The Euler International Mathematical Institute

St.Petersburg Department of Steklov Institute of Mathematics of the Russian Academy of Sciences

$$C = \lim_{n\to+\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x}\right) dx$$

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

Polynomial Computer Algebra

$$e^{i\pi} = -1$$

$$\frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$$

April 15-20, 2024

$$d^2 = R^2 - 2rR$$

$$\int_0^{\infty} x^{a-1} e^{-x}\, dx$$

$$\int_a^b F(x, y, y')\, dx \to \max$$

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2) =$$
$$= (ap+bq+cr+ds)^2 + (aq-bp+cs-dr)^2 +$$
$$+ (ar-bs-cp+dq)^2 + (as+br-cq-dp)^2$$

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{\frac{3k^2+k}{2}}$$

$$\sqrt{ax^2+bx+c} = t - \sqrt{a}x$$
$$\sqrt{ax^2+bx+c} = xt + \sqrt{c}$$
$$\sqrt{ax^2+bx+c} = t(x-\lambda)$$

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, dx$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2 n^2}\right)$$

$$V - E + F = 2$$

Saint Petersburg

Russian Academy of Sciences
St.Petersburg Department
of Steklov Mathematical Institute
Euler International Mathematical Institute
St.Petersburg Electrotechnical University "LETI"

International Conference

Polynomial Computer Algebra

International Conference on Polynomial Computer Algebra

St. Petersburg, April, 2024

Санкт Петербург
2024

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The Euler top and the Lagrange top as two special cases of the Galois top

Semjon Adlaj

The MacCullagh ellipsoid of inertia is transformed if we relocate its centre along the Galois axis [7, 8, 9, 11, 13, 14, 15]. The two principal axes, corresponding to extreme moments of inertia would rotate (with respect to rigid body reference frame), whereas the direction of the principal axis, corresponding to the intermediate moment of inertia is preserved (remaining orthogonal to the Galois axis). The latter (newly emerging, rotated and rescaled)[1] MacCullagh ellipsoid of inertia shares the same Galois axis with the former ellipsoid. We have thus ensured that the Galois top, that is, a heavy top in a uniform gravitational field which fixed point lies at a Galois axis, is well-defined.

The Euler top and the Lagrange top are thereby seen as two special cases of the Galois top. The Euler top corresponds to the special case of the Galois top which fixed point coincides with its centre of mass, whereas the Lagrange top corresponds the special case of the Galois top which Galois axis coincides with a principal axis of inertia, that is, the case where two principal moments of inertia did coincide one with the other.[2]

The Galois essential and alternative elliptic functions, which were discussed in [1, 2, 3, 4, 5, 6, 10, 12, 14], provide the natural means for constructing explicit solutions to both Euler top and Lagrange top, as well as, vividly exhibiting their symmetries.

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[1] Recall that the length of a principal axis of MacCullagh ellipsoid is proportional to the square root of the corresponding principal moment of inertia.

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On multidimensional analogs of Euler (Tait-Bryan) angles and Grassmanians.

M. V. Babich, L. A. Bordag, A. Khvedelidze and D. Mladenov

Abstract. The position of a frame in $\mathbb{R}^3$ can be parametrized by the famous Euler angles: *precession* represents a rotation around the z axis, *nutation* represents a rotation around the $\mathcal{N}$ or x' axis, and *intrinsic rotation* represents a rotation around the Z or z'' axis.

Another angles were introduced in last of XIX by P. G. Tait and G. H. Bryan. The angles are called *yaw*, *pitch*, and *roll* and are used in the aerospace technique now. Yaw axis z is a vertical axis, pitch axis y is a horizontal axis running from left to right parallel to the wings of a winged aircraft, and roll axis is a longitudinal axis x, an axis drawn from tail to nose.

I will demonstrate that it is the Tait-Bryan parametrization that can be generalized on the case of the spliting of a space $\mathbb{R}^N$ or $\mathbb{C}^N$ on the orthogonal subspaces of arbitrary dimensions.

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On parametrization of orthogonal symplectic matrices and its applications

Alexander Batkhin and Alexander Petrov

Abstract. The method of computing the parametric representation of an orthogonal symplectic matrix is considered. The dimension of the family of such matrices is calculated. The general structure of matrices of small even dimensions up to 8 is discussed in detail. A conjecture on the structure of a skew symmetric matrix generating a generic orthogonal symplectic transformation is formulated. The problem of constructing an orthogonal symplectic matrix of dimension 4 by a given vector is solved. The application of this transformation to the study of families of periodic solutions of an autonomous Hamiltonian system with two degrees of freedom is discussed.

1. Introduction

While studying the phase flow of a non-integrable Hamiltonian system, it is usually assumed that there is some information about its invariant varieties: equilibrium positions, periodic solutions or invariant tori of different dimensions. In this case, one can compute the normal form of the system near the corresponding variety and use it to obtain information on the stability of this variety, local integrability in its vicinity, the nature of bifurcations at small changes of parameters and, under certain conditions, asymptotically integrate the normalized system of equations.

For studying dynamics near invariant varieties of dimension greater than zero, the normalization technique is less well developed. Here, either a Poincaré mapping reduces a continuous-time Hamiltonian system to a mapping that preserves the phase volume, or a special coordinate transformation is performed to simplify the study of phase flow. Successful continuation along the family requires the computation of normal and tangent displacements. Previously, such displacements were computed by reducing the system to Birkhoff normal form, which implied additional computational cost. Then variants of the method appeared, when at each

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step of integration along the periodic solution an orthogonal-symplectic transformation was performed [1, 2], or integration was performed in the Fresné basis [3]. Later, in a series of papers by Kreisman [4, 5], it was shown that it is sufficient to do such a transformation once after the monodromy matrix of the periodic solution has been computed. In presented paper we provide a general algorithm of computation of an generic *orthogonal symplectic matrix* (or simply OSM) of any even dimension and gives more precise description of their structure dimension equals to 4.

2. General form of orthogonal-symplectic matrix

Hereafter, $\mathbf{B}_n$ denotes a square real matrix of dimension $2n \times 2n$; $\mathbf{E}_n$ and $\mathbf{E}$ are unit matrices of dimension $n \times n$ and $2n \times 2n$, respectively. The sign $^\top$ denotes the transpose operation of a matrix or vector.

Let us state a theorem that allows to parametrize an OSM of a general form

$$\mathbf{A}_n^\top \mathbf{A}_n = \mathbf{E}, \quad \mathbf{A}_n^\top \mathbf{J} \mathbf{A}_n = \mathbf{J}.$$

Theorem 1 ([6])**.** *A matrix $\mathbf{A}_n$ is symplectic if and only if the matrix $\mathbf{\Psi}_n$*

$$\mathbf{\Psi}_n = -2\mathbf{J}(\mathbf{E} + \mathbf{A}_n)^{-1}(\mathbf{A}_n - \mathbf{E}), \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{E}_n \\ -\mathbf{E}_n & \mathbf{0} \end{pmatrix} \tag{1}$$

is symmetric.

In fact, Theorem 1 allows us to constructively build a matrix $\mathbf{A}_n$ that is both orthogonal and symplectic. Such a class of matrices turns out to be very useful in the study of families of periodic solutions of Hamiltonian systems, and for critical solutions it allows to determine the type of bifurcation of the family (for details see [5, 7]).

According to Theorem 1 the computations can organized as follows.

1. Define an arbitrary skew symmetric matrix $\mathbf{K}_n$ of size $2n \times 2n$, which is uniquely defined by $n(2n-1)$ elements.
2. Compute the orthogonal matrix $\mathbf{A}_n$ by the Cayley-like formula
$$\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)\,(\mathbf{E} - \mathbf{K}_n)^{-1}, \tag{2}$$
which is always possible due to the fact that a skew symmetric matrix $\mathbf{K}_n$ does not have non-zero real eigenvalues.
3. By (1) we get the expression of the matrix $\mathbf{\Psi}_n$ through the matrix $\mathbf{K}_n$.
4. Using the symmetry condition $\mathbf{\Psi}_n^\top = \mathbf{\Psi}_n$, we obtain a system of relations between the elements of the matrix $\mathbf{K}_n$.
5. According to Theorem 1 the matrix $\mathbf{A}_n$ is symplectic.

Theorem 2. *The number of independent elements of the matrix $\mathbf{K}_n$, which defines by Cayley's formula* (2) *the symplectic matrix $\mathbf{A}_n$, is equal to n^2. The number of relations between the elements of the matrix $\mathbf{K}_n$ is $n(n-1)$.*

3. Representation of OSMs for $n = 2$

Let the matrix $\mathbf{K}_2$ be of the form $\mathbf{K}_2 = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & k_4 & k_5 \\ -k_2 & -k_4 & 0 & k_6 \\ -k_3 & -k_5 & -k_6 & 0 \end{pmatrix}$. Carrying out the calculations of items 1–3, we obtain that for the matrix $\mathbf{A}_2$ to be symplectic, two conditions should be fulfilled:

$$k_4 = k_3, \quad k_6 = k_1. \tag{3}$$

Hence, we obtain a matrix $\mathbf{K}_2$ of the form

$$\mathbf{K}_2 = \begin{pmatrix} 0 & k_1 & k_2 & k_3 \\ -k_1 & 0 & k_3 & k_5 \\ -k_2 & -k_3 & 0 & k_1 \\ -k_3 & -k_5 & -k_1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{pmatrix}. \tag{4}$$

Here $\mathbf{B}$ is a skew symmetric 2×2 matrix and $\mathbf{C}$ is a symmetric 2×2 matrix.

According to Theorem 2 there is a four-parameter family of OSMs $\mathbf{A}_2$. Their structure can be described as follows. Let us denote by $\mathbf{A}_2^{(j)}$, $j = 1, \ldots, 4$, the jth column of the matrix $\mathbf{A}_2$. Then the following conditions hold.

1. Each column of $\mathbf{A}_2^{(j)}$ is a unit vector.
2. All columns are pairwise orthogonal.
3. $\mathbf{A}_2^{(j)} = \mathbf{J}\mathbf{A}_2^{(j+2)}$ or $\mathbf{A}_2^{(j+2)} = -\mathbf{J}\mathbf{A}_2^{(j)}$ for $j = 1, 2$.
4. Let us put the following notations: $Q \stackrel{\text{def}}{=} k_1^2 - k_2 k_5 + k_3^2$, $R^2 \stackrel{\text{def}}{=} 2k_1^2 + k_2^2 + 2k_3^2 + k_5^2$, then the first two columns are

$$\begin{aligned} \mathbf{A}_2^{(1)} &= \frac{1}{d^2} \begin{pmatrix} 1 + k_5^2 - k_2^2 - Q^2 \\ -2k_1(1+Q) - 2k_3(k_5 + k_2) \\ -2k_2 + 2k_5 Q \\ -2k_3(1+Q) + 2k_1(k_5 + k_2) \end{pmatrix}, \\ \mathbf{A}_2^{(2)} &= \frac{1}{d^2} \begin{pmatrix} 2k_1(1+Q) - 2k_3(k_5 + k_2) \\ 1 - k_5^2 + k_2^2 - Q^2 \\ -2k_3(1+Q) - 2k_1(k_5 + k_2) \\ -2k_5 + 2k_2 Q \end{pmatrix} \end{aligned} \tag{5}$$

   where $d^2 = (k_2 + k_5)^2 + (1+Q)^2 = R^2 + Q^2 + 1$. Here $d^2 = \det(\mathbf{E} - \mathbf{K}_2)$.
5. Columns $\mathbf{A}_2^{(3)}$ and $\mathbf{A}_2^{(4)}$ are obtained according to the property 3.

Consider the following particular problem.

Problem. *Let a monodromy matrix $\mathbf{M}_2$ of a periodic solution $\mathbf{z}(t, \mathbf{z}^0)$ with period T of an autonomous Hamiltonian system with two degrees of freedom be known. Find such an orthogonal-symplectic transformation given by the matrix $\tilde{\mathbf{A}}_2$ that reduces the matrix $\mathbf{M}_2$ to a simpler form.*

Applying Gröbner basis technique one can deduce from (5) that the first two columns of the matrix $\tilde{\mathbf{A}}_2$ take the form of

$$\tilde{\mathbf{A}}_2^{(1)} = \frac{1}{\tilde{d}}\begin{pmatrix} 1-\tilde{Q} \\ -2k_1 \\ -2k_2 \\ -2k_3 \end{pmatrix}, \quad \tilde{\mathbf{A}}_2^{(2)} = \frac{1}{\tilde{d}}\begin{pmatrix} 2k_1 \\ 1-\tilde{Q} \\ -2k_3 \\ 2k_2 \end{pmatrix}, \quad \tilde{d} = 1+\tilde{Q}. \tag{6}$$

Let some non degenerate periodic solution $\mathbf{z}(t, \mathbf{z}^0)$ of a family with initial condition $\mathbf{z}^0$ and period T be known. In a generic case, there is a periodic solution $\mathbf{z}(t) + \delta\mathbf{z}(t)$ with period $T + \delta T$ near the generic case. Decomposing the left-hand side of the periodicity condition $\mathbf{z}(T + \delta T, \mathbf{z}^0 + \delta\mathbf{z}) = \mathbf{z}^0 + \delta\mathbf{z}$ into a Taylor series, leaving in the expansion terms not higher than the first order of smallness for $\delta\mathbf{z}$ and δT, we obtain that small additives of $\delta\mathbf{z}$ and δT should satisfy a linear homogeneous system

$$(\mathbf{M}_2 - \mathbf{E})\delta\mathbf{z}(T) + \mathbf{v}^0\delta T = 0, \tag{7}$$

where $\mathbf{v}^0 = \mathbf{J}\,\mathrm{grad}\,H(\mathbf{z}(T, \mathbf{z}^0))$. The set of solutions to this system is determined by the structure of the monodromy matrix $\mathbf{M}$.

