/r_{23} = -\frac{\sqrt{m_1m_2(m_1+m_2+m_3)}\tan\varphi/2 + m_2\sqrt{m_3}}{\sqrt{m_1m_2(m_1+m_2+m_3)}\tan\varphi/2 - m_1\sqrt{m_3}}$$

$$
\begin{aligned}
&(m_3+m_2)z^5 + (2m_3+3m_2)z^4 + (m_3+3m_2)z^3\\
&\qquad - (m_3+3m_1)z^2 - (2m_3+3m_1)z - (m_3+m_1) = 0.
\end{aligned}
$$

Motion in invariable configurations

$$V = \frac{C_1}{\sqrt{\rho}} \,.$$

Lagrange-Jacobie identity

$$\ddot{I} = 2\left(\frac{C_1}{\sqrt{I}} + 2h\right),$$

reduces ($r^2 = I, \frac{dt}{d\tau} = r = \sqrt{I}$) to

$$r'' = 2hr + C_1,$$

The solution for $h < 0$

$$r = \sqrt{I} = A\cos(n\tau - \vartheta) - \frac{C_1}{2h}, \quad n = \sqrt{-2h}.$$

Equations of motion

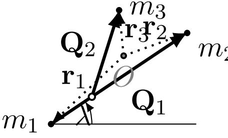
Kinetic energy:

$$T = \frac{\dot\xi_1^2+\dot\xi_2^2+\dot\xi_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} + \frac{\xi_3^2\dot\xi_2^2+\xi_2^2\dot\xi_3^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)^2} -$$

$$- \frac{\xi_2\dot\xi_2\xi_3\dot\xi_3}{4\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)^2} + \frac{\dot\lambda^2\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}{2} + \frac{\dot\lambda(\xi_2\dot\xi_3-\xi_3\dot\xi_2)}{2(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)}$$

V does not depend on λ and

$$\frac{\partial T}{\partial\lambda} = \sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\,\dot\lambda + \frac{\xi_2\dot\xi_3-\xi_3\dot\xi_2}{2(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)} = J$$



coincide with angular moment constant J.
Rauth function

$$\begin{aligned} R &= T+V-J\dot\lambda = \\ &= \frac{\dot\xi_1^2+\dot\xi_2^2+\dot\xi_3^2-4J^2}{8\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}} + \frac{J(\xi_2\dot\xi_3-\xi_3\dot\xi_2)}{2\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}(\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}+\xi_1)} + V \end{aligned}$$

and generalized energy integral:

$$\frac{\dot\rho^2+\rho^2\dot\theta^2+\rho^2\cos^2\theta\dot\varphi^2+4J^2}{8\rho} - \frac{D(\phi,\theta)}{\sqrt{\rho}} = h.$$

$2D$ General 3BP, Region of Possible Motion

Hamiltonian

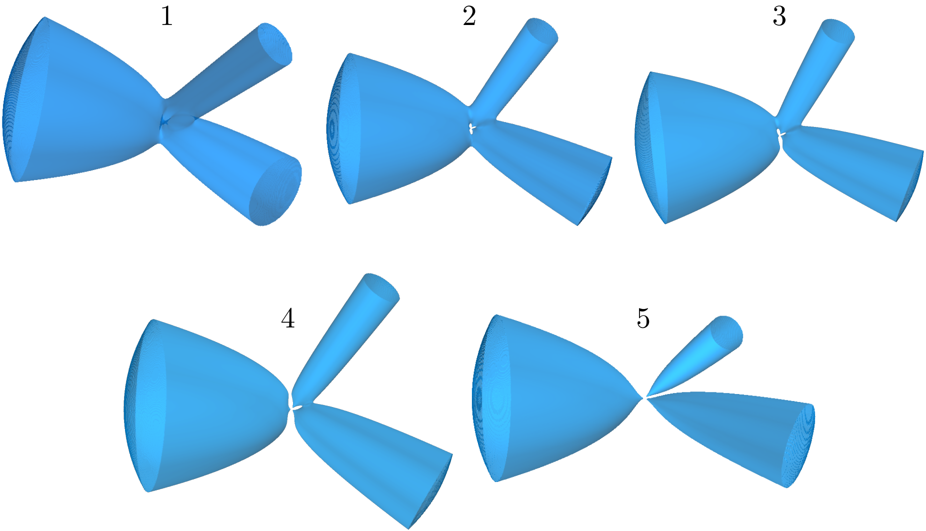
$$H = T - U = \frac{4J^2 + \dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}{8\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} - U(\xi_1, \xi_2, \xi_3) = h$$

$$U(\xi_1, \xi_2, \xi_3) = \sum_{i,j=1,i<j}^{3} \frac{m_i m_j}{r_{ij}} = \frac{1}{\sqrt[4]{\xi_1^2 + \xi_2^2 + \xi_3^2}} \, W(\varphi, \theta),$$

