Periodic orbits in the shape space

Vladimir Titov

Abstract. Periodic orbits of the general three bodies problem with linear symmetry and $2-1$-symmetry are considered in the shape space. The moment of inertia of the orbits under consideration does not change much during the period, so it is enough to look these orbits in the shape sphere or in the space of angular coordinates.

Introduction

It is shown in [1] that the finite symmetry groups of the general planar three-body problem are exhausted by 10 groups. Two of these groups served as the basis for the search for symmetric periodic solutions [2]. The found trajectories can be mapped into the shape space, such transformation is unique up to the rotation of the original barycentric system. Since the distance from the origin varies little in the shape space for these trajectories (within $\pm 10\%$), then for qualitative analysis we can limit ourselves to their projection onto the shape sphere.

1. The shape space

The shape space is the space of congruent triangles, configurations of the general three body problem. This space is obtained by reducing the configuration space of the problem by translations, and then by reducing by rotations. The first reduction is performed by moving to the Jacobi coordinates

$$\begin{aligned}\mathbf{Q}_1 &= \mathbf{r}_2 - \mathbf{r}_1,\\ \mathbf{Q}_2 &= \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2},\end{aligned}$$

the second is the Hopf map: considering the coordinates $\mathbf{Q}_1$ and $\mathbf{Q}_2$ as points of a complex space, we introduce

$$\begin{aligned}\xi_1 &= \frac{1}{2}\mu_1|\mathbf{Q}_1|^2 - \frac{1}{2}\mu_2|\mathbf{Q}_2|^2,\\ \xi_2 + i\xi_3 &= \sqrt{\mu_1\mu_2}\,\mathbf{Q}_1\bar{\mathbf{Q}}_2.\end{aligned} \tag{1}$$

The three-dimensional space $\Xi = (\xi_1, \xi_2, \xi_3)$ is the space of oriented congruent triangles, each point of this space represents a class of such oriented congruent triangles. This space is called the *shape space*, and it is in this space that we will study the properties of the solutions to the three-body problem under consideration.

TABLE 1. Orbits with line symmetry

$m_1 = 0.99$, $m_2 = 1.01$, $m_3 = 1.0$					
A	E	$\lvert J\rvert$	ω	$[I_{\min}, I_{\max}]$	Stab
11.42286	-0.606002	1.36301	1/4	[10.095,10.108]	+
12.04740	-0.639135	1.19429	1/3	[7.508,7.557]	+
12.06332	-0.639979	1.17690	1/3	[7.489,7.538]	+
12.07962	-0.640844	1.15915	1/3	[7.471,7.520]	+
13.15385	-0.697833	0.92132	1/2	[5.474,5.590]	+
13.15566	-0.697930	0.93926	1/2	[5.474,5.590]	+
13.15748	-0.698026	0.95484	1/2	[5.534,5.591]	+
14.06146	-0.745984	0.83708	1/3	[5.156,5.393]	-
14.08066	-0.747002	0.85327	1/3	[5.114,5.347]	-
14.09948	-0.748001	0.86909	1/3	[5.098,5.332]	-
14.55725	-0.772286	0.88706	1/4	[5.253,5.574]	-
16.64808	-0.883208	1.19288	1/3	[3.830,3.964]	-
16.76479	-0.889400	1.37020	1/3	[6.487,6.492]	+
17.80747	-0.944715	2.06327	1/4	[3.189,3.517]	-
20.59152	-1.09242	1.45497	1/3	[6.276,6.278]	+

In this space the moment of inertia I is given by simple expression:

$$I = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}. \tag{2}$$

that is, in the shape space it represents the distance of a point (ξ_1, ξ_2, ξ_3) to the origin of coordinates and, if you enter in space Ξ spherical coordinates ρ, φ, θ, then the coordinate ρ is naturally considered the size of a triangle, and φ, θ are angular variables that determine its shape. Naturally, the coordinate ρ coincides with the moment of inertia. In the shape space, all properties related to the moment of inertia of the system are naturally related to the size of the triangle. You can take the square root of ρ for the size of the triangle, then the unit of such value will coincide with the unit of the length. In any case, the points of a sphere of any fixed radius, for example, $\rho = 1$ or $\rho = 1/2$, will be responsible precisely for the shape of a triangle, such a sphere is called *shape sphere*, and the entire shape space is a cone above this sphere with a vertex at the point of triple collision $(0, 0)$. Thus, a point on the shape sphere is a class of similar triangles, all points on the ray in the shape space, beginning in the origin correspond to similar configurations of three bodies and differ only by size (see, for example [3].