The vectors $\mathbf{v}^0$ and $-\mathbf{J}\mathbf{v}^0$ are, respectively, the right and left eigenvectors of the matrix $\mathbf{M}_2$, then the transformation $\mathbf{M}_2 \to \tilde{\mathbf{A}}_2^\top \mathbf{M}_2 \tilde{\mathbf{A}}_2$ reduces [5] the matrix $\mathbf{M}$ to the symplectic matrix $\mathbf{N}_2 = \begin{pmatrix} 1 & n_{12} & n_{13} & n_{14} \\ 0 & n_{22} & n_{23} & n_{24} \\ 0 & 0 & 1 & 0 \\ 0 & n_{42} & n_{43} & n_{44} \end{pmatrix}$. If we now substitute the variables $\delta\boldsymbol{\zeta} = \tilde{\mathbf{A}}_2\delta\mathbf{z}$, then the system (7) is written as $(\mathbf{N}_2 - \mathbf{E})\delta\boldsymbol{\zeta}(T) + v^0\delta T\tilde{\mathbf{A}}_2^{(1)} = 0$, where v^0 is the magnitude of the phase velocity at the initial point $\mathbf{z}^0$ of the periodic solution. This system has a general solution in the form $\delta\boldsymbol{\zeta} = c_1\tilde{\mathbf{A}}_2^{(1)} + \delta\boldsymbol{\zeta}'$, $\delta T = -\frac{1}{v}\sum_{j=2}^{4} n_{1j}\delta\zeta_j'$, and the vector $\delta\boldsymbol{\zeta}'$ is orthogonal to the vector $\tilde{\mathbf{A}}_2^{(1)}$. So $\delta\boldsymbol{\zeta}'$ specifies the displacement along the family of periodic orbits and $\tilde{\mathbf{A}}_2^{(1)}$ specifies the displacement along the periodic solution.

4. General structure of an arbitrary OSM

Generalizing the computations performed for cases $n = 1, 2, 3, 4$, we can formulate the following conjecture.

Conjecture. *If a skew symmetric matrix $\mathbf{K}_n$ has the form*

$$\mathbf{K}_n = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C}^\top & \mathbf{D} \end{pmatrix}, \quad \mathbf{B}^\top = -\mathbf{B}, \quad \mathbf{D}^\top = -\mathbf{D},$$

then the matrix $\boldsymbol{\Psi}_n = -2\mathbf{J}(\mathbf{E} + \mathbf{A}_n)^{-1}(\mathbf{E} - \mathbf{A}_n)$ has the following form $\boldsymbol{\Psi}_n = -2\begin{pmatrix} \mathbf{C}^\top & \mathbf{D}^\top \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$, where $\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)(\mathbf{E} - \mathbf{K}_n)^{-1}$.

It follows from the conjecture that in order for the matrix $\boldsymbol{\Psi}_n$ to be symmetric, the conditions $\mathbf{C} = \mathbf{C}^\top$ and $\mathbf{B} = \mathbf{D}$ should be satisfied. Thus, according to Theorem 1, the following statement is obtained:

Statement. *If the matrix* $\mathbf{K}_n$ *is a block matrix* $\mathbf{K}_n = \begin{pmatrix} \mathbf{B} & \mathbf{C} \\ -\mathbf{C} & \mathbf{B} \end{pmatrix}$, *with* $\mathbf{B}$ *is a skew symmetric* $n \times n$ *matrix of* $n(n-1)/2$ *independent elements and* $\mathbf{C}$ *is a symmetric* $n \times n$ *matrix of* $n(n+1)/2$ *independent elements, then the matrix* $\mathbf{A}_n = (\mathbf{E} + \mathbf{K}_n)(\mathbf{E} - \mathbf{K}_n)^{-1}$ *is a generic* ***orthogonal symplectic matrix*** *of* n^2 *independent elements.*

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Teaching Fractals with Technology

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Abstract. We present our project on teaching fractals with technology and give some examples.

Introduction

Modern technology changes radically the way of teaching natural sciences. Using technology, same course could be taught on different levels of complexity. We present our project on teaching fractals with technology. Fractals are chosen as an example since they are very attractive visually and are encountered in many branches of natural sciences and applications.

Using computer tools (from basic demo software and turtle graphics to CAS and JAVA/Python codes), course with the same topics can be taught on different levels, from primary school to graduate and post-graduate level. It is also possible to present material to students with wide spectrum of age and mathematical background.

We give some examples and discuss our experience of teaching/demonstrating fractals to primary school students and to school teachers.

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Compact first-order differential approximations: a case study of the Korteweg-de Vries equation

Yuri A. Blinkov

Abstract. On the example of the first-order differential approximation, a qualitative study was conducted for the Crank-Nicolson-type scheme for the Korteweg-de Vries equation. This made it possible to qualitatively assess the method's truncation error and propose simple criteria for selecting the time step and spatial step during calculations. The presented methods make it possible to carry out effective calculations using computer algebra systems. For the research, author's programs for working with the first-order differential approximation, implemented in the computer algebra system SymPy, were used.

In the 1960s of the last century, N. N. Yanenko and Yu. I. Shokin [1] formulated the method of differential approximations for the difference scheme. First-order differential approximation (FDA) for partial differential equations of evolutionary type and, in particular, the Korteweg-de Vries equation using computer algebra systems is considered in [2], and for the Navier-Stokes equations in [3].

In this work, the Korteweg-de Vries equation (1)

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

is chosen as a demonstrator of FDA for investigating difference schemes. Soliton solutions of the Korteweg-de Vries equation are those that describe wave propagation in nonlinear media. The soliton solution (2) represents a traveling wave $\xi = k(x-4k^2t)$ depending on the parameter k, which propagates without changing its shape and amplitude

$$u = \frac{2k^2}{\cosh^2 \xi} \tag{2}$$

In this work, we will investigate a second-order scheme with respect to h [2]

$$\frac{u_j^{n+1}-u_j^n}{\tau}+\frac{3}{4h}\left(\left({u^2}_{j+1}^{n+1}-{u^2}_{j-1}^{n+1}\right)+\left({u^2}_{j+1}^{n}-{u^2}_{j-1}^{n}\right)\right)+ \\ +\frac{1}{4h^3}\left(\left(u_{j+2}^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}-u_{j-2}^{n+1}\right)+ \right. \\ \left. +\left(u_{j+2}^{n}-2u_{j}^{n+1}+2u_{j-1}^{n}-u_{j-2}^{n}\right)\right)=0. \quad (3)$$

Constructing FDA uses only algebraic operations and can be effectively implemented algorithmically using computer algebra tools. The author's program is implemented in the open-source computer algebra system SymPy (`https://www.sympy.org`) and can be downloaded at `https://github.com/blinkovua/sharing-blinkov/blob/master/KDV_FDA_Crank-Nicolson.ipynb`.

Algorithmically, the application of the Gröbner basis construction algorithm for constructing FDA can be represented as working with an infinite module with the *POT* ordering (*position over term* - ordering first by dependent variables and then by independent variables), where the role of *position* is played by time steps τ and space steps h. In this case, calculations are carried out up to the first non-zero members of the series in τ and h.

Lexicographic ordering. Initially, the FDA method was a decomposition of the difference scheme (3) into a Taylor series expansion in the central point of the difference scheme template at the point $(\tau/2, 0)$ which would have the following form

$$6uu_x+u_t+u_{xxx}+h^2\left(uu_{xxx}+\frac{u_{xxxxx}}{4}+3u_{xx}u_x\right)+ \\ +\tau^2\left(\frac{3uu_{ttx}}{4}+\frac{u_{ttt}}{24}+\frac{u_{ttxxx}}{8}+\frac{3u_{tt}u_x}{4}+\frac{3u_{tx}u_t}{2}\right)+\ldots=0 \quad (4)$$

Since the formal Taylor series (4) is equal to zero, its linear combination with itself and its differential consequences will also be equal to zero. This can be taken advantage of and brought to a canonical form that can be used to study the properties not only of difference schemes for linear equations but also for nonlinear ones [1].

In the works [1], evolutionary type equations were studied. This allowed the transformation of the formal Taylor series (4) by replacing all derivatives with respect to time through spatial derivatives

$$6uu_x+u_t+u_{xxx}+h^2\left(uu_{xxx}+\frac{u_{xxxxx}}{4}+3u_{xx}u_x\right)+ \\ +\tau^2\left(18u^3u_{xxx}+9u^2u_{xxxxx}+162u^2u_{xx}u_x+\frac{3uu_{xxxxxxx}}{2}+ \right. \\ +63uu_{xxxx}u_x+99uu_{xxx}u_{xx}+108uu_x^3+\frac{u_{xxxxxxxxx}}{12}+6u_{xxxxxx}u_x+ \\ \left. +\frac{27u_{xxxxx}u_{xx}}{2}+21u_{xxxx}u_{xxx}+81u_{xxx}u_x^2+99u_{xx}^2u_x\right)+\ldots=0 \quad (5)$$

The canonical form (5) obtained by replacing all derivatives with respect to time through spatial derivatives allows, firstly, to precisely determine the order of the difference scheme, and secondly, it allows to draw certain conclusions about such properties of difference schemes as stability, approximation, accuracy, monotonicity, conservativeness, group properties, etc. [1].

Replacing all derivatives with respect to time through spatial derivatives corresponds to the construction of Groebner basis with lexicographical ordering first by t and then by x ($t \succ x$).

Since knowing the Taylor series expansion in a selected point, one can use it to *formally* recalculate the derivatives of the sought function (in this case, the differential equation) in another selected point. Therefore, the choice of the point for the FDA expansion does not matter, which is required for the canonical representation. For example, let us consider the expansion of the difference scheme (3) into a Taylor series at the point $(\tau, -h)$

$$\begin{aligned}
&6uu_x + u_t + u_{xxx} + h\left(6uu_{xx} + u_{tx} + u_{xxxx} + 6u_x^2\right) + \\
&\quad + h^2\left(4uu_{xxx} + \frac{u_{txx}}{2} + \frac{3u_{xxxxx}}{4} + 12u_{xx}u_x\right) + \\
&\quad + \tau\left(-3uu_{tx} - \frac{u_{tt}}{2} - \frac{u_{txxx}}{2} - 3u_tu_x\right) + \\
&\quad + \tau h\left(-3uu_{txx} - \frac{u_{ttx}}{2} - \frac{u_{txxxx}}{2} - 6u_{tx}u_x - 3u_tu_{xx}\right) + \\
&\quad + \tau^2\left(\frac{3uu_{ttx}}{2} + \frac{u_{ttt}}{6} + \frac{u_{ttxxx}}{4} + \frac{3u_{tt}u_x}{2} + 3u_{tx}u_t\right) + \ldots = 0 \quad (6)
\end{aligned}$$

In the expansion (6), it is seen that the members of the series at τ, τh are simple differential consequences of the original equation (1). At higher-order terms of the expansion, besides the original equation, the lower-order members of the original expansion may also be involved. The construction of the Groebner basis reduces the series (6) to the form (5).

The choice of the point for the expansion affects only the amount of calculations, taking into account the nonlinearity of the equations, the high order of the applied derivatives, and symbolic parameters of the problem. A much greater reduction in the amount of calculations can be achieved by performing the calculation not in lexicographical ordering, but in ordering by total degree, and then in reverse lexicographical (degrevlex – degree reverse lexicographic).

Degrevlex ordering. The initial Taylor series expansion does not depend on the ordering, but depends only on the point of expansion. Therefore, using expansion (4) or expansion (6) in the degrevlex ordering, we obtain a more compact,

compared to (5), version of FDA.

$$6uu_x + u_t + u_{xxx} + h^2\left(3u^2u_x + \frac{uu_t}{2} - \frac{u_{txx}}{4} - \frac{3u_{xx}u_x}{2}\right) + \\ + \tau^2\left(-\frac{u_{ttt}}{12}\right) + \ldots = 0 \quad (7)$$

In this specific case, we can obtain this form even in lexicographical ordering by swapping the order of variables $x \succ t$, but the approach based on the use of degrevlex ordering in most cases gives the most compact form of FDA or close to it, and it is less dependent on the choice of variable order.

Symbolic experiments. Despite their very cumbersome appearance, especially in lexicographical ordering, the first differential approximations can be effectively checked against exact solutions. More precisely, FDA, when substituted with the exact solution, allows us to evaluate the scheme itself without programming it and conducting computational experiments to check it.

Let's substitute the exact solution (2) into FDA (5) or (7) to get the following form of FDA

$$h^2\left(-8k^7\left(\tanh\xi - 1\right)\left(\tanh\xi + 1\right) \times \right. \\ \left. \times \left(15\tanh^4\xi - 16\tanh^2\xi + 3\right)\tanh\xi\right) + \tau^2\left(-256k^{11}\left(\tanh\xi - 1\right) \times \right. \\ \left. \times \left(\tanh\xi + 1\right)\left(3\tanh^2\xi - 2\right)\tanh\xi/3\right) + \ldots = 0 \quad (8)$$

Since ξ represents a running wave, and the value of hyperbolic tangent lies in the interval from -1 to 1, the main contribution to the truncation error of the difference scheme (3) on the exact solution (2) can be represented in the form of $\mathcal{O}(\tau^2k^{11}, h^2k^7)$.

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Asymptotic expansions of a manifold near its curve of singular points

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Abstract. In [1–3] parametric expansions near 5 singular points and 3 curves consisting of singular points were computed for a two-dimensional algebraic manifold Ω. Here we present general methods for computing the expansions of a manifold near its curve of singular points and their application to a single curve $\mathcal{F}$.

1. Introduction

In [4–8] the study of the three-parameter family of special homogeneous spaces in terms of the normalized Ricci flow was started. Ricci flows give the evolution of Einstein metrics on a manifold. The equation of the normalized Ricci flow reduces to a system of two ordinary differential equations with three parameters a_1,a_2 and a_3:

$$\begin{aligned} \frac{dx_1}{dt} &= \tilde{f}_1(x_1, x_2, a_1, a_2, a_3), \\ \frac{dx_2}{dt} &= \tilde{f}_2(x_1, x_2, a_1, a_2, a_3), \end{aligned} \tag{1}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are some concrete functions.