The surface of zero velocity (in $\xi_1\xi_2\xi_3$-space):

$$U(\xi_1, \xi_2, \xi_3) + h - \frac{J^2}{2\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}} = 0.$$

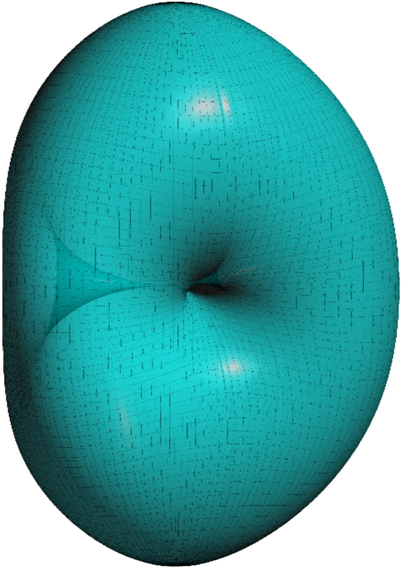
Zero velocity surfaces ($2D$ General Three-Body Problem)



Topology of available motion space

1. The available region is bounded by a surface with three branches. Motion is possible inside this space with the exception of the punctured point, the origin of coordinates. As J grows the outer surface with three branches decreases, and the punctured point becomes a surface whose cross-section with the equator plane resembles a trefoil, and the cross-section with the meridian plane has the shape «frigole»,
2. with growth of J, the outer and inner surfaces join together and a hole forms in the outer surface,
3. with further growth of J, two branches are first separated, the space of possible motion is still connected,
4. but as J increases, one branch separates from the other two, and finally,
5. beginning from a certain value of J, we have three separate areas of possible movement.

Inner surface



Symmetries

Scale symmetry

$$\begin{aligned} \boldsymbol{\rho}_i(t) &= \lambda \mathbf{r}_i(\lambda^{-3/2}t) \\ \dot{\boldsymbol{\rho}}_i(t) &= \lambda^{-1/2}\mathbf{v}_i(\lambda^{-3/2}t), \end{aligned}$$

with this

$$\begin{aligned} h' &= \lambda^{-1}h \\ J' &= \lambda^{1/2}J, \quad \lambda \in \mathbb{R}^+ \end{aligned}$$

Finite symmetry groups. 10 finite symmetry groups exist in planar three-body problem. Only three of them are considered: dihedral symmetry group (D_{12}, simple choreography), $2-1$-choreography and linear symmetry.

Periodic orbits

Periodic orbits are determined as minimizers of action functional

$$\mathcal{A} = \int_{t_1}^{t_2} L(\mathbf{q}_i, \dot{\mathbf{q}}_i, t)\, dt.$$

in the form

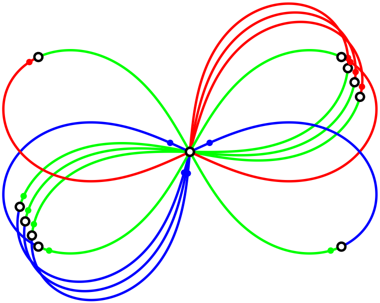
$$\begin{aligned} x_j(t) &= C_x^0 + \sum_{i=1} C_{x\,i}^j \cos\, it + S_{x\,i}^j \sin\, it \\ y_j(t) &= C_y^0 + \sum_{i=1} C_{y\,i}^j \cos\, it + S_{y\,i}^j \sin\, it, \end{aligned}$$

Figure-"Eight"

$$
\begin{aligned}
x_j(t) &= \sum_{i=1} C_{xi} \cos i(t + 2(j-1)\pi/3) + S_{xi} \sin i(t + 2(j-1)\pi/3) \\
y_j(t) &= \sum_{i=1} C_{y_i} \cos i(t + 2(j-1)\pi/3) + S_{y_i} \sin i(t + 2(j-1)\pi/3),
\end{aligned}
$$

If we use a baricentric coordinate system, we have

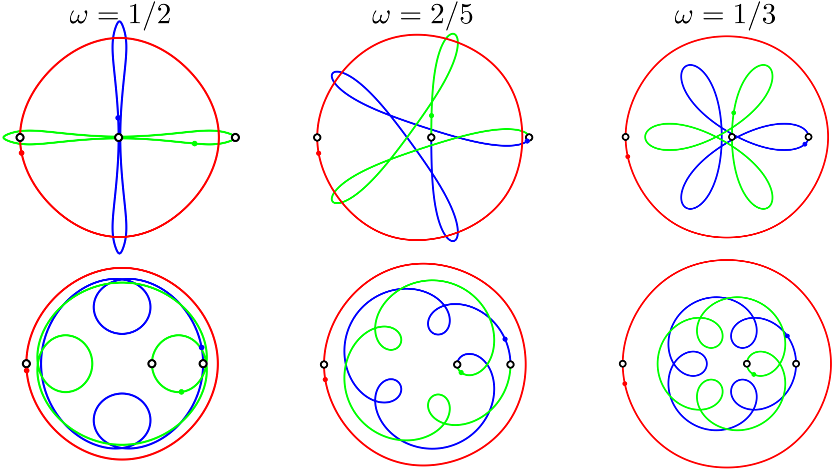
$$C_{xi} = S_{xi} = 0 \text{ if } i = 3m,\ m = 0, \ldots$$