Evidently, the equator of the sphere of shapes (and the plane $\xi_3 = 0$) correspond to collinear configurations ($\mathbf{Q}_1 \times \mathbf{Q}_2 = 0$). Thus, all points of double

TABLE 2. Orbits with $2-1$-symmetry and "figure-eight"

A	E	$\lvert C\rvert$	ω	$[I_{\min}, I_{\max}]$	Stab
$m_1 = m_2 = 0.95$, $m_3 = 1.1$					
10.61083	-0.562922	1.73204	1/5	[13.520,18.706]	+
11.87886	-0.630193	1.34061	1/3	[7.646,7.695]	+
12.41405	-0.658586	1.22094	2/5	[6.446,6.518]	+
12.43822	-0.850687	3.17929	1/5	[13.037,13.062]	+
13.13826	-0.697007	1.09433	1/2	[5.463,5.580]	+
14.90941	-0.790968	2.76171	1/3	[6.779,6.847]	+
16.03507	-0.850687	2.61695	2/5	[5.352,5.457]	+
16.57031	-0.879082	2.44831	1/3	[3.869,3.957]	-
17.61955	-0.934746	2.43060	1/2	[3.967,4.154]	-
19.78460	-1.049610	1.57727	1/3	[6.501,6.503]	+
21.89957	-1.161810	2.58582	1/3	[6.441,6.443]	+
25.74992	-1.366082	1.65989	1/3	[6.380,6.381]	+
27.53447	-1.460752	2.51159	1/3	[6.362,6.363]	+
$m_1 = m_2 = 1.05$, $m_3 = 0.9$					
12.20094	-0.647280	0.98928	1/3	[7.276,7.323]	+
15.79177	-0.837779	2.68412	1/3	[6.275,6.341]	+
16.61662	-0.881539	2.33447	1/3	[3.779,3.943]	-
Figure-eight: $m_1 = m_2 = m_3 = 1.0$					
24.37193	-1.29297	0	-	[1.973,1.982]	+

collisions at any mass values lie on the equator, in the space of shapes, three rays correspond to double collisions (in the plane of the equator).

2. Periodic orbits

To find a periodic solution to the three-body problem, it is enough to find a (local) minimizer of the functional of the action of the problem in the space of 2π-periodic functions, periodic solutions with a period other than 2π can be found using the scale symmetry and you can search for a solution in the form

$$\begin{aligned} x_j(t) &= C_x^0 + \sum_{i=1} C_{x\,i}^j \cos it + S_{x\,i}^j \sin it \\ y_j(t) &= C_y^0 + \sum_{i=1} C_{y\,i}^j \cos it + S_{y\,i}^j \sin it, \end{aligned} \tag{3}$$

where j is the body number.

Barutello et al. [1] showed that all finite symmetry groups of the Lagrangian action functional in the planar three-body problem contain only ten elements. The corresponding 2π-periodic solutions for three groups from this list, the ¡¡linear¿¿ group, $2-1$-choreographies and the dihedral group D_{12}, are obtained. These solutions are shown in the table 1 and table 2.

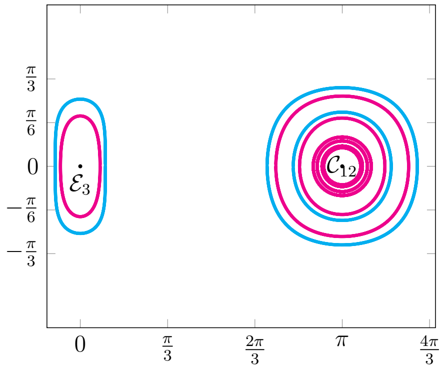

FIGURE 1. Trajectories with $2-1$-symmetry, $\omega = 1/3$

Due to the scale symmetry the solutions of the three-body problem $\mathbf{r}(t)$ can be reduced to solutions with a fixed value ($h = -1/2$ if the energy constant is negative) $\mathbf{r}(t) = \lambda\mathbf{r}(\lambda^{-1/2}t)$: $\lambda = h/h' = 2E$.

As can be seen from the tables, the moment of inertia of the obtained periodic orbits changes little, usually less than a few percent, so it is interesting to look at the obtained trajectories on the shape sphere. Figure **??** shows projections of trajectories from the table 2 with $\omega = 1/3$ on the plane of φ, θ.

Conclusion

Periodic orbits are determined by the variational method, these ones on the shape space, more precisely on the shape sphere, look in the case of $2-1$-symmetry look like small quasi-circles around the point of double collisions $\mathcal{C}_{12}$, or around the point of the corresponding Eulerian configuration $\mathcal{E}_3$; in the case of linear symmetry, where all three masses are arbitrary, quasi-circles can be located both around the points $\mathcal{C}_{12}$, $\mathcal{C}_{13}$, $\mathcal{C}_{23}$, and around the points $\mathcal{E}_1$, $\mathcal{E}_2$ or $\mathcal{E}_3$.

References

[1] Barutello, V., Ferrario D., and Terracini S.: 2008, ArRMA 190, 2

[2] Titov, V.: 2006. Few-Body Problem: Theory and Computer Simulations. University of Turku, Finland. Annales Universitatis Turkuensis, Series 1A. Astronomica–Chemica–Physica–Mathematica. 358, 9–15.

[3] Titov, V. B.: 2022. Zapiski nauchnyh seminarov POMI, 517, 1, 225–249.

Vladimir Titov
Department of Celestial Mech., St. Petersburg State University, St. Petersburg, Russia
e-mail: `tit@astro.spbu.ru`