The singular points of this system correspond to Einstein invariant metrics. At a singular (fixed) point x_1^0, x_2^0 the system (1) has two eigenvalues λ_1 and λ_2. If at least one of them is equal to zero, the singular point x_1^0, x_2^0 is called degenerate. In [4–8] a theorem is proved that the set Ω of values of parameters a_1, a_2, a_3, at which the system (1) has at least one degenerate singular point is described by the equation

$$\begin{aligned}Q(s_1,s_2,s_3)\overset{\text{def}}{\equiv}&(2s_1+4s_3-1)\left(64s_1^5-64s_1^4+8s_1^3+240s_1^2s_3-1536s_1s_3^2-\right.\\&\left.-4096s_3^3+12s_1^2-240s_1s_3+768s_3^2-6s_1+60s_3+1\right)-\\&-8s_1s_2(2s_1+4s_3-1)(2s_1-32s_3-1)(10s_1+32s_3-5)-\\&-16s_1^2s_2^2\left(52s_1^2+640s_1s_3+1024s_3^2-52s_1-320s_3+13\right)+\\&+64(2s_1-1)s_2^3(2s_1-32s_3-1)+2048s_1(2s_1-1)s_2^4=0,\end{aligned}$$

where s_1, s_2, s_3 are elementary symmetric polynomials, equal, respectively, to

$$s_1=a_1+a_2+a_3,\quad s_2=a_1a_2+a_1a_3+a_2a_3,\quad s_3=a_1a_2a_3.$$

In [9] for symmetry reasons, from coordinates $\mathbf{a}=(a_1,a_2,a_3)$ authors passed to the coordinates $\mathbf{A}=(A_1,A_2,A_3)$ by linear substitution

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=M\cdot\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix},\quad M=\begin{pmatrix}(1+\sqrt{3})/6 & (1-\sqrt{3})/6 & 1/3\\(1-\sqrt{3})/6 & (1+\sqrt{3})/6 & 1/3\\-1/3 & -1/3 & 1/3\end{pmatrix}$$

Definition 1. *Let $\varphi(X)$ be some polynomial, $X=(x_1,\ldots,x_n)$. Point $X=X^0$ of the set $\varphi(X)=0$ is called a* ***singular point of*** *k-**order**, if in this point all partial derivatives of the polynomial $\varphi(X)$ by $x_1,\ldots,x_n$ go to zero up to k-th order and at least one partial derivative of order $k+1$ does not go to zero.*

In [9] all singular points of the manifold Ω were found in coordinates $\mathbf{A}=(A_1,A_2,A_3)$. Five third-order points,

Name	**Coordinates A**
$P_1^{(3)}$	$(0,0,3/4)$
$P_2^{(3)}$	$(0,0,-3/2)$
$P_3^{(3)}$	$\left(-\frac{1+\sqrt{3}}{2},\frac{\sqrt{3}-1}{2},\frac{1}{2}\right)$
$P_4^{(3)}$	$\left(\frac{\sqrt{3}-1}{2},-\frac{1+\sqrt{3}}{2},\frac{1}{2}\right)$
$P_5^{(3)}$	$(1,1,1/2)$

three second-order points,

Name	**Coordinates A**
$P_1^{(2)}$	$\left(\frac{1+\sqrt{3}}{4},\frac{1-\sqrt{3}}{4},\frac{1}{2}\right)$
$P_2^{(2)}$	$\left(\frac{1-\sqrt{3}}{4},\frac{1+\sqrt{3}}{4},\frac{1}{2}\right)$
$P_2^{(3)}$	$(1,1,1/2)$

and three more algebraic curves of singular points of first order

$$\mathcal{F} = \left\{a_1 = a_2, 16a_1^3 + 16a_1^2a_3 - 4a_1 - 2a_3 + 1 = 0\right\},$$

$$\mathcal{I} = \left\{A_1 + A_2 + 1 = 0, A_3 = \frac{1}{2}\right\},$$

$$\mathcal{K} = \left\{A_1 = -\frac{9}{4}th\left(t\right),\ A_2 = -\frac{9}{4}h\left(t\right),\ A_3 = \frac{3}{4},\ h\left(t\right) = \frac{t^2+1}{(t+1)(t^2-4t+1)}\right\}.$$

In this case, the points $P_3^{(3)}$, $P_4^{(3)}$ and $P_5^{(3)}$ are of the same type, they pass into each other at rotation around the origin of the plane A_1, A_2 by an angle $2\pi/3$, just as all points $P_1^{(2)}$, $P_2^{(2)}$, $P_3^{(2)}$. The curves $\mathcal{F}$, $\mathcal{I}$, $\mathcal{K}$ correspond to two more curves of the same type. Therefore, it is enough to study the manifold Ω in the neighborhoods of the points $P_1^{(3)}$, $P_2^{(3)}$, $P_5^{(3)}$, $P_3^{(2)}$ and curves $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{K}$. Moreover, in [9] the cross sections of the manifold Ω by the planes $A_3 = \text{const}$, were calculated and it was shown that in a finite part of the space $\mathbb{R}^3 = \{A_1, A_2, A_3\}$ the manifold Ω consists of one-dimensional branches F_1, F_2, F_3, and two-dimensional branches G_1, G_2, G_3 which are broken into parts $F_i^{\pm}$, $G_i^{\pm}$ with boundaries at the plane $A_3 = 1/2$.

The structure of the manifold Ω near singular points $P_i^{(3)}$ and $P_i^{(2)}$ was considered in [1,2]. The structure of the manifold Ω near three algebraic curves $\mathcal{I}$, $\mathcal{K}$, $\mathcal{F}$ of singular points of the first order was considered in [3]. For this study, we use the following algorithm consisting of 8 steps.

2. Calculation scheme

Step 1: Introduce local coordinates $X = (x_1, x_2, x_3)$. If we consider a straight line consisting of singular points (as $\mathcal{I}$), then one coordinate x_1 is directed along the line and coordinates x_2,x_3 describe deviations from the line. If the curve is located on a plane, we introduce the coordinate x_3, normal to this plane, coordinates x_1,x_2 of the curve on the plane are parameterized $x_1 = b_1(t)$, $x_2 = b_2(t)$ and a coordinate $y_2 = x_2 - b(t)$ of the deviation from this curve.

Step 2: The original polynomial $R(\mathbf{A})$ write in local coordinates as

$$g(t, y_2, x_3) = \sum \varphi(t)_{pq} y_2^p x_3^q, \tag{2}$$

and compute its support $\mathbf{S} = \{(p, q) : \varphi_{pq} \not\equiv 0\}$. Let the support $\mathbf{S}$ consists of points (p_i, q_i), $i = 1, \ldots, k$.

Step 3: Newton's polygon $\Gamma(g)$ is calculated as a convex hull of the support $\mathbf{S}$:

$$\Gamma(g) = \left\{(p, q) = \sum_{i=1}^{k} \lambda_i(p_i, q_i),\ \lambda_i \geqslant 0,\ i = 1, \ldots, k,\ \sum_{i=1}^{k} \lambda_i = 1\right\}.$$

Boundary $\partial\Gamma$ of polygon $\Gamma(g)$ consists of its vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$ which we call as generalized faces. Here j is the number of the generalized face $\Gamma_j^{(d)}$. Each face $\Gamma_j^{(d)}$ corresponds to its truncated polynomial

$$\hat{g}_j^{(d)}(Y) = \sum g_{(p,q)} y_2^p x_3^q \text{ over } (p,q) \in \mathbf{S} \cap \Gamma_j^{(d)}$$

and the normal cone $\mathbf{U}_j^{(d)}$, consisting of all normals to the face $\Gamma_j^{(d)}$, which are the external normals to the polygon Γ. For their computation we use PolyhedralSets of the computer algebra system (CAS) Maple package [10].

Step 4: Select the faces $\Gamma_j^{(1)}$ with normals $N_j \leqslant 0$ and corresponding truncated polynomials $\hat{g}_j^{(1)}(t, y_2, x_3)$.

Step 5: For each selected truncated polynomial $\hat{g}_j^{(1)}(t, y_2, x_3)$, we calculate the corresponding power transformations

$$(\ln y_2, \ln x_3) = (\ln z_1, \ln z_3)\, \alpha, \tag{3}$$

where α is such a unimodular matrix 2×2, that

$$\hat{g}_j^{(1)}(t, y_2, x_3) = h(z_1, t) z_3^l \tag{4}$$

with a multiplier z_3^l.

Step 6: We make the power transformation (3) in the polynomial (2) itself and write it in the following form

$$g(Z) = T(z_1, t, z_3) = z_3^l \sum_{k=0}^{m} T_k(z_1, t) z_3^k,$$

with some natural number m. The polynomial $T_k(z_1, t)$ is calculated by the command `coeff(T,z[k],m)` in CAS `Maple`, and $T_0(z_1, t) = h(z_1, t)$ from Equality (4).

Step 7: If $T_0(z_1(t), t) \not\equiv 0$, then we substitute in the polynomial $T(z_1, t, z_3) z_3^{-l}$

$$z_1 = b_1(t) + \varepsilon, \quad z_2 = b_2(t) + \varepsilon \tag{5}$$

and obtain the function $u(\varepsilon, t, z_3) = T(z_1, z_2, z_3) z_3^{-l}$. Now we apply to the equation $u(\varepsilon, t, z_3) = 0$ Theorem 1 [1] on the generalized implicit function and obtain the parametric expansion

$$\varepsilon = \sum_{k=1}^{\infty} c_k(t) z_3^k. \tag{6}$$

Step 8: Calculate several terms of expansion (6) and substitute them into (5). The result is substituted into the power transformation (3) and we obtain the parametric expansion of Ω into a power series by z_3, with coefficients which are rational functions of the t.

3. Structure of the manifold Ω near the Curve $\mathcal{F}$ of singular points

Theorem 1. *The curve $\mathcal{F}$ consists of branches $F_1^{\pm}$,$F_2^{\pm}$,$F_3^{\pm}$. On them two-dimensional branches $G_1^{\pm}$,$G_2^{\pm}$,$G_3^{\pm}$ of the manifold Ω meet (but do not intersect).*

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New cases of integrability of the Euler–Poisson system

Alexander Bruno and Alexander Batkhin

Abstract. In the classical problem of motion of a rigid body around a fixed point described by the Euler–Poisson system, new cases of global integrability are found. For one of these cases, generalizing the Kovalevskaya case, a fourth global integral is proposed.

1. Introduction

The Euler-Poisson equations (1750) are a real autonomous system of six ordinary differential equations (ODEs).

$$\begin{aligned} &Ap' + (C-B)qr = Mg\,(y_0\gamma_3 - z_0\gamma_2)\,,\\ &Bq' + (A-C)pr = Mg\,(z_0\gamma_1 - x_0\gamma_3)\,,\\ &Cr' + (B-A)pq = Mg\,(x_0\gamma_2 - y_0\gamma_1)\,,\\ &\gamma_1' = r\gamma_2 - q\gamma_3, \quad \gamma_2' = p\gamma_3 - r\gamma_1, \quad \gamma_3' = q\gamma_1 - p\gamma_2, \end{aligned} \tag{1}$$

with dependent variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$ and parameters A, B, C, x_0, y_0, z_0, satisfying the triangle inequalities

$$0 < A \leqslant B + C, \quad 0 < B \leqslant A + C, \quad 0 < C \leqslant A + B. \tag{2}$$

Here, the prime sign $'$ indicates differentiation over independent variable time t, Mg is the weight of the body, A, B, C are the principal moments of inertia of the rigid body, x_0, y_0, z_0 are the coordinates of the center of gravity of the rigid body, $\gamma_1, \gamma_2, \gamma_3$ are the vertical directional cosines.

The system (1) describes the motion of a spinner around a fixed point (Golubev, [1]) and has three first integrals: energy, momentum, and geometric:

$$\begin{aligned} &I_1 \stackrel{\text{def}}{=} Ap^2 + Bq^2 + Cr^2 - 2Mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) = h = \text{const},\\ &I_2 \stackrel{\text{def}}{=} Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 = l = \text{const},\\ &I_3 \stackrel{\text{def}}{=} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{aligned} \tag{3}$$

The system is integrable if there is a fourth general integral I_4. So far, 4 cases of global integrability are known:

Case 1. Euler-Poinsot: $x_0 = y_0 = z_0 = 0$ and $I_4 \stackrel{\text{def}}{=} A^2p^2 + B^2q^2 + C^2r^2 = \text{const}$.

Case 2. Lagrange-Poisson: $B \neq C$, $x_0 =\neq 0$, $y_0 = z_0 = 0$, and $I_4 \stackrel{\text{def}}{=} p = \text{const}$.

Case 3. Kovalevskaya (1890): $A = B = 2C$, $x_0 \neq 0$, $y_0 = z_0 = 0$, and

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1\right)^2 + (2pq + c\gamma_2)^2 = \text{const}, \tag{4}$$

where $c = Mgx_0/C$.

Case 4. Kinematic symmetry: $A = B = C$ and $I_4 \stackrel{\text{def}}{=} x_0p + y_0q + z_0r = \text{const}$. It is derived from case 2.

2. Results

We found the following new cases of integrability of the system (1).

Case 5: $A = B = 2C$, $x_0 \neq 0$, $y_0 \neq 0$, $z_0 = 0$. Then the fourth integral I_4 has the form

$$I_4 \stackrel{\text{def}}{=} \left(p^2 - q^2 + c\gamma_1 - d\gamma_2\right)^2 + (2pq + d\gamma_1 + c\gamma_2)^2 = \text{const}, \tag{5}$$

where $c = Mgx_0/C$, $d = Mgy_0/C$. This is a generalization of Kovalevskaya's case 3 and her fourth integral (4). As for cases 1–4 the fourth integral (5) is independent of the integrals (3).

Case 6: $B = C$, $A^2\,(A - 2B)\,x_0^2 = B\,(2A - B)^2\,y_0^2$, $z_0 = 0$.

For case 6, the inequalities of triangle (2) are not satisfied. For case 6, the additional fourth integral I_4 was not written out, and local integrability was checked near the corresponding fixed points for third-order resonances. According to (Bruno, [2, Section 5.3]) the coefficients of the resonance terms of the normal form at $2:1$ resonance should be zero in integrable cases. In this case they are zero.

3. Theory

We have found some general property of integrable cases 1–4, which is formulated below as Hypothesis 2. So we have to compute all those values of parameters A, B, C, x_0, y_0, z_0 for which this property is satisfied. And then, by computing the

resonance terms of the normal form of the system (1), to extract from them those values at which the system (1) is integrable.

Hypothesis 1 (Edneral, [3]). *If an autonomous polynomial ODE system is locally integrable in the neighborhood of all its stationary points, then it is globally integrable.*

Therefore, to find global integrability, we must first find all the stationary points of the ODE system, and then find out whether the system is locally integrable in their neighborhoods.

Let $X = (p, q, r, \gamma_1, \gamma_2, \gamma_3)$, the point $X = X^0$ is a stationary point of the system (1) and M is a matrix of the linear part of the system (1) near the point X^0. The characteristic polynomial $\chi(\lambda)$ of the matrix M is $\chi(\lambda) = \lambda^6 + a_4\lambda^4 + a_2\lambda^2$. Its discriminant

$$D_\lambda(\chi) = a_4^2 - 4a_2 \tag{6}$$

is a rational function $D = G/H$, where G and H are polynomials.

A stationary point is locally integrable (Bruno, [2]) if $a_2 < 0$ or $D_\lambda(\chi) < 0$. But this property is not satisfied for definite values of the system parameters (1).

The stationary points of the system (1) form one-dimensional and two-dimensional families $\mathcal{F}_j^l$ in $\mathbb{R}^6$.