$$
\begin{aligned}
x(t) &= \quad 0.4082 \sin t + 0.1973 \sin 2t \\
&\quad +0.0326 \sin 4t - 0.0094 \sin 5t \\
&\quad -0.0022 \sin 7t - 0.0017 \sin 8t \\
&\quad -0.0005 \sin 10t + 0.0002 \sin 11t, \\
y(t) &= -0.6409 \sin t + 0.1256 \sin 2t \\
&\quad +0.0208 \sin 4t + 0.0148 \sin 5t \\
&\quad +0.0034 \sin 7t - 0.0011 \sin 8t \\
&\quad -0.0003 \sin 10t - 0.0002 \sin 11t \\
&\quad -0.0001 \sin 13t.
\end{aligned}
$$

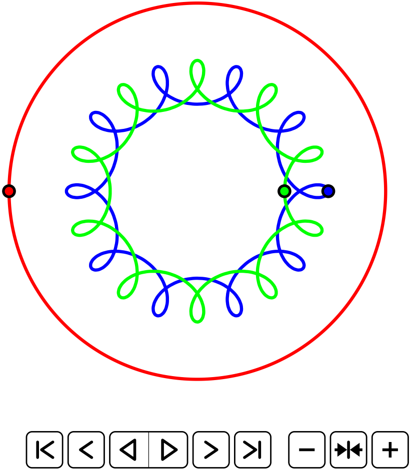
$2 - 1$ choreography

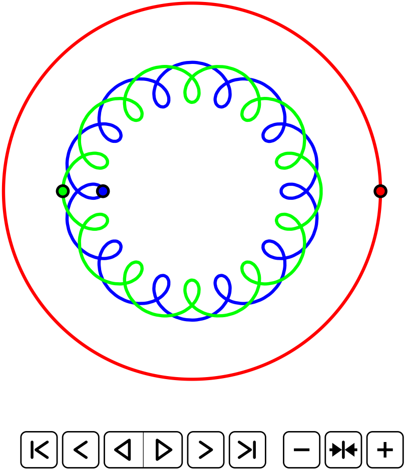
In this case two bodies of equal mass move along the same trajectory with a shift by π. A group of type R. An orbit in an inertial system is a minimizer found in a system rotating with angular velocity ω.



Close binary

$$\omega = 1/3,\ k = 5$$



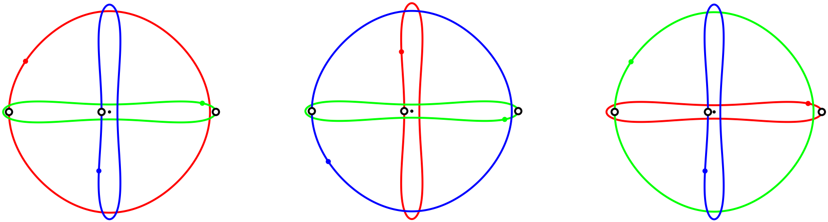


$2-1$-symmetry orbits

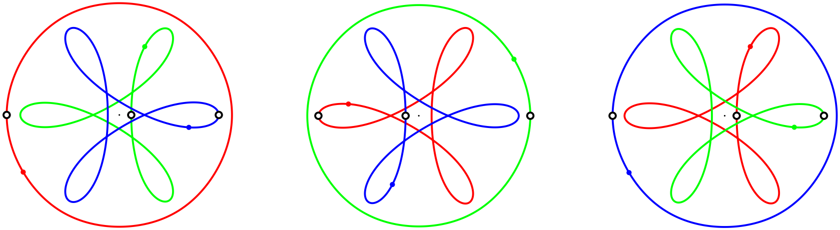
	$m_1 = m_2 = 0.95,\ m_3 = 1.1$				
A	E	$\lvert C\rvert$	ω	$[I_{\mathsf{min}}, I_{\mathsf{max}}]$	Stab.
10.61083	-0.562922	1.73204	1/5	[13.520,18.706]	+
11.87886	-0.630193	1.34061	1/3	[7.646,7.695]	+
12.41405	-0.658586	1.22094	2/5	[6.446,6.518]	+
12.43822	-0.850687	3.17929	1/5	[13.037,13.062]	+
13.13826	-0.697007	1.09433	1/2	[5.463,5.580]	+
14.90941	-0.790968	2.76171	1/3	[6.779,6.847]	+
16.03507	-0.850687	2.61695	2/5	[5.352,5.457]	+
16.57031	-0.879082	2.44831	1/3	[3.869,3.957]	-
17.61955	-0.934746	2.43060	1/2	[3.967,4.154]	-
19.78460	-1.049610	1.57727	1/3	[6.501,6.503]	+
21.89957	-1.161810	2.58582	1/3	[6.441,6.443]	+
25.74992	-1.366082	1.65989	1/3	[6.380,6.381]	+
27.53447	-1.460752	2.51159	1/3	[6.362,6.363]	+
$m_1 = m_2 = 1.05,\ m_3 = 0.9$					
12.20094	-0.647280	0.98928	1/3	[7.276,7.323]	+
15.79177	-0.837779	2.68412	1/3	[6.275,6.341]	+
16.61662	-0.881539	2.33447	1/3	[3.779,3.943]	-
Figure-eight: $m_1 = m_2 = m_3 = 1.0$					
24.37193	-1.29297	0	-	[1.973,1.982]	+