Hypothesis 2. *If near a stationary point X^0 of the family $\mathcal{F}_j^l$ the system (1) is locally integrable, then at these parameter values the second discriminant $\Delta\left(\mathcal{F}_j^l\right)$ of the numerator G of the first discriminant $D_\lambda(\chi)$ on the parameter of the family $\mathcal{F}_j^l$ is zero.*

Considering Hypothesis 1, now the search for integrable cases consists of the following 5 steps.

- **Step 1:** Fix the number l of non-zero parameters x_0, y_0, z_0 and find all families $\mathcal{F}_j^l$ of stationary points.
- **Step 2:** Compute the discriminants (6) $D_\lambda(\chi)$ on the families $\mathcal{F}_j^l$.
- **Step 3:** On families $\mathcal{F}_j^l$, compute the second discriminants $\Delta\left(\mathcal{F}_j^l\right)$ of the numerators G of the first discriminants D.
- **Step 4:** Find the values of the parameters of the system (1) at which all $\Delta\left(\mathcal{F}_j^l\right) = 0$ at fixed l.
- **Step 5:** Check the obtained parameter values for integrability by computing the normal forms of the system (1) near stationary points or by finding the fourth integral.

4. Computations

4.1. Case $l = 0 : x_0 = y_0 = z_0 = 0$

Then the system (1) has 3 families of stationary points:

$\mathcal{F}_1^0$: $\{q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0\}$, p is a parameter;

$\mathcal{F}_2^0$: $\{p = r = 0,\ \gamma_2 = q/k = \pm 1,\ \gamma_1 = \gamma_3 = 0\}$, q is a parameter;
$\mathcal{F}_3^0$: $\{p = q = 0,\ \gamma_3 = r/k = \pm 1,\ \gamma_1 = \gamma_2 = 0\}$, r is a parameter.

They all have $\Delta\left(\mathcal{F}_j^0\right) \equiv 0$ on them, so the system (1) is integrable (**Case 1**).

4.2. **Case** $l = 1 : x_0 \neq 0, y_0 = z_0 = 0$

Then the system (1) has 4 families of stationary points:

$\mathcal{F}_1^1$: $\{q = r = 0,\ \gamma_1 = p/k = \pm 1,\ \gamma_2 = \gamma_3 = 0,\ B \neq C\}$, p is a parameter;

$\mathcal{F}_2^1$: $\left\{p = \dfrac{x_0}{k(C-A)}, q = 0,\ \gamma_1 = \dfrac{p}{k}, \gamma_2 = 0, \gamma_3 = \dfrac{s}{k}, \gamma_1^2 + \gamma_3^2 = 1, A \neq C \neq B\right\}$, r is a parameter;

$\mathcal{F}_3^1$: $\left\{p = \dfrac{x_0}{k(B-A)}, r = 0,\ \gamma_1 = \dfrac{p}{k}, \gamma_2 = \dfrac{q}{k}, \gamma_3 = 0, \gamma_1^2 + \gamma_2^2 = 1, A \neq B \neq C\right\}$, q is a parameter;

$\mathcal{F}_4^1$: $\left\{p = \dfrac{x_0}{k(B-A)},\ \gamma_1 = \dfrac{p}{k}, \gamma_2 = \dfrac{q}{k}, \gamma_3 = \dfrac{r}{k}, \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, A \neq B = C\right\}$, q, r are parameters.

All the second discriminants for these families are zero when:

1. $B = C$ – **Case 2**;
2. $A = B = 2C$ – **Case 3**;
3. $A = 2C$, $B = 3C$;
4. $A = 2C$, $B = \delta C$, where δ is the root of the equation $\delta^3 - 12\delta^2 + 33\delta - 24 = 0$, i.e., $\delta_1 \approx 1.194$, $\delta_2 \approx 2.387$, $\delta_3 \approx 8.419$.

But the check shows that there is no local integrability in items 3 and 4.

4.3. **Case** $l = 2 : x_0 \neq 0, y_0 \neq 0, z_0 = 0$

Then the system (1) has 2 families of stationary points:

$$\mathcal{F}_1^2 : \left\{p = \frac{x_0}{k(C-A)}, q = \frac{y_0}{k(C-B)},\ \gamma_1 = \frac{p}{k}, \gamma_2 = \frac{q}{k}, \gamma_3 = \frac{r}{k}, \right.$$
$$\left. \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, A \neq C \neq B\right\}, r \text{ is a parameter;}$$

$$\mathcal{F}_2^2 : \left\{p = -\frac{x_0}{k(A+T)}, q = -\frac{y_0}{k(B+T)}, r = 0,\ \gamma_1 = \frac{p}{k}, \gamma_2 = \frac{q}{k}, \gamma_3 = 0, \right.$$
$$\left. \gamma_1^2 + \gamma_2^2 = 1\right\}, T \text{ is a parameter;}$$

The second discriminants $\Delta(\mathcal{F}_1^2)$ and $\Delta(\mathcal{F}_2^2)$ were obtained during the computation, but we could not factorize $\Delta(\mathcal{F}_2^2)$ because it contains several hundred thousand monomials. Therefore, we computed $\Delta(\mathcal{F}_2^2)$ on the zeros of $\Delta(\mathcal{F}_1^2)$. We get the following results.

With $B = C$ both Δ are zero at

- $A = B = C$ (**Case 4**),
- $B = C$ and $A^2(A - 2B)x_0^2 = B(2A - B)^2 y_0^2$ (new **Case 6**),
- $A = 2B = 2C$.

Checking shows that the last case is non-integrable.

With $A = B$ we get that $\Delta(\mathcal{F}_2^2) = -4A^2C^3(A - 2C)$. Therefore, when $A = B = 2C$, both Δ are zero. This is the *new* **Case 5**.

4.4. Case $l = 3 : x_0 \neq 0, y_0 \neq 0, z_0 \neq 0$

Then the system (1) has one family of stationary points:

$$\mathcal{F}_1^3 : \left\{ p = -\frac{x_0}{k(A+T)}, q = -\frac{y_0}{k(B+T)}, r = -\frac{z_0}{k(C+T)}, \right.$$
$$\left. \gamma_1 = \frac{p}{k}, \gamma_2 = \frac{q}{k}, \gamma_3 = \frac{r}{k}, \gamma_1^2 + \gamma_2^2 = 1 \right\},$$

T is a parameter;

The first discriminant of $D_\lambda(\chi)$ is a 10th degree polynomial of T. It is impossible to compute its discriminant on T in the generic case, but when $A = B = C$ (**Case 4**) it is zero. When $A = B = 2C$, the second discriminant $\Delta(\mathcal{F}_1^3) = 384A^2 \left(x_0^2 + y_0^2\right)^4 \neq 0$. According to Hypothesis 2, this is a non-integrable case.

So here also as for the family $\mathcal{F}_2^2$ we should look for other methods of computing discriminants or more powerful computers.

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Analytic solving any equation of polynomial type on variables and derivatives

Alexander Bruno

Abstract. A calculus [1] has been developed which allows one to calculate analytically asymptotic expansions of solutions to equations which are polynomials on variables and their derivatives, as well as to systems of such equations. This calculus is applied to equations of any type: algebraic [2, 3], ordinary differential [4] and partial differential [5], as well as to their systems. The calculus is based on algorithms of power geometry: (a) selection of truncated equations consisting of all leading terms, as well as (b) power transformations, (c) logarithmic and (d) normalizing coordinate transformations. The required software for this calculus has already been developed.

1. Introduction

For a single equation, the sequence of calculations is as follows:

I. First, the truncated equations are selected and the regions where they are first approximations of the original equation are specified.
II. Each truncated equation is then simplified using power and logarithmic coordinate transformations, possibly repeatedly, to an equation that has a simple solution.
III. This is augmented to the solution of the truncated equation.
IV. If its perturbation in the full equation has a linear part, we obtain the solution of the original equation by the normalizing transformation.
V. If this perturbation does not have a linear part, we repeat this process for it, i.e., we again separate the truncated equations and simplify them until we come to situation IV, i.e., to a perturbation with a linear part, for which we find a solution.

The methods of applying this calculus to equations of different types are described below.

The article [1] outlines the objects and sequences of calculus for:

1. One algebraic equation.
2. One ordinary differential equation (ODE) of order n.
3. An autonomous system of n ODEs.
4. One partial differential equation.

A brief review of applications is also given there.

Here we give algorithms of nonlinear analysis for cases of one equation and discuss levels of power geometry.

2. Levels of Power Geometry

Everything that has been told in [1] refers to the zero level of power geometry, for there it has been «sealed» that

$$\operatorname{ord} y' = \operatorname{ord} y - 1. \tag{1}$$

But this is not always the situation. By rejecting this property, we get a wider set of solutions. Let's discuss this in more detail.

In the algebraic equation

$$f(X) \stackrel{\text{def}}{=} \sum a_Q X^Q = 0,$$

with $X \in \mathbb{R}^n$ or $\mathbb{C}^n$ to each monomial $a_Q X^Q$ can be assigned a point $\check{Q} = \{Q, \ln|a_Q||\}$ in $\mathbb{R}^{n+1}$. Their set forms the *supersupport* $\check{S}(f)$, and its convex hull $\check{H}(f)$ is the *Adamar polyhedron* [6].

We build truncated equations on its faces. They are simpler than the truncated equations corresponding to the faces of Newton's polyhedron, and allow us to study cases where Newton's polyhedron fails.

For a single ODE, one can search for solutions that have $\operatorname{ord} y - \operatorname{ord} y' \neq 1$ by introducing a new coordinate for the order of the derivative y'. This was done in [7] and allowed us to obtain solutions in the form of power expansions whose coefficients are trigonometric or elliptic functions.

We can consider solutions where $\operatorname{ord} y^{(k)} - \operatorname{ord} y^{(k+1)}$ is arbitrary, or several such differences are arbitrary, and obtain new types of solutions. For details see [8, 9, 10].

A similar thing could be done with partial differential equations, but it has not been done yet.

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QUBO formulations of particle tracking algorithms

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Abstract. SPD (Spin Physics Detector) is a planned future experiment on the NICA megascience project developed in Dubna. Based on modeling data of the SPD experiment, this work is the first attempt to use the Hopfield network approach to formulate a QUBO problem and use simulated annealing to estimate the feasibility of the future use of quantum annealing to speed up present SPD particle tracking approaches.

Introduction

One of the key stages of data processing from particle physics experiments is the reconstruction of trajectories (tracks) of interacting particles from measurement data. In many future experiments, such as the High-Luminosity Large Hadron Collider (HL-LHC) or the SPD experiment planned at the NICA collider in Dubna, a special difficulty will be caused by the extremely high frequency of interactions.

In SPD, a very high data acquisition rate of 20 GB/sec resulting from 3 MHz collision frequency implies that tracks of several events will be overlapped and recorded in a single time-slice. Besides this, a strong contamination of data by fake measurements due to the specifics of used track detectors [2] will further raise the bar for track reconstruction (tracking) algorithms performance.

In our recent study [1], methods based on the Hopfield neural network for tracking simulated events of the SPD experiment were investigated. The minimum of the network energy function, corresponding to the solution of the problem, was obtained via simulated annealing.

However, it has been shown in recent works [7, 8] that combinatorial optimization problems can be successfully solved using quantum annealing techniques. For this purpose, the track reconstruction problem is formulated as quadratic unconstrained binary optimization (QUBO) and can be natively solved by quantum annealers, such as the commercially available D-Wave machines. Although the quantum speed-up potential is not yet clear, the anneal time of $\approx 20\mu s$, independently of the size of the problem, promises an acceleration that deserves to be

explored. So far, current D-Wave hardware yields results very similar to classical solvers - the anneal needs to be run multiple times, as noise, thermal fluctuations and other external factors may interfere with the process. Further complications arise due to various overhead costs and the necessity to split large QUBOs into small instances that fit the hardware [7].

We should also point out an interesting possibility of applying algorithms for gate-based quantum computers, like the Quantum Approximate Optimization Algorithm (QAOA), the Variational Quantum Eigensolver (VQE) and the Harrow-Hassadim-Lloyd (HHL) algorithm, which, when applied to the optimization of the Hopfield network, can serve to further accelerate the search for the global minimum of the proposed matrix representing the network energy function.

The Hopfield network approach

A track with n hits (3d coordinates from detectors) can be regarded as a set of $n-1$ consecutive lines ("track segments") with a smooth shape and without bifurcation [6]. Based on methods developed in the late eighties (Denby 1988 [3]; Peterson 1989 [4]) and the beginning of nineties (Stimpfl-Abele and Garrido [6]), the Hopfield network [5] approach uses for track reconstruction a method that optimizes an energy function for which we chose the following form:

$$E = -\gamma \sum_{i,j,k} \left(\frac{\cos^{\lambda}(\theta_{ijk})}{(r_{ij}+r_{jk})^{\eta}} \right) v_{ij} v_{jk} + \alpha \left(\sum_{j \neq k} v_{ij} v_{ik} + \sum_{i \neq j} v_{ik} v_{jk} \right) + \beta \sum_{i,j} v_{ij} \,, \quad (1)$$

where θ_{ijl} is the angle between possible track segments v_{ij} and v_{jk} (equal to one when active and zero otherwise) of length r_{ij} and r_{jk}, respectively. The second term is a penalty for an undesired track bifurcation, the third one is a constant inhibition term which helped us make the energy matrix more sparse.

This way, we obtain a segment classification task, where each term of the energy function E is designed for geometric rewards and penalties weighted by parameters $\gamma, \lambda, \alpha, \beta$, such that tracks composed of short track segments (doublets) that lie on a smooth curves with no bifurcations are biased, cf. [8].

Results for SPD modeling data

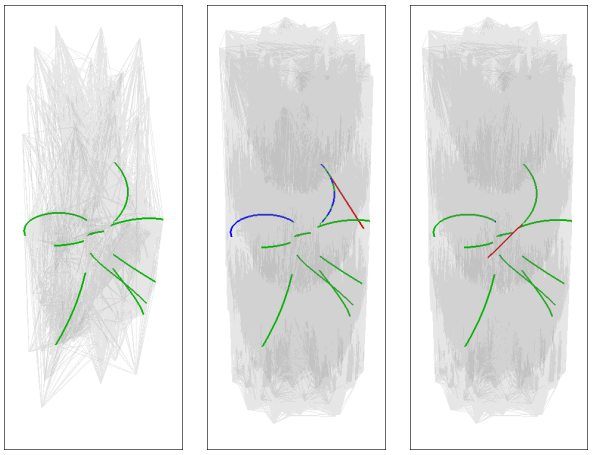
An example of the results of our method for an event with 10 tracks with different number of noise hits and sets of optimized parameters is shown in Fig. 1. Due to the large number of detector layers, tracks consist of a large number of short segments, which facilitates their reconstruction. However, it can be seen that a larger amount of noise hits decreases the tracking quality.

QUBO formulation of the particle tracking problem

Energy function (1) resembles a QUBO, which is defined as

$$\min_{\boldsymbol{x} \in \{0,1\}^n} \boldsymbol{c}^{\intercal} \boldsymbol{x} + \boldsymbol{x}^{\intercal} Q \boldsymbol{x} \,, \quad (2)$$

where the minimum is taken over the collection of binary vectors $\boldsymbol{x}$ of length n, $\boldsymbol{c} \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix. QUBO does not, by definition,



(a) (b) (c)

FIGURE 1. Results of tracking of an event with 10 tracks. True positive, true negative, false positive and false negative segments are shown. (a) 100 noise hits (b) 1750 noise hits, minimization of false-positive segments prioritized (c) 1750 noise hits, TrackML [9] metric prioritized [1, 8].

contain constraints, so any potential constraints have to be incorporated to the objective function as penalties through reformulation techniques. A drawback of such an approach is that such squared penalty will result in a QUBO with $O(n^2)$ quadratic terms. However, there exist approaches to design more efficient ways to represent such constraints for optimization with quantum annealers such as D-Wave (e.g. [10]).

When QUBO is solved on a quantum annealer, each linear coefficient c_i of QUBO is mapped as a bias onto a distinct qubit i, and each quadratic coefficient q_{ij} is encoded as a weight of a link between qubits i and j, called a coupler [10].

The mapping from the QUBO problem to the graph, describing the interconnection between the qubits in the hardware ("chimera" graph structure in the current D-Wave architecture) is a limitation of this approach, and is currently dealt with through so-called minor-embedding [12]. More on QUBO/Ising formulations of NP problems can be found in [11].

To improve the performance of the algorithm, we plan to formulate QUBO to identify the best pairs of triplets, instead of doublets. A triplet T_i is a set of three hits (a, b, c) or a pair of consecutive doublets (a, b and b, c). Two triplets of hits (a, b, c) and (d, e, f), can be combined to form a quadruplet if $b = d \wedge c = e$ or a quintet if $c = d$. The objective function (2) to minimize has two components: a linear term that weighs the quality of individual triplets and a quadratic term

used to express relationships between pairs of triplets:

$$\sum_{i=1}^{N} a_i T_i + \sum_{i}^{N} \sum_{j<i}^{N} b_{ij} T_i T_j \,, \tag{3}$$

where T are all potential *triplets*, a_i are the *bias weights*, and b_{ij} the *coupling strengths* computed from the relation between the triplets T_i and T_j [7].

Conclusion and outlook

We attempted to apply several modifications of the algorithm to the simulation of SPD data with the presence of fake hits. The method showed good results, but under rather simple conditions. Fake and noise hits pose notable difficulties for tracking by different methods. We need to study the impact of events where fake hits are generated more correctly in terms of the geometry of the experimental setup. An improvement of the energy function, which gives us the QUBO model, has to be worked out. More advanced segment filtering methods are needed (e.g. using triplets [7]), which would possibly reduce the impact of noise hits, and also allow the method to be tested on TrackML [9] data. Finally, a method to assess the timing performance of the algorithm needs to be developed.

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Block-Fermat numbers in modular arithmetic

Benjamin Chen and Eugene V. Zima

Abstract. Conversion to a modular representation or residue number system (RNS) is a popular technique to accelerate arithmetic in computer algebra systems. We describe and analyse the strategy of selecting moduli in modular arithmetic using so called Block-Fermat numbers of the form $2^n + 1$ and compare this with Mersenne type numbers of the form $2^n - 1$. We show that conversion to RNS for Block-Fermat numbers has the same complexity as for Mersenne type numbers of the same size, however reconstruction based on Block-Fermat numbers is much faster.

1. Introduction

Conversion to a modular representation or residue number system (RNS) is a popular technique to accelerate arithmetic of computer algebra systems. An overhead of using modular approach is related to the need of conversion of the input involving large integers to RNS and the need of reconstruction of the result from multiple modular images using he Chinese remainder theorem [6, 8]. The choice of specific values moduli can influence significantly the time complexity of both computations in RNS and reconstruction of the result. One popular approach is to choose many moduli that fit machine word and use hardware arithmetic for the simultaneous conversion to and from an RNS [1, 3]. Another approach is related to the choice of moduli with special shape (bit pattern) that accelerates reduction modulo m_i and (possibly) reconstruction. One of the oldest examples of such approach is described in [9, 8] (see also [10]). We consider combination of these approaches.

In [3] an algorithm for simultaneous conversions between a given set of integers and their modular representations based on fast linear algebra is described. A highly optimized implementation of the algorithm in FFLAS-FFPACK library [4] that exploits the computational features of modern CPUs is provided. This implementation performance on the benchmark of matrix multiplication starts to deteriorate when the size of entries of randomly selected integer matrices becomes very large (2^{18} or more bits). Authors discuss two possible workarounds for these cases. In [2] we described an alternative approach based on two-layer modular

arithmetic. The idea is to select large moduli allowing faster than usual reduction/reconstruction on the first layer, and reduce the problem to several problems with entries bit-size amenable for FFLAS-FFPACK implementation. On the second layer simultaneous conversion [3] is used. Results from the second layer are used to reconstruct the final answer using accelerated reconstruction with specially selected moduli.

2. Block-Fermat numbers

Consider pairwise co-prime natural numbers $m_0, m_1, \ldots, m_k$ and assume we work with non-negative representatives in each class of residues modulo m_i. Modular reconstruction problem is: given non-negative $x_0, x_1, \ldots, x_k$ ($x_i < m_i$), find non-negative $X < m_0 m_1 \ldots m_k$ such that $X = x_i \bmod m_i$ for $i = 0, 1, \ldots, k$. Standard modular reconstruction algorithms [6] precompute products of moduli $M_i = \prod_{j=0}^{i-1} m_j$ and inverses $M_i^{-1} \bmod m_i, i = 1, \ldots, k$. Reconstruction involves several multiplications by these quantities which provide significant contribution to the total complexity. When deciding a particular approach to the choice of moduli m_i one can try to satisfy the following natural requirements:

1. $\gcd(m_i, m_j) = 1$ for $i \neq j$;
2. reduction modulo m_i is "simpler" than division with remainder;
3. products of moduli and their inverses have bit pattern (preferably scalable) that allows accelerated multiplication by those quantities;
4. bit length of moduli m_i is balanced.

Several strategies to select moduli of the form $2^n \pm 1$ are considered in [10]. They all improve modular reduction time (replacing division with remainder by shifts and additions/subtractions). Shift-based [10] moduli of the form $m_i = 2^{a2^i} + 1, \quad i = 0, 1, \ldots, k$, also satisfy requirement 3, as

$$\left(\prod_{j=0}^{i-1} m_j \right)^{-1} \bmod m_i = 2^{a2^i - 1} - 2^{a-1} + 1, \quad i = 1, 2, ..., k \tag{1}$$

for arbitrary natural a. However, such choice of moduli does not satisfy requirement 4. In fact, the bit length of m_i is larger than the bit length of product $m_0 m_1 \ldots m_{i-1}$.

There are several strategies of selection of balanced moduli of Fermat-type. Some of strategies were first discussed in [10]. We show that one of strategies produces the set of moduli which satisfies all aforementioned natural requirements. Consider moduli $a = 2^{2^n - 2^k} + 1$ and $b = 2^{2^n - 2^\ell} + 1$ with $2 \leq k < n$, $2 \leq \ell < n$ (we call them Block-Fermat moduli). They are relatively prime for $k \neq \ell$ (see [10]) and more balanced in size, compared to shift-based moduli m_i in (1). In fact, for fixed value of n the largest modulus requires at most twice as many bits as the smallest one. The quantity $(a^{-1} \mod b)$, which is used in reconstruction from modular images, has a sparse bit pattern and also is scalable. This means

that if one scales the exponents in a and b by the same natural scaling factor u (i.e., considers moduli $\tilde{a} = 2^{u(2^n-2^k)} + 1$ and $\tilde{b} = 2^{u(2^n-2^\ell)} + 1$) then $(\tilde{a}^{-1} \mod \tilde{b})$ will have the same sparsity as $(a^{-1} \mod b)$. For example, for $a = 2^{224} + 1$ and $b = 2^{192} + 1$ we have

$$a^{-1} \mod b = 2^{191} + 2^{159} + 2^{127} + 2^{95} + 2^{63} + 2^{31} + 1.$$

Now, if we scale moduli by factor $u = 100$ and consider $\tilde{a} = 2^{22400} + 1$ and $\tilde{b} = 2^{19200} + 1$, then

$$\tilde{a}^{-1} \mod \tilde{b} = 2^{19199} + 2^{15999} + 2^{12799} + 2^{9599} + 2^{6399} + 2^{3199} + 1.$$

Multiplication by such sparse inverses reduces to additions and shifts, which can be used to accelerate reconstruction of the result from modular images in upper layer of two-layer modular arithmetic [2].

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Estimates for roots of a polynomial in the field of multiple formal fractional power series in zero characteristic

Alexander L. Chistov

Let k be a ground field of zero–characteristic with algebraic closure $\overline{k}$. We assume that k is finitely generated over its primitive subfield. Let $f \in k[X_1, \ldots, X_n, Z]$ be a polynomial of degree $\deg_{Z,X_1,\ldots,X_n} \leqslant d$ for an integer $d \geqslant 2$.

Consider $f \in k(X_1, \ldots, X_n)[Z]$ as a polynomial in one variable Z with coefficients in $k(X_1, \ldots, X_n)$. Then the roots $Z = z_\alpha$ of the polynomial f belong to the field of multiple formal fractional power series in $X_1, \ldots, X_n$, i.e. to the union by all integers $\nu_1, \ldots, \nu_n \geqslant 1$ of the fields of multiple formal fractional power series:

$$\bigcup_{\nu_1,\ldots,\nu_n \geqslant 1} \overline{k}((X_1^{1/\nu_1}))((X_2^{1/\nu_2})) \ldots ((X_n^{1/\nu_2})). \tag{1}$$

This field is algebraically closed. The aim of this talk is to attract the atention to the problem of estimating and constructing the roots z_α in the field (1) (of course one needs to estimate the sizes of coefficients from $\overline{k}$ of z_α in the field (1)). This problem is solved for $n = 1$ in [1]. To our knowledge for the case case $n > 1$ no estimates have been obtained so far.

The problem for an arbitrary n is reduced to the case $\nu_1 = \ldots = \nu_n = 1$. Hence now $z_\alpha \in \overline{k}((X_1))((X_2)) \ldots ((X_n))$. Further, for all $1 \leqslant j \leqslant n$ put $X_j' = X_j/(X_1^{\mu_{j,1}} \cdot \ldots \cdot X_{j-1}^{\mu_{j,j-1}})$ for some integers $\mu_{j,i} \geqslant 0$ (so $X_1' = X_1$). Then one can choose integers $\mu_{j,i}$ such that

$$z_\alpha \in \overline{k}[[X_1', \ldots, X_n']], \tag{2}$$

i.e., z_α are formal power series in $X_1', \ldots, X_n'$ with coefficients from $\overline{k}$. Good upper bounds for the coefficients of formal power series in (2) can be obtained using the results of [2], [3].

Now it remains to estimate the least possible $\mu_{j,i}$. This can be done applying the results of [1] or [3] recursively. The direct application of [1] or [3] gives double–exponential in n upper bounds for $\mu_{j,i}$. But we hope to improve the estimates from

[3] and obtain upper bounds for $\mu_{j,i}$ which are subexponential in the number of coefficients of the polynomial f, i.e., upper bounds polynomial in $d^{n^{O(1)}}$.

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On the integrability of two- and three-dimensional dynamical systems with a quadratic right-hand side in cases of resonances in linear parts and in cases of general position

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Аннотация. We use a heuristic method that allows one to a prior determine the cases of integrability of an autonomous dynamical systems with a polynomial right-hand side. We demonstrate the capabilities of the method using examples of two and three-dimensional dynamic systems with quadratic nonlinearity. A significant advance relative to our previous works is the possibility of studying systems of a general form, without resonances in the linear parts.

Introduction

In previous works [1, 2] a technique was described for constructing systems of algebraic equations for the parameters of an ODE system with resonance in its linear part. It was experimentally shown that with relations on the parameters obtained as a result of solving such a system, it is usually possible to find explicit expressions for the first integrals of the ODE.

This report discusses the use of this technique to search for first integrals of two and three-dimensional systems with quadratic nonlinearity on the right-hand sides.

1. Two-dimensional case

For the center case

$$\begin{aligned} \dot{x} &= \quad y + a_1 x^2 + a_2 x\, y + a_3 y^2, \\ \dot{y} &= -x + b_1 x^2 + b_2 x\, y + b_3 y^2, \end{aligned} \tag{1}$$

the first integrals can be calculated for the systems:

1	$\dot{x} = y + a_1x^2 - 2\,b_3xy + a_3y^2,$	$\dot{y} = -x + b_1x^2 - 2\,a_1xy + b_3y^2;$
2	$\dot{x} = y + a_1x^2 + a_2xy - a_1y^2,$	$\dot{y} = -x - b_3x^2 + b_2xy + b_3y^2.$
3	$\dot{x} = y + a_2xy,$	$\dot{y} = -x + b_1x^2 + b_3y^2;$
4	$\dot{x} = y + a_1x^2 - 2\,b_3xy,$	$\dot{y} = -x + b_1x^2 - 2\,a_1xy + b_3y^2;$
5	$\dot{x} = y + a_1x^2 - 2b_3xy - a_1y^2,$	$\dot{y} = -x + b_1x^2 - 2a_1xy + b_3y^2;$
6	$\dot{x} = y + a_1x^2 + a_2xy - a_1y^2,$	$\dot{y} = -x - b_3x^2 - 2a_1xy + b_3y^2.$
7	$\dot{x} = y + a_2xy - b_2y^2,$	$\dot{y} = -x - a_2x^2 + b_2xy;$
8	$\dot{x} = y + a_1x^2 + \frac{1}{5}\,(-3a_1 + b_2)y^2,$	$\dot{y} = -x + b_2xy;$
9	$\dot{x} = y + \frac{1}{5}\,b_2y^2,$	$\dot{y} = -x + b_2xy;$
10	$\dot{x} = y - b_2xy + \frac{1}{5}\,b_2y^2,$	$\dot{y} = -x - \frac{1}{5}\,b_2x^2 + b_2xy;$
11	$\dot{x} = y + b_2xy + \frac{1}{5}\,b_2y^2,$	$\dot{y} = -x + \frac{1}{5}\,b_2x^2 + b_2xy;$
12	$\dot{x} = y + a_1x^2 - \frac{5}{6}\,a_1y^2,$	$\dot{y} = -x - \frac{7}{6}\,a_1xy;$
13	$\dot{x} = y + a_1x^2 - \frac{2}{3}\,a_1y^2,$	$\dot{y} = -x - \frac{4}{3}\,a_1xy.$

In case 2 above we can prove the existence of an integrating factor, but the first integral is too cumbersome to handle. We are grateful to Prof. M.V. Demina for the calculation of the first integral in case 6.