Line symmetries orbits

Three linear symmetry orbits (for $\omega = 1/2$ and $\omega = 1/3$) corresponding to the cyclic permutation of masses m_1, m_2, m_3.

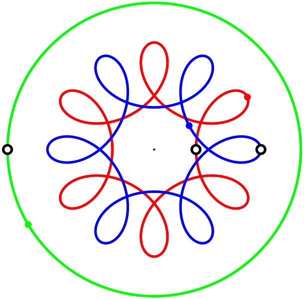


$m_1 = 0.99$, $m_2 = 1.01$, $m_3 = 1.0$

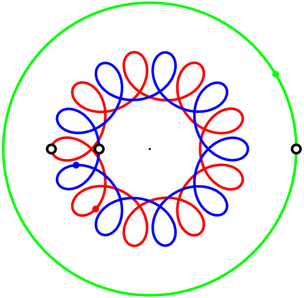


Orbits with close binary

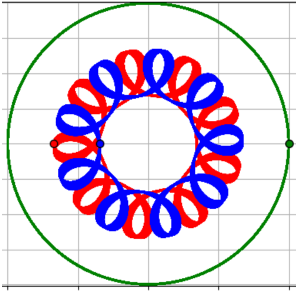
$\omega = 1/3$



$k = 2$



$k = 3$

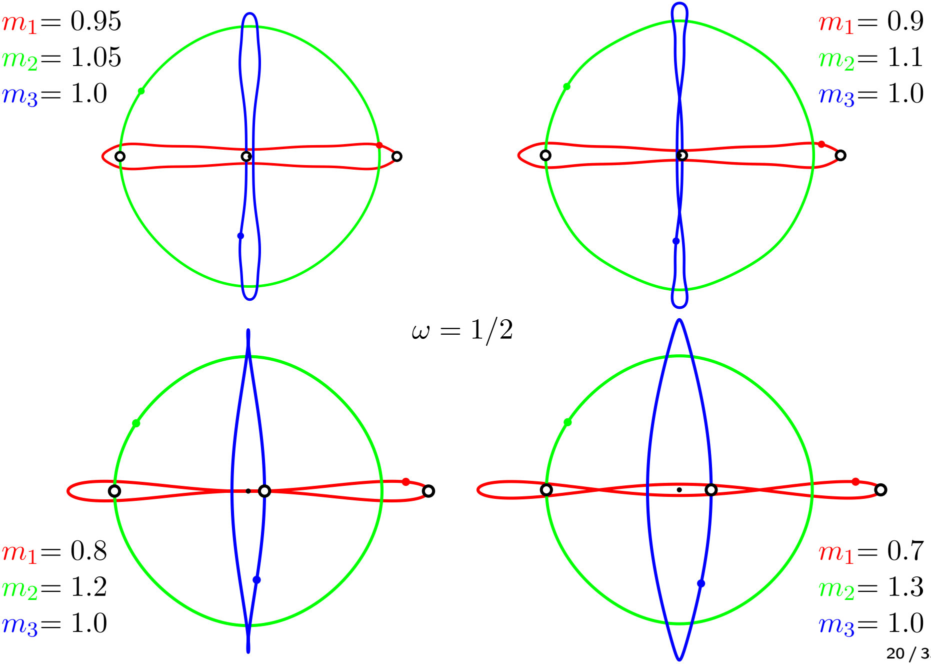


Numerical integration
500 periods

Line symmetries orbits

$m_1 = 0.99,\ m_2 = 1.01,\ m_3 = 1.0$					
A	E	$\|J\|$	ω	$[I_{\text{min}}, I_{\text{max}}]$	Stab.
11.42286	-0.606002	1.36301	1/4	[10.095,10.108]	+
12.04740	-0.639135	1.19429	1/3	[7.508,7.557]	+
12.06332	-0.639979	1.17690	1/3	[7.489,7.538]	+
12.07962	-0.640844	1.15915	1/3	[7.471,7.520]	+
13.15385	-0.697833	0.92132	1/2	[5.474,5.590]	+
13.15566	-0.697930	0.93926	1/2	[5.474,5.590]	+
13.15748	-0.698026	0.95484	1/2	[5.534,5.591]	+
14.06146	-0.745984	0.83708	1/3	[5.156,5.393]	-
14.08066	-0.747002	0.85327	1/3	[5.114,5.347]	-
14.09948	-0.748001	0.86909	1/3	[5.098,5.332]	-
14.55725	-0.772286	0.88706	1/4	[5.253,5.574]	-
16.64808	-0.883208	1.19288	1/3	[3.830,3.964]	-
16.76479	-0.889400	1.37020	1/3	[6.487,6.492]	+
17.80747	-0.944715	2.06327	1/4	[3.189,3.517]	-
20.59152	-1.09242	1.45497	1/3	[6.276,6.278]	+

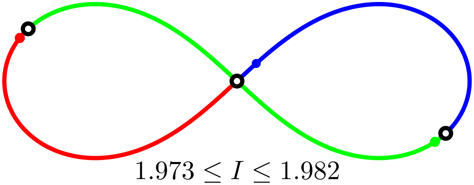
Masses variation

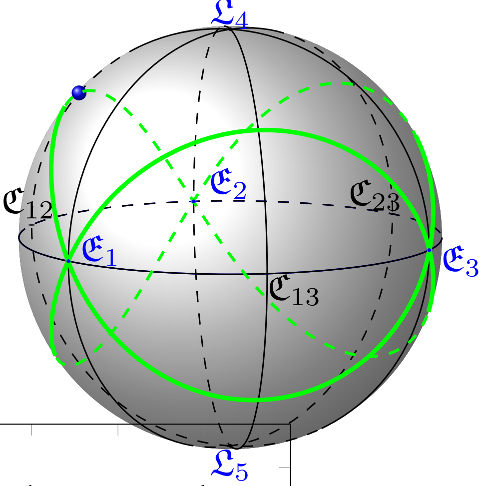


Various masses

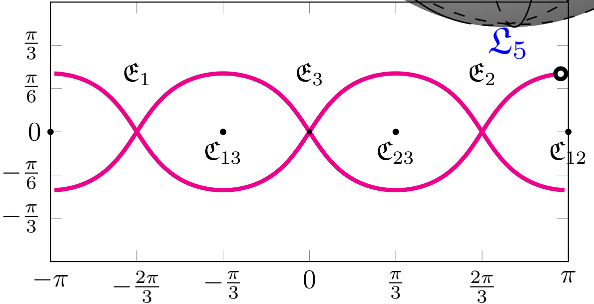
m_1	$m_1 + m_2 = 2$, $m_3 = 1.0$, $\omega = 1/2$				
	A	E	$\|J\|$	$[I_{\text{min}}, I_{\text{max}}]$	Stab.
0.99	13.15748	-0.698026	0.95484	[5.534,5.591]	+
0.95	13.15312	-0.697795	1.01964	[5.470,5.588]	+
0.9	13.12580	-0.696348	1.09648	[5.456,5.575]	+
0.8	12.99779	-0.689554	1.23654	[4.722,5.568]	+
0.7	12.77091	-0.677518	1.35872	[4.054,5.800]	+

Figure-Eight on shape sphere

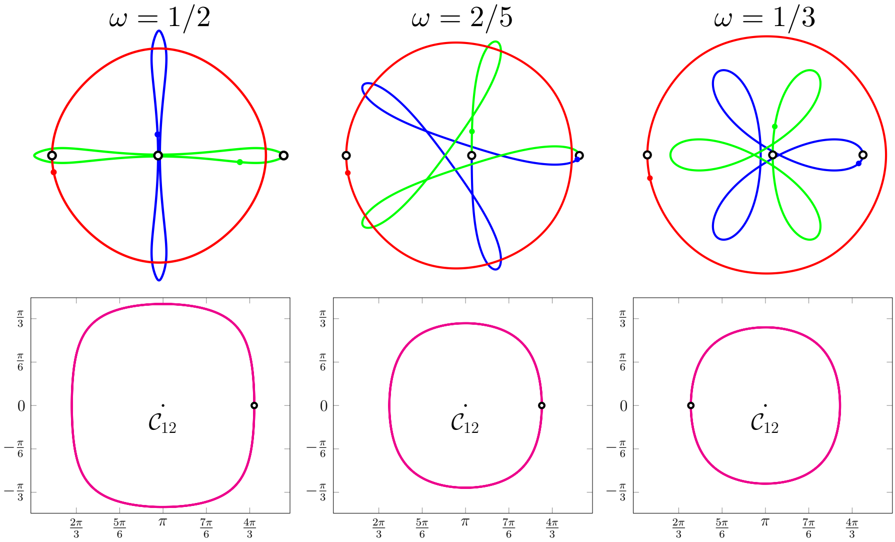




In φ, θ-plane:



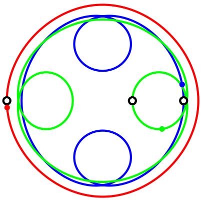
$2 - 1$ choreography

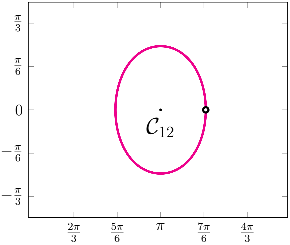


$5.462 \leq I \leq 5.580$ | $6.446 \leq I \leq 6.518$ | $5.462 \leq I \leq 5.580$ $(m_1 = m_2 = 1.05, m_3 = 0.9)$