For the saddle case

$$\begin{aligned} \dot{x} &= \alpha x + a_1x^2 + a_2x\,y + a_3y^2, \\ \dot{y} &= -y + b_1x^2 + b_2x\,y + b_3y^2, \end{aligned} \tag{2}$$

at the resonance 1:1, i.e. at $\alpha = 1$ we got the first integrals for the systems:

1	$\dot{x} = x + b_2b_3/a_2\,x^2 + a_2xy + a_3y^2,$	$\dot{y} = -y + a_3b_2^3/a_2^3\,x^2 + b_2xy + b_3y^2;$
2	$\dot{x} = x + a_1x^2 + a_3y^2,$	$\dot{y} = -y + b_1x^2 + b_3y^2;$
3	$\dot{x} = x - \frac{1}{2}\,b_2x^2 + a_2xy + a_3y^2,$	$\dot{y} = -y + b_1x^2 + b_2xy - \frac{1}{2}\,a_2y^2;$
4	$\dot{x} = x + a_1x^2,$	$\dot{y} = -y + b_1x^2 + b_2xy;$
5	$\dot{x} = x + 2b_2x^2 + a_2xy + a_3y^2,$	$\dot{y} = -y + \frac{a_2b_2}{a_3}\,x^2 + b_2xy + 2a_2y^2;$
6	$\dot{x} = x - b_2/2x^2 + a_2xy,$	$\dot{y} = -y + b_2xy - a_2/2y^2;$
7	$\dot{x} = x + 2b_2x^2 + a_2xy,$	$\dot{y} = -y + b_2xy + 2a_2y^2.$

For the resonance $2:1$, i.e. at $\alpha = 2$ we got the first integrals for the systems:

1	$\dot{x} = 2x + \frac{2b_2b_3x^2}{a_2+b_3} + a_2xy,$	$\dot{y} = -y + b_2xy + b_3y^2;$
2	$\dot{x} = 2x - \frac{2}{3}\,b_2x^2 - 4b_3xy,$	$\dot{y} = -y + b_1x^2 + b_2xy + b_3y^2;$
3	$\dot{x} = 2x + a_1x^2 - b_3xy,$	$\dot{y} = -y + b_1x^2 + b_3y^2;$
4	$\dot{x} = 2x + a_1x^2,$	$\dot{y} = -y + b_1x^2 + b_2xy;$
5	$\dot{x} = 2x + \frac{4}{3}\,b_2x^2 + \frac{1}{2}\,b_3xy,$	$\dot{y} = -y + b_2xy + b_3y^2;$
6	$\dot{x} = 2x + \frac{1}{2}\,b_3xy,$	$\dot{y} = -y + b_1x^2 + b_3y^2;$
7	$\dot{x} = 2x + a_3y^2,$	$\dot{y} = -y.$

The results above were got by solving the algebraic systems on the system parameters. Each of these algebraic system was calculated for a certain resonance, i.e. for the fixed natural parameter α. But the form of all these equations, their variables, are the same. The idea arises to look for a general solution of systems for several resonances. We solved a combined system for 1:1, 2:1 and 3:1 resonances.

For each set of parameters obtained as a result of solving this unified system, it was possible to calculate the first integrals of system (2) for an arbitrary (symbolic) α. These systems are:

$$
\begin{array}{lll}
1 & \dot{x} = \alpha x + a_1 x^2, & \dot{y} = -y + b_1 x^2 + b_3 y^2; \\
2 & \dot{x} = \alpha x + a_1 x^2, & \dot{y} = -y + b_1 x^2 + b_2 xy; \\
3 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y + b_3 y^2; \\
4 & \dot{x} = \alpha x + 2b_2 x^2 + a_3 y^2, & \dot{y} = -y + b_2 xy; \\
5 & \dot{x} = \alpha x + a_2 xy, & \dot{y} = -y + b_2 xy; \\
6 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y; \\
7 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y - \frac{1}{2} a_2 y^2; \\
8 & \dot{x} = \alpha x + b_2 x^2 + a_2 xy, & \dot{y} = -y + b_2 xy + a_2 y^2; \\
9 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y + a_2 y^2; \\
10 & \dot{x} = \alpha x + a_2 xy, & \dot{y} = -y + b_1 x^2 + 2a_2 y^2; \\
11 & \dot{x} = \alpha x + a_2 xy + a_3 y^2, & \dot{y} = -y + 2a_2 y^2.
\end{array}
$$

of the MATHEMATICA-11 system or by hand using the Darboux method.
The first integrals for the systems above were calculated by the procedure DSolve

Three dimension case

First we considered resonant cases of the system

$$
\begin{aligned}
\dot{x} &= \ \ M_x\, x + a_2\, x\, y + a_4\, x\, z + a_5\, y\, z, \\
\dot{y} &= -M_y\, y + b_2\, x\, y + b_4\, x\, z + b_5\, y\, z, \\
\dot{z} &= \ \ -z + c_2\, x\, y + c_4\, x\, z + c_5\, y\, z
\end{aligned}
\tag{3}
$$

with natural M_x, M_y on the square table $\{1,2,3\} \times \{1,2,3\}$

In the two-dimensional case, we struggled to evaluate each integral. But here we limited ourselves to calculations only using the DSolve procedure of the MATHEMATICA 13.3.1.0 system. The result is in the table

N	M_x	M_y	Alg.solutions	Integrals	% Success
8	1	1	23	19	83
8	1	2	16	12	75
8	1	3	25	19	76
8	2	1	57	49	86
8	2	2	34	29	85
8	2	3	43	35	81
9	3	1	60	51	85
9	3	2	63	58	92
10	3	3	43	38	88

"N" here is the normal form order, "Alg.solutions" is a number of rational solutions of the corresponding algebraic system, the "Integrals" is a number of success solutions by the MATHEMATICA and "Success" is the percentage of successfully integrated cases to the total number of solutions of the corresponding algebraic system.

Then we solved the unified algebraic system of 329 equations, found its 10 solutions, and found that system MATHEMATICA-11 solves all corresponding

systems of ODEs of the form (3) except one. These systems with arbitary M_x and M_y are:

1	$\dot{x} = M_x x + a_2 x\cdot y + a_4 x\cdot z + a_5 y\cdot z,$	$\dot{y} = -M_y y + b_5 y\cdot z,$	$\dot{z} = -z + c_5 y\cdot z;$
2	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_2 x\cdot y + b_4 x\cdot z,$	$\dot{z} = -z + c_4 x\cdot z;$
3	$\dot{x} = M_x x + a_2 x\cdot y + a_4 x\cdot z + a_5 y\cdot z,$	$\dot{y} = -M_y y + a_4 y\cdot z,$	$\dot{z} = -z - a_2 y\cdot z;$
4	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_2 x\cdot y,$	$\dot{z} = -z + c_4 x\cdot z;$
5	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_4 x\cdot z,$	$\dot{z} = -z + c_4 x\cdot z;$
6	$\dot{x} = M_x x,$	$\dot{y} = -M_y y,$	$\dot{z} = -z + c_4 x\cdot z + c_5 y\cdot z;$
7	$\dot{x} = M_x x,$	$\dot{y} = -M_y y + b_2 x\cdot y + b_5 y\cdot z,$	$\dot{z} = -z;$
8	$\dot{x} = M_x x + a_4 x\cdot z,$	$\dot{y} = -M_y y + b_4 x\cdot z + a_4 y\cdot z,$	$\dot{z} = -z;$
9	$\dot{x} = M_x x + a_5 y\cdot z,$	$\dot{y} = -M_y y + b_2 x\cdot y,$	$\dot{z} = -z - b_2 x\cdot z;$ Nonintegrable?
10	$\dot{x} = M_x x,$	$\dot{y} = -M_y y,$	$\dot{z} = -z + c_4 x\cdot z.$

2. The general three-dimension system

Finally, we considered the general case of a three-dimensional system with 20 parameters

$$\begin{aligned}
\dot{x} &= \quad M_x x + a_1 x^2 + a_2 x\cdot y + a_3 y^2 + a_4 x\cdot z + a_5 y\cdot z + a_6 z^2,\\
\dot{y} &= -M_y y + b_1 x^2 + b_2 x\cdot y + b_3 y^2 + b_4 x\cdot z + b_5 y\cdot z + b_6 z^2,\\
\dot{z} &= \quad -z + c_1 x^2 + c_2 x\cdot y + c_3 y^2 + c_4 x\cdot z + c_5 y\cdot z + c_6 z^2.
\end{aligned}$$

Calculating the normal form up to 6th order for 4 pairs $\{M_x, M_y\}= \{1,1\}$, $\{1,2\}$, $\{2,1\}$ and $\{2,2\}$, we got a system of 121 equations with 18 parameters. We calculated 174 of its solutions. For 109 of them the MATHEMATICA-13 system calculated solutions of the corresponding ODEs.

3. Conclusion

There are many cases of integrability of dynamical systems, and the corresponding exact solutions can be very useful in applications, for example, in problems of chemical kinetics [3].

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Teaching Statistics with Technology

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Abstract. We discuss usage of computer technology in teaching the introductory statistics course in St. George's University.

Introduction

Computers have become indispensable teaching tools, providing convenient means for preparing classes, demonstrations, etc. Computers have also freed students from boring and time-consuming calculations by completing work quickly and efficiently.

Computer Algebra Systems (CAS), such as Mathematica or Maple besides being easy to program basic formulas, provide a set of ready statistical tools ranging from basic descriptive statistics to fitting the models, cluster analysis, hypothesis testing and time-series analysis.

Here, we discuss computer tools used in teaching the course on Introductory Statistics. While CAS are convenient for teachers to prepare for the classes, preparing demonstrations and assignments, we find that students need first of all practical tools that can and will be used in the future for real statistical analysis. So for lab exercises students are recommended to use Excel, RCommander or JASP.

Excel provides the means for basic statistical analysis and visualizations. As a spreadsheet it also is convenient as a replacement for "old-fashion" way of calculations using frequency tables, etc. RCommander is recommended since it provides a set of professional utilities for statistical analysis and visualization. It also can be used as a first step in learning and using R. Typical problem students have with using RCommander is related to installation of it. JASP is easy to install and use and provides most of the tools needed for the course.

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Doubly-periodic string comparison

Nikita Gaevoy, Boris Zolotov and Alexander Tiskin

Abstract. The longest common subsequence (LCS) problem is a fundamental algorithmic problem. Given a pair of strings, the problem asks for the length of the longest string that is a subsequence in both input strings. In previous works, the third author developed an algebraic framework for the LCS problem and its relatives, based on the Hecke monoid. Among the many algorithmic problems that can be approached efficiently by this technique, there is the natural problem of obtaining the LCS length for a pair of strings, one or both of which have a periodic structure. The case where one of the input strings is periodic has been considered before; in this work, we develop an efficient algorithm for the more general case where both input strings are periodic. This algorithm for doubly-periodic LCS has been engineered by the first author; the resulting code can process doubly-periodic inputs of sizes far beyond the reach of ordinary and singly-periodic LCS algorithms.

Introduction

The longest common subsequence (LCS) problem is a fundamental algorithmic problem. Given a pair of strings, the problem asks for the length of the longest string that is a subsequence in both input strings. The LCS and the equivalent problem of computing the *edit distance* between two strings have found many applications such as computing DIFF of two texts in the corresponding Linux utility, or, generally, the best weighted alignment of two strings, which is widely used in bioinformatics.

LCS-related problems display unexpected ties with semigroup algebra, computational geometry and transposition networks. In previous works, the third author developed an algebraic framework for the LCS problem and its relatives, based on the Hecke monoid. Surprisingly, multiplication in the Hecke monoid is found to reflect closely the behavior of the LCS structure under string concatenation.

Among the many algorithmic problems that can be approached efficiently by this technique, there is the natural problem of obtaining the LCS length for a pair of strings, one or both of which have a periodic structure. The case where one of

the input strings is periodic has been considered before: it is solved by reduction in the affine counterpart of the Hecke monoid.

In this work, we develop an efficient algorithm for the more general case where both input strings are periodic. It works in time $O(mn)$, under mild assumptions, where m and n are the lengths of the periods of these strings. This algorithm for doubly-periodic LCS has been engineered by the first author; the resulting code can process doubly-periodic inputs of sizes far beyond the reach of ordinary and singly-periodic LCS algorithms.

1. The framework

In our previous work [3, 5], we established a connection between string comparison and sticky multiplication of permutations, expressed as $P \boxdot Q = R$. A subquadratic sticky multiplication algorithm (called Steady Ant Algorithm) is given in [5].

Traditionally, the monoid of permutations considered under sticky multiplication is known as the *Hecke monoid* $\mathbb{H}_n$. An element of $\mathbb{H}_n$ can be represented by a *sticky braid* which is analogous to classical braids.

This work extends this connection to infinite periodic strings and affine permutations [2]. Sticky multiplication is generalised directly to affine permutations and expressed as $\tilde{P} \cdot \tilde{Q} = \tilde{R}$. The resulting monoid is known as *affine Hecke monoid* $\tilde{\mathbb{H}}_n$.

An (affine) sticky braid is called *reduced*, if every pair of its strands cross at most once. A sticky braid (finite or affine) can also be viewed as a *transposition network* (introduced as *primitive sorting networks* in [1]).

For a finite permutation P (affine permutation $\tilde{P}$) we choose an arbitrary reduced braid (or, equivalently, a transposition network) realizing this permutation and denote it by $\mathcal{B}(P)$ ($\mathcal{B}(\tilde{P})$, respectively).

Affine replication operator repl_n allows to extend a function $P\colon [0:n) \to \mathbb{Z}$ to an affine permutation $\tilde{P}$ of order n, provided all images of P are pairwisely non-congruent modulo n.

The $\Gamma\Phi$- or $\Phi\Gamma$-decomposition is a representation of an affine permutation $\tilde{P}$ of order n as a product of two affine permutations, one of which is a repl_n of a permutation in S_n, and the other is monotone increasing on $[0:n)$.

2. Affine sticky multiplication

We reduce sticky multiplication of affine permutations $\tilde{P}, \tilde{Q}$ to sticky multiplication of finite permutations. We compute the sticky product of three consecutive periods of $\tilde{P}$, $\tilde{Q}$, taken with respect to the codomain of $\tilde{P}$, which is also the domain of $\tilde{Q}$. We then cut out and replicate the middle period of the resulting product, obtaining $\tilde{P} \boxdot \tilde{Q}$; see Algorithm 1.

We prove the correctness of our algorithm by showing that the injection $Q^* = Q_{(P)}|_{[n:2n)} \cdot \Gamma'$ can be replicated to form an affine permutation, and the result

Algorithm 1 Affine Sticky Multiplication

Input: affine permutations $\tilde{P}$, $\tilde{Q}$ of order n.
Output: affine permutation $\tilde{P} \boxdot \tilde{Q}$.
Description:

1. Compute $\Gamma\Phi_{3n}$-decomposition $\tilde{P} = \Gamma\Phi$ and $\Phi\Gamma_{3n}$-decomposition $\tilde{Q} = \Phi'\Gamma'$. Let $P = \Phi|_{[0:3n)}$, $Q = \Phi'|_{[0:3n)}$.
2. Compute sticky product $P \boxdot Q$. Compute $Q_{(P)} = P^{-1} \cdot (P \boxdot Q)$.
3. Compute $Q^* = Q_{(P)}|_{[n:2n)} \cdot \Gamma'$. This is an injection $[n:2n) \to \mathbb{Z}$. We have $\tilde{Q}_{(\tilde{P})} = \mathrm{repl}_n\, Q^*$.
4. Output the affine permutation $\tilde{P} \cdot \tilde{Q}_{(\tilde{P})}$.

of the replication provides the correct value for $\tilde{Q}_{(\tilde{P})}$. Equivalently, Q^* coincides with a single period of $\tilde{Q}_{(\tilde{P})}$, that is, $Q^* = \tilde{Q}_{(\tilde{P})}|_{[n:2n)}$ as functions $[n:2n) \to \mathbb{Z}$.

We do this by showing that $\mathcal{B}(\tilde{Q})$ can simulate the behavior of $\mathcal{B}(Q)$ up to a subsequent decompression by Γ', if it is prohibited from untangling intersections between certain trajectories, while keeping the images of elements within $[n:2n)$ unchanged.

The time complexity of this algorithm is $O(n \cdot \log n)$ as it uses the Steady Ant Algorithm for computing the sticky product of two finite permutations, and otherwise only sorts preimage-image pairs of permutations within a period and takes standard compositions of permutations.

3. Algorithm for doubly-periodic LCS

Let $A = a^r$, $B = b^s$ be finite periodic strings of lengths $M = rm$, $N = sn$ with periods a, b, respectively. To find $\mathrm{LCS}(a, b)$, we:

1. extend B to an infinite n-periodic string b^∞,
2. obtain a skew-affine embedded braid for a vs. b^∞ by placing crosses and elbow joints on the grid in a straightforward manner,
3. reduce this braid using wraparound combing [4], compute the corresponding affine permutation $\tilde{P}$ describing implicitly the LCS of a vs. b^∞,
4. compute $\tilde{P}^{\boxdot r}$ by binary exponentiation using Algorithm 1 as a subroutine,
5. count the elements of $\tilde{P}^{\boxdot r}$ that fall within the bounds of B respecting the affine structure of the braid.

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Binomial Coefficients as Functions of their Denominator; Another Primality Criteria for Natural Integers

Nikita Gogin and Vladislav Shubin

Abstract. We prove that an odd positive integer n is prime iff denominator of the rational number $\langle\binom{n}{n-1}\rangle$ is n^{n-1}, where $\langle\binom{n}{x}\rangle = \mathcal{B}_n(x)$ is interpolation polynomial on x for the set of binomial coefficients $\{\binom{n}{r}\}_{r=0,1,\ldots n}$ and $x \in [0,n] \subset \mathbb{R}$.

Keywords. Prime numbers, Binomial coefficients, Interpolation polynomial, Newton interpolation formula, Krawtchouk polynomials.

1. Introduction and Preliminaries

Binomial coefficients have
surprisingly great expressive power . . .
Yu. V. Matiyasevich [2]

In this paper we use generally accepted definition of the (generalized) binomial coefficients as polynomials on the (real) variable x:

$$\binom{x}{m} = \frac{x(x-1)\ldots(x-m+1)}{m!} \tag{1}$$

where m is a nonnegative integer, $\binom{x}{0} = 1$. [3]
However in joint publication [4], among other things, we proved that the identity

$$\binom{n}{k} = (-2)^n \sum_{i=0}^{n} \binom{\frac{i-1}{2}}{n} K_i^{(n)}(k) \tag{2}$$

is valid for all integer $k, 0 \le k \le n$, where $K_i^{(n)}(k)$ are the Krawtchouk polynomials of order n. [3]

The right side of this equality is a polynomial on k and this allows us to accept it as the definition of the symbol $\langle\binom{n}{x}\rangle$ where x stands for k and can be treated

now as an element of any (not necessarily commutative) algebra over a field of zero characteristic and n is a (fixed) nonnegative integer. In particular if $x \in [0,n] \subset \mathbb{R}$ the polynomial $\mathcal{B}_n(x) = \langle \binom{n}{x} \rangle$ is of course the ordinal interpolation polynomial for the set of binomial coefficients $\{\binom{n}{r}\}_{r=0,1,\dots n}$. expanded by the (orthogonal) basis of Krawtchouk polynomials. Since $\mathcal{B}_n(x)$ is interpolation polynomial, we can use its notation in any form convenient for our purposes. Here we take the explicit Newton interpolation formula for equidistant nodes with a step $h = 1$ [5]:

Let $f(k), k = 0, 1, \dots, n$ be a tuple of values of a real function f. Then the following formula for interpolation polynomial $P_n(f;x)$ is s valid:

$$P_n(f;x) = \sum_{m=0}^{n} \binom{x}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(k), \tag{3}$$

where $x \in [0,n] \subset \mathbb{R}$.

2. Some Auxiliary Formulas

Applying formula (3) to $x = n^{-1}$ and $f(k) = \binom{n}{k}$ we get the equality

$$\mathcal{B}_n(n^{-1}) = \left\langle \binom{n}{n^{-1}} \right\rangle = \sum_{m=0}^{n} \binom{n^{-1}}{m} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{n}{k}. \tag{4}$$

Let now $A^{(n)}$ and $B^{(n)}$ be two auxiliary arrays :

$A^{(n)} = \{a_m\}_{0 \le m \le n}$, with $a_0 = 1$
and for $1 \le m \le n$

$$a_m = \binom{n^{-1}}{m} = \frac{n^{-1}(n^{-1}-1)(n^{-1}-2)\dots(n^{-1}-(m-1))}{m!} = \frac{\lambda_m}{n^m}, \tag{5}$$

where

$$\lambda_m = \frac{\prod_{r=0}^{m-1}(1-nr)}{\prod_{s=1}^{m} s}, \tag{6}$$

and

$B^{(n)} = \{b_m\}_{0 \le m \le n}$ with $b_0 = 1$ and for $1 \le m \le n$

$$b_m = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{n}{k} = \#_{t^m}[(1-t)^m (1+t)^{(n+m)-m}] = K_m^{(n+m)}(m). \tag{7}$$

In particular if $m = n$ and n is odd then $b_n = 0$ [6]. Formula (4) evidently can be written as a scalar product:

$$\mathcal{B}_n(n^{-1}) = \left\langle A^{(n)}, B^{(n)} \right\rangle \tag{8}$$

.

From the equalities (4),(7) and (8) we get the "duality" formula:

$$\binom{n}{n^{-1}} = \sum_{m=0}^{n} \binom{n^{-1}}{m} K_m^{(n+m)}(m). \tag{9}$$

Lemma 1. 1. (a) *If $1 < s \leq m$ is an index of the denominator of formula* (6) such that gcd(s,n)=1 then there exists a unique index r , $0 \leq r \leq m-1$ in its numerator such that $s|(1-nr)$;
(b) In particular if $n = p$ is an odd **prime** integer then all numbers λ_m are **integers** with $\gcd(\lambda_m, n = p) = 1$ for all $m < n = p$.
2. (a) $a_n = \frac{a_{n-1}(1-n(n-1))}{n}$ *for any n;*
(b) *in particular if $n = p$ then $\lambda_p = 1 - p(p-1) \equiv 1 \bmod(p)$;*
3. (a) $A^{(p)} = \left\{ \{a_m = \frac{\lambda_m}{p^m}\}_{0 \leq m \leq (p-1)}, a_p = \frac{\lambda_{p-1}(1-p(p-1))}{p^{p+1}} \right\}$;
(b) *Denominators of $A^{(p)}$ are*

$$\begin{cases} p^m & for\ 0 \leq m \leq p-1 \\ p^{p+1} & for\ m = p \end{cases}. \tag{10}$$

Proof. 1. $s|(1-nr) \Longleftrightarrow r = n^{-1} (\bmod\ s)$ is unique because $\gcd(s, n) = 1$;
The special case $n = p$ is obvious from formula (5);
2. (a) is clear from formula (5);
(b) is evident;
3. is obvious from 2. (a) and (b).

□

Lemma 2. *If $n = p$ is an odd* **prime** *integer then*

$$b_p = 0;\ b_m \equiv (-1)^m (mod\ p)\ for\ 0 \leq m \leq p-1. \tag{11}$$

Proof. For the first equality see formula (7) above.
For the second congruence we get from formula (7):

$$b_m = (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{p}{k}. \tag{12}$$

But $\binom{p}{k}(mod\ p) \equiv 0$ excepting $k = 0$ and $k = p$ when it is $\equiv 1$. Thus here in (12) we need only $k = 0$ and in this first case $b_m \equiv (-1)^m (mod\ p)$ for all $0 \leq m \leq p-1$. □

Theorem 1. *An odd natural number n is prime iff.*

$$\text{Denominator}\left\langle \binom{n}{n^{-1}} \right\rangle = \text{Denominator} \sum_{m=0}^{n} \binom{n^{-1}}{m} K_m^{(n+m)}(m) = n^{n-1}. \tag{13}$$

Proof. 1. If $n = p$ (prime) then by f. (8) we have $\left\langle \binom{p}{p-1} \right\rangle = \langle A^{(p)} | B^{(p)} \rangle$ and recollecting item (3) of Lemma 1 and formula (10) we get $\left\langle \binom{p}{p-1} \right\rangle = \sum_{m=0}^{p-1} a_m b_m + a_p b_p = \sum_{m=0}^{p-1} \frac{\lambda_m}{p^m} b_m + \frac{\lambda_p}{p^{p+1}} b_p = \frac{Q}{p^{p+1}}$ where $Q - \lambda_p b_p = Q - \lambda_p \cdot 0 = Q = p^2(\lambda_0 p^{p-1} b_0 + \ldots + \lambda_{p-1} b_{p-1})$ hence $\left\langle \binom{p}{p-1} \right\rangle = \frac{Q}{p^{p-1}}$. This fraction is evidently irreducible because $\lambda_{p-1} \cdot b_{p-1} \equiv 1 (mod\ p)$ and so $\text{Denominator}\left\langle \binom{p}{p-1} \right\rangle = p^{p-1}$

2. Otherwise, if n is not prime and index s, $1 < s < n$ is such that $\gcd(n, s) > 1$ then n is not inversible modulo s (compare the proof of Lemma 1, item 1) and this s becomes an '*extramultiplier*' (besides n^{n-1}) in the denominator of $\left\langle \binom{n}{n-1} \right\rangle$ and hence this denominator cannot be equal to n^{n-1}.

□

3. Concuding Remark

At the present moment we consider the Neville's algorithm as the most convenient tool for evaluation of the $\text{Denominator}\left\langle \binom{n}{n-1} \right\rangle$ *(See theorem 1). The complexity of this algorithm can be estimated as* $O(n^2)$ [7].

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Investigation of the Periodic Planar Oscillations of a Two-Body System in an Elliptic Orbit Using the Polynomial Algebra Methods

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Abstract. Computer algebra methods are used to investigate the planar oscillations of a system of two bodies connected by a spherical hinge that moves along an elliptic orbit under the action of gravitational torque in the plane of the orbit. The two-body system motion on an elliptic orbit is described by the second order system of differential equations with the periodic coefficients. Applying the perturbation techniques the periodic solution of the equations of motion is constructed in the form of power series in a small parameter. Using the proposed approach it is shown that the motion of the two-body system is described by periodic oscillations in the plane of an elliptic orbit. All the relevant symbolic computations are performed with the help of computer algebra systems.

Introduction

This work presents the results of investigation the dynamics of a two-body system (satellite and stabilizer) connected by a spherical hinge that moves in gravitational field in the plane of an elliptical orbit using polynomial algebra methods. The dynamics of various schemes for satellite-stabilizer gravitational orientation systems on a circular orbit was discussed in many papers, some review of them can be found in papers [1, 2, 3].

Since the problem is very complicated, in the previous works we studied the equilibrium orientations of the system on a circular orbit only in the simplest cases when the spherical hinge is located at the intersection of the satellite and stabilizer principal central axis of inertia and in the case where the spherical hinge is positioned on the line of intersection between two planes formed by the principal central axes of inertia of the satellite and stabilizer [3, 4, 5, 6, 7]. The application of computer algebra makes it possible to find the solutions of this problem.

On a circular orbit, there are spatial oscillations of a system of two connected bodies at the vicinity of equilibria. In paper [8], the eigenoscillations of a system of two bodies were studied and the parameters of the system, optimal in terms of speed, were found for the transition of the system to equilibrium. A detailed study of the oscillations of a satellite (a rigid body) in the plane of an elliptical orbit and the conditions for their stability were carried out in [9].

The works devoted to the study of planar oscillations of a system of two coupled bodies on an elliptic orbit were carried out only for simple cases, when the centers of mass of the first and second bodies coincide [10], [11]. Here, we consider the planar oscillations of a two-body system on an elliptic orbit in case when the spherical hinge is located at the intersection of the first and second body principal central axis of inertia. Applying the perturbation techniques and appropriate symbolic computations with the help of computer algebra system Wolfram Mathematica [12], we construct the periodic solution in the form of a power series in a small parameter.