$2 - 1$ choreography

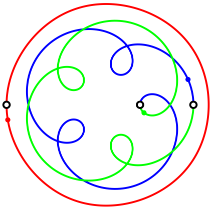
$\omega = 1/2$

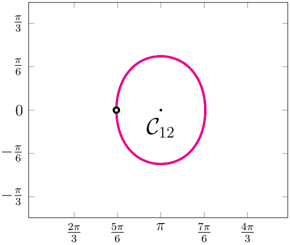




$3.967 \leq I \leq 4.154$

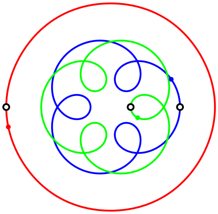
$\omega = 2/5$

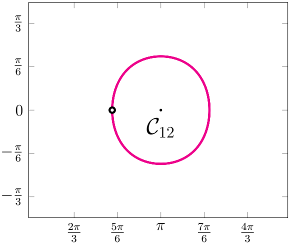




$5.352 \leq I \leq 5.457$

$\omega = 1/3$

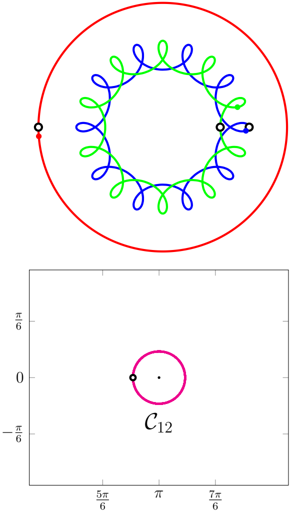




$6.275 \leq I \leq 6.341$
$(m_1 = m_2 = 1.05, m_3 = 0.9)$

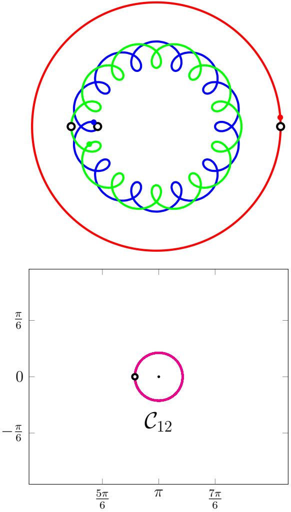
$2-1$ choreography, close binary

$\omega = 1/3$



$$6.380 \leq I \leq 6.381$$

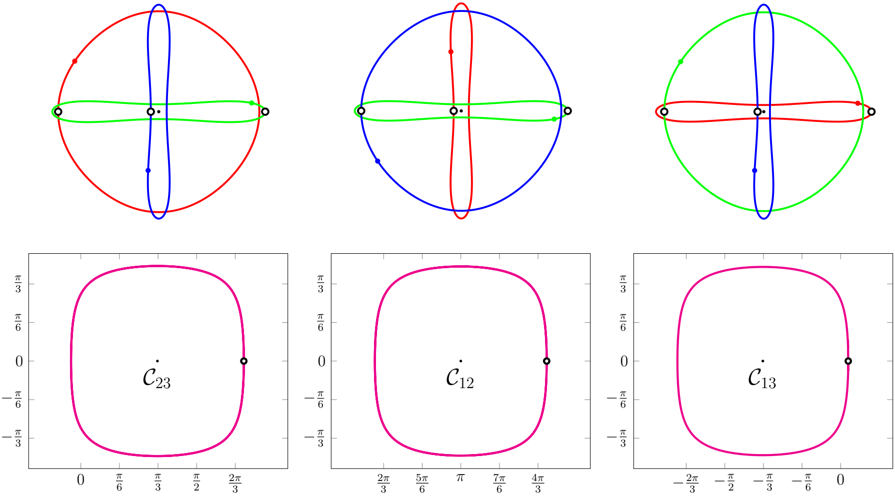
$k = 5$



$$6.362 \leq I \leq 6.363$$

Line symmetry orbits

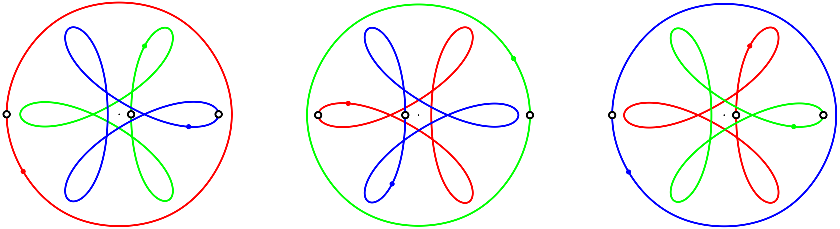
Three linear symmetry orbits corresponding to the cyclic permutation of masses m_1, m_2, m_3, $\omega = 1/2$.

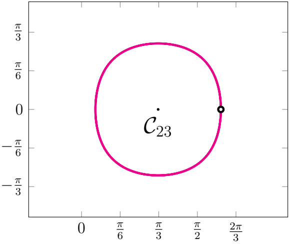


$5.474 \leq I \leq 5.590$ $\quad$ $5.474 \leq I \leq 5.590$ $\quad$ $5.534 \leq I \leq 5.591$

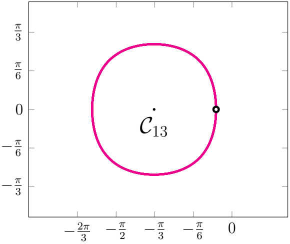
Line symmetry orbits

Three linear symmetry orbits corresponding to the cyclic permutation of masses m_1, m_2, m_3, $\omega = 1/3$.

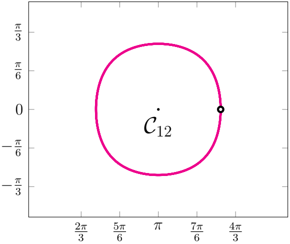




$7.471 \leq I \leq 7.520$



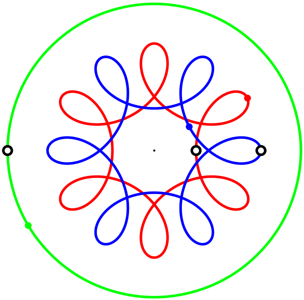
$7.508 \leq I \leq 7.557$



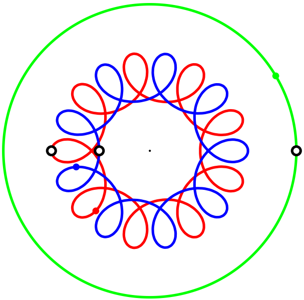
$7.489 \leq I \leq 7.538$

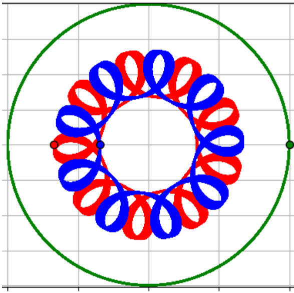
Orbits with close binary, $\omega = 1/3$

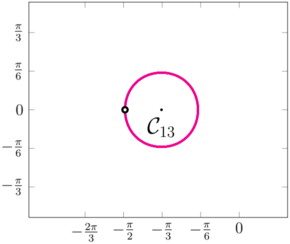
$k = 2$



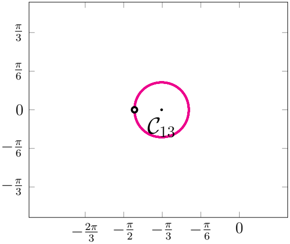
$k = 3$







$6.487 \leq I \leq 6.492$

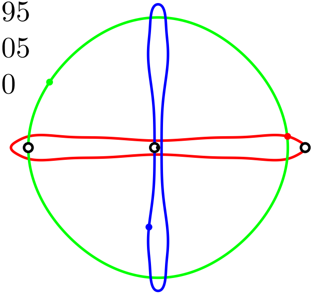


$6.276 \leq I \leq 6.278$

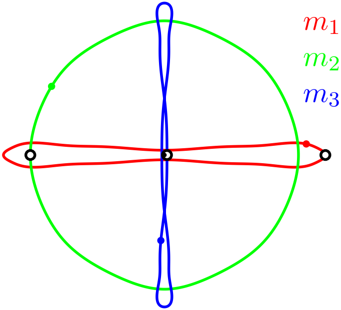
Numerical integration
500 periods

Different masses

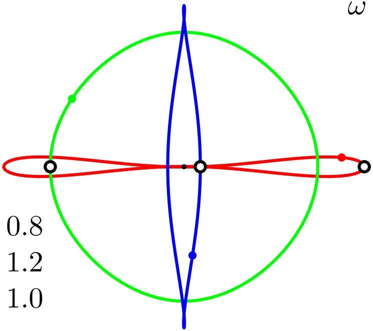
$m_1 = 0.95$
$m_2 = 1.05$
$m_3 = 1.0$



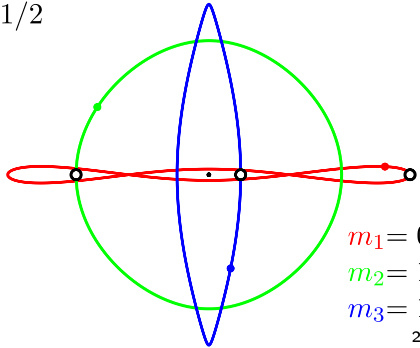
$m_1 = 0.9$
$m_2 = 1.1$
$m_3 = 1.0$



$\omega = 1/2$



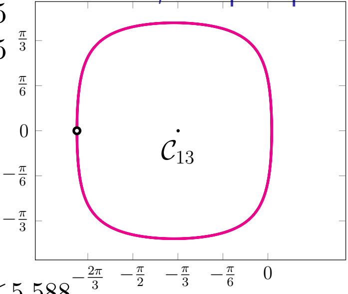
$m_1 = 0.8$
$m_2 = 1.2$
$m_3 = 1.0$



$m_1 = 0.7$
$m_2 = 1.3$
$m_3 = 1.0$

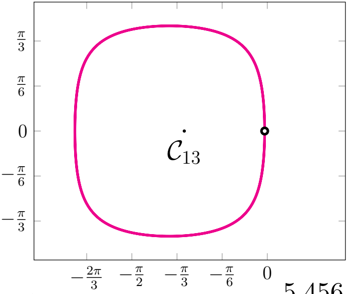
Different masses, shape sphere

$m_1 = 0.95$
$m_2 = 1.05$
$m_3 = 1.0$



$5.470 \leq I \leq 5.588$

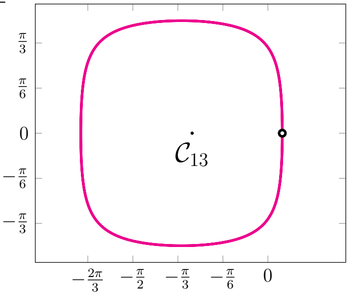
$m_1 = 0.9$
$m_2 = 1.1$
$m_3 = 1.0$



$5.456 \leq I \leq 5.575$

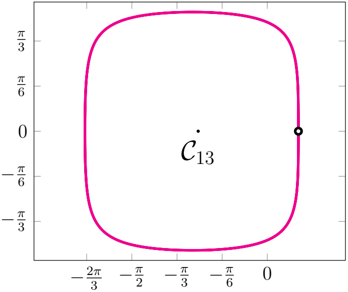
$\omega = 1/2$

$4.722 \leq I \leq 5.568$



$m_1 = 0.8$
$m_2 = 1.2$
$m_3 = 1.0$

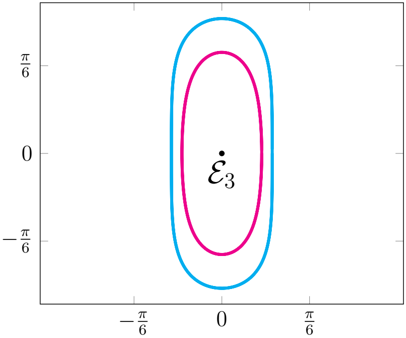
$4.054 \leq I \leq 5.800$

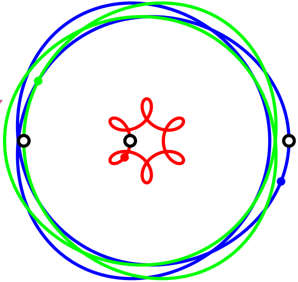


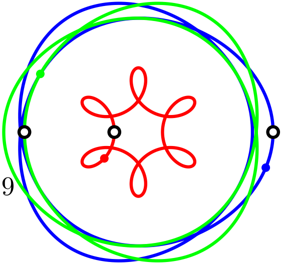
$m_1 = 0.7$
$m_2 = 1.3$
$m_3 = 1.0$

$\mathcal{E}_2,\ \omega = 1/3$

$3.869 \leq I \leq 3.957$



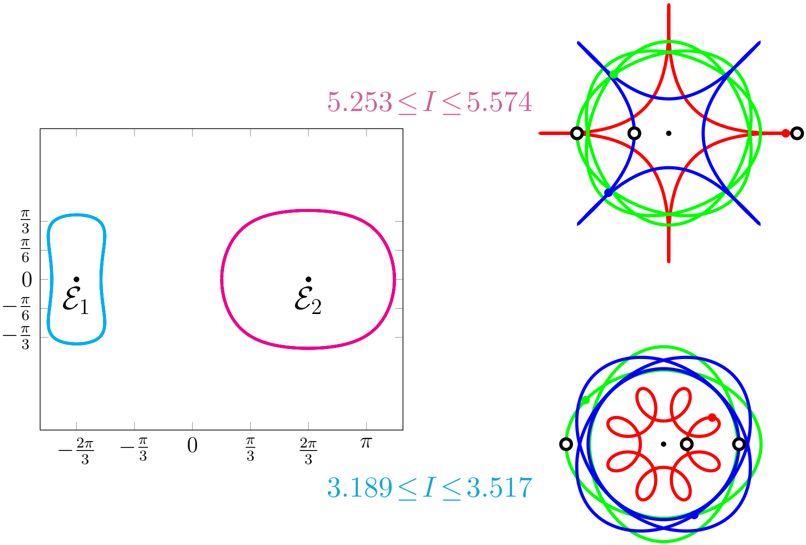




$3.779 \leq I \leq 3.943$

$m_1 = m_2 = 1.05, m_3 = 0.9$

$\mathcal{E}_i,\ \omega = 1/4$



THANKS!!!