1. Equations of Motion

We consider the problem of two bodies connected by a spherical hinge that move on an elliptic orbit. To write the equations of motion of two-body system, we introduce the following right-handed Cartesian coordinate systems: $OXYZ$ is the orbital coordinate system, the OZ axis is directed along the radius vector connecting the Earth center of mass C and the center of mass O of the two-body system, the OX axis is directed along the linear velocity vector of the center of mass O, and the OY axis coincides with the normal to the orbital plane. The axes of coordinate systems $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$, are directed along the principal central axes of inertia of the first and the second body, respectively. The orientation of the coordinate system $O_ix_iy_iz_i$ with respect to the orbital coordinate system is determined by the aircraft angles α_i (pitch), β_i (yaw), and γ_i (roll) (see [3]).

Suppose that (a_i, b_i, c_i) are the coordinates of the spherical hinge P in the body coordinate system $Ox_iy_iz_i$, A_i, B_i, C_i are principal central moments of inertia; $M_1M_2/(M_1 + M_2) = M$; M_i is the mass of the ith body; ω is the angular velocity for the center of mass of the two-body system moving along an elliptic orbit. Then we use the expressions for kinetic energy of the system in the case when $b_1 = b_2 = c_1 = c_2 = 0$ and the coordinates of the spherical hinge P in the body coordinate systems are $(a_i, 0, 0)$ and when the motions of the two-body system are located in the plane of the elliptic orbit ($\alpha_1 \neq 0, \alpha_2 \neq 0, \beta_1 = \beta_1 = 0$, $\gamma_1 = \gamma_2 = 0$, $\dot{\alpha_1} = d\alpha_1/dt$, $\dot{\alpha_2} = d\alpha_2/dt$, where t is time) in the form [1]

$$\begin{aligned} T &= 1/2\big(B_1 + Ma_1^2\big)(\dot{\alpha_1} + \omega)^2 + 1/2\big(B_2 + Ma_2^2\big)(\dot{\alpha_2} + \omega)^2 \\ &- Ma_1a_2\cos(\alpha_1 - \alpha_2)(\dot{\alpha_1} + \omega)(\dot{\alpha_2} + \omega). \end{aligned} \tag{1}$$

The force function, which determines the effect of the Earth gravitational field on the system of two connected by a hinge bodies, is given by [1]

$$\begin{aligned} U &= -3\mu/(2\rho^3)\big((A_1-C_1)\sin^2\alpha_1+(A_2-C_2)\sin^2\alpha_2\big) \\ &+ 3/2M\mu/\rho^3\big(a_1\sin\alpha_1-a_2\sin\alpha_2\big)^2+M\mu/\rho^3a_1a_2\cos(\alpha_1-\alpha_2). \end{aligned} \quad (2)$$

Here ρ is a radial distance between the center of mass of the Earth C and center of mass of the system O; $\mu = fM_0$, where f is a gravitational constant, and M_0 is the mass of the Earth; $\omega = \frac{d\vartheta}{dt} = \omega_0(1+e\cos\vartheta)^2$; $\frac{\mu}{\rho^3} = \omega_0^2(1+e\cos\vartheta)^3$; ϑ is the true anomaly and e is the orbital eccentricity. On the circular orbit $\omega = \omega_0$, $\frac{\mu}{\rho^3} = \omega_0^2$, $\vartheta = \omega_0 t$.

By using the kinetic energy expression (1) and the expression (2) for the force function, the equations of motion for this system can be written as Lagrange equations of the second kind by applying symbolic differentiation in the Wolfram Mathematica system [12], [13]

$$\frac{d}{dt}\frac{\partial T}{\partial\dot\alpha_i}-\frac{\partial T}{\partial\alpha_i}-\frac{\partial U}{\partial\alpha_i}=0, \quad i=\overline{1,2}, \quad (3)$$

in the form of a system of second-order ordinary differential equations in variables α_1 and α_2 [1]

$$\begin{aligned} &(B_1+Ma_1^2)(\ddot\alpha_1+\dot\omega)-Ma_1a_2(\ddot\alpha_2+\dot\omega)\cos(\alpha_1-\alpha_2) \\ -\;&Ma_1a_2\big((\dot\alpha_2+\omega)^2-\mu/\rho^3\big)\sin(\alpha_1-\alpha_2) \\ +\;&3\mu/\rho^3\big((A_1-C_1-Ma_1^2)\sin\alpha_1+Ma_1a_2\sin\alpha_2\big)\cos\alpha_1=0, \\ -\;&Ma_1a_2(\ddot\alpha_1+\dot\omega)\cos(\alpha_1-\alpha_2)+(B_1+Ma_1^2)(\ddot\alpha_2+\dot\omega) \\ +\;&Ma_1a_2\big((\dot\alpha_1+\omega)^2-\mu/\rho^3\big)\sin(\alpha_1-\alpha_2) \\ +\;&3\mu/\rho^3\big((A_2-C_2-Ma_2^2)\sin\alpha_2+Ma_1a_2\sin\alpha_1\big)\cos\alpha_2=0, \end{aligned} \quad (4)$$

which determine the oscillations of the system in the plane of the elliptic orbit in the orbital coordinate system. In (4), the dot denotes differentiation with respect to time t. One can easily check that the system (4) has the stationary solution

$$\alpha_1=\alpha_2=0. \quad (5)$$

Our goal is to obtain the periodic solution of the equations of motion (4) in the form of a power series in a small parameter e ($e \ll 1$) in the neighborhood of the stationary solution (5) with the help of computer algebra system.

2. Periodic solutions

To perform the calculations we assume that the oscillations are small and replace the sine and cosine in (4) by their expansions in power series. Doing the substitution $dt = d\vartheta/(\omega_0(1+e\cos\vartheta)^2)$ in (4) we change the independent variable from t to ϑ and reduce the system to the form

$$
\begin{aligned}
&- \ (1+e\cos\vartheta)\alpha_2'' + 2e\alpha_2'\sin\vartheta + (B_1+Ma_1^2)/(Ma_1a_2)\big((1+e\cos\vartheta)\alpha_1{}''\\
&- \ 2e\alpha_1'\sin\vartheta\big) - e(1+e\cos\vartheta)(\alpha_2'+1)^2 + e\big(2\sin\vartheta(1-(B_1+Ma_1^2)/Ma_1a_2)\\
&+ \ (4+3\big((A_1-C_1)-Ma_1^2\big)/(Ma_1a_2))\big) = 0, \qquad (6)\\
&- \ (1+e\cos\vartheta)\alpha_1'' + 2e\alpha_1'\sin\vartheta + (B_2+Ma_2^2)/(Ma_1a_2)\big((1+e\cos\vartheta)\alpha_2{}''\\
&- \ 2e\alpha_1'\sin\vartheta\big) + e(1+e\cos\vartheta)(\alpha_1'+1)^2 + e\big(2\sin\vartheta(1-(B_2+Ma_2^2)/Ma_1a_2)\\
&+ \ (2+3\big((A_2-C_2)-Ma_2^2\big)/(Ma_1a_2))\big) = 0.
\end{aligned}
$$

The prime in (6) denotes differentiation with respect to ϑ. It is possible to check that a general solution of nonlinear system (6) cannot be found in analytic form. It is convenient for application of the perturbation techniques [14] and symbolic algorithms proposed in paper [15]. However, we can seek for an approximate solution in the form of power series in the small parameter e:

$$\alpha_i(\vartheta) = e\alpha_i{}^{(1)}(\vartheta) + e^2\alpha_i{}^{(2)}(\vartheta) + ..., \qquad (7)$$

Computation of unknown functions $\alpha_i(\vartheta)$ in (7) is done in accordance with the techniques proposed in [14] and [15] requires quite tedious symbolic computations. In this paper symbolic computations are performed using Wolfram Mathematica functions: Expand, TrigExpand, Series, Normal, Replace, DSolve, NDSolve.

Substituting (7) into (6) and collecting coefficients of equal powers of e, we obtain the set of systems of linear differential equations which can be solved in succession. For example, using in (7) only the first linear elements we obtain the corresponding periodic solutions in the form

$$\alpha_1^{(1)}(\vartheta) = \bar{a}_1\sin(\vartheta) + \bar{b}_1\cos(\vartheta), \quad \alpha_2^{(1)}(\vartheta) = \bar{a}_2\sin(\vartheta) + \bar{b}_2\cos(\vartheta), \qquad (8)$$

where the coefficients $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2$ can be defined from the linear algebraic system. The amplitudes of the oscillations of the first and the second bodies have the expressions

$$R_1^2 = (\bar{a}_1^2+\bar{b}_1^2)e^2 = 4\frac{e^2b^2}{d^2}, \quad R_2^2 = (\bar{a}_2^2+\bar{b}_2^2)e^2 = 4\frac{e^2\bar{b}^2}{d^2}, \qquad (9)$$

where

$$
\begin{aligned}
b &= (B_1+Ma_1(a_1-a2))(3(A_2-C_2)-B_2) - 4Ma_2(a_1B_2+a_2B_1),\\
\bar{b} &= (B_2+Ma_2(a_1-a2))(3(A_1-C_1)-B_1) - 4Ma_1(a_1B_2+a_2B_1), \qquad (10)\\
d &= (3(A_1-C_1)-B_1)(3(A_2-C_2)-B_2) - 4Ma_1^2(3(A_2-C_2)-B_2)\\
&- \ 4Ma_2^2(3(A_1-C_1)-B_1).
\end{aligned}
$$

In the present work, we have considered the first approximation of the planar oscillations of a system of two bodies connected by a spherical hinge that moves along an elliptic orbit. We have found the expressions of the periodic motion of the system in the linear approximation. All the relevant computations in this work are performed with the computer algebra system Wolfram Mathematica. At the next

step we will do the quadratic and cubic approximation of the periodic solutions which have very cumbersome expressions.

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The zeros of random sections of real vector bundles

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Abstract. We define integral geometric analogues of the Chern classes for real vector bundle on a smooth real variety. More precisely, we define the Chern densities of a real bundle. These densities are analogues of the Chern forms of a complex vector bundle and inherit some of their properties.

The zeros of random systems of functions

We begin with a theorem on the number of common zeros of random systems of functions from [1]. Let V be a finite-dimensional space of smooth functions on an n-dimensional differentiable manifold X. Consider a random system of equations

$$f_1 = \ldots = f_n = 0,\ 0 \neq f_i \in V \tag{1}$$

Denote by $N(f_1, \ldots, f_n)$ the number of solutions of the system (1). We define the randomness of the system using a certain scalar product in V as follows: we consider the functions f_i as independent random vectors in V with respect to the Gaussian measure chosen in V according to the chosen scalar product. The situation with a more general choice of probability distribution in V is described in [2]. Let $\mathfrak{M}(V)$ denote the expected value of the random variable $N(f_1, \ldots, f_n)$. Next, for the calculation of $\mathfrak{M}(V)$, we will need the notion of a Banach set on X, as well as the notion of a volume of the Banach set.

Definiton 1. Let T^*X be a cotangent bundle of X, and $\mathcal{E}(x)$ be a convex centrally symmetric compact set in the cotangent space T^*_xX of X at the point x. The collection $\mathcal{E} = \{\mathcal{E}(x) \subset T^*_xX \,|\, x \in X\}$ is called a *Banach set* in X.

Definiton 2. Consider the domain $\bigcup_{x\in X} \mathcal{E}(x) \subset T^*(X)$. It's volume relative to the standard symplectic structure in T^*X is called the *volume of Banach set* and is denoted by $\mathrm{vol}(\mathcal{E})$.

For $x \in X$ let's define the linear functional $\theta(x)$ on V as $\theta(x)(f) = f(x)$. Next we assume that $\forall x \in X,\ \exists f \in V\colon\ f(x) \neq 0$. That's why the set $\theta(X)$ in V^* does not contain 0.

Definiton 3. Let's define the mapping $\Theta\colon X \to V^*$ as $\Theta(x) = \theta(x)/\sqrt{\langle\theta(x), \theta(x)\rangle_*}$, where $\langle *, *\rangle_*$ is the scalar product in the space V^* associated with the scalar product $\langle *, *\rangle$ in V. Let $d\Theta_x\colon T_xX \to V^*$ be a differential of Θ at the point x. Denote by $d^*\Theta_x\colon V \to T_x^*X$ the adjoint linear operator, and define the Banach set $\mathcal{E}_V$ by $\mathcal{E}_V(x) = d^*\Theta_x(B)$, where B is the unit ball in V centered at the origin. The compact set $\mathcal{E}_V(x)$ is an ellipsoid. We note that in a more general context discussed in [Ka1], arbitrary Banach sets on X can arise.

Theorem 1. $\mathfrak{M}(V) = n!/(2\pi)^n \operatorname{vol}(\mathcal{E}_V)$

Example 1. Let X be the unit circle S^1, V_m the space of trigonometric polynomials $f(\theta) = \sum_{k\leq m} a_k \cos(k\theta) + b_k \sin(k\theta)$ of degree m. Then (see [6])

$$\mathfrak{M}(V_m) = \sqrt{\frac{m(m+1)}{3}}$$

For trigonometric polynomials in many variables see [7].

Now let's state a similar theorem in the case where we consider n spaces V_i and equations $f_1 = \ldots = f_n = 0$, where $f_i \in V_i$. For this, we will need the concept of the mixed volume of Banach sets. Using Minkowski sum and homotheties, we can form linear combinations of convex sets with non-negative coefficients. The linear combination of Banach sets is defined by

$$(\sum_i \lambda_i \mathcal{E}_i)(x) = \sum_i \lambda_i \mathcal{E}_i(x).$$

For n Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ the volume of $\lambda_1\mathcal{E}_1 + \ldots + \lambda_n\mathcal{E}_n$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_n$. Its coefficient at $\lambda_1 \cdot \ldots \cdot \lambda_n$ divided by $n!$ is called the mixed volume of Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$. The mixed volume of Banach sets $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is denoted by $\operatorname{vol}(\mathcal{E}_1, \ldots, \mathcal{E}_n)$.

Theorem 2. *Let $\mathfrak{M}(V_1, \ldots, V_n)$ denote the expectation of the random variable $N(f_1, \ldots, f_n)$. Then it holds that*

$$\mathfrak{M}(V_1, \ldots, V_n) = \frac{n!}{(2\pi)^n} \operatorname{vol}(\mathcal{E}_{V_1}, \ldots, \mathcal{E}_{V_n})$$

The ring of Banach sets

Next we need a concept of the ring of Banach sets. It arises as an analogue of the well-known concept of a ring of convex bodies, first defined in [3]. There are several different versions of this concept. Here we construct an analogue of the definition from [4]. We call the formal difference $\mathcal{E} - \mathcal{B}$ of Banach sets the *virtual Banach set*. Virtual Banach sets form a vector space, where multiplication by negative numbers is defined by $(-1)\cdot(\mathcal{E} - \mathcal{B}) = \mathcal{B} - \mathcal{E}$.

The following notations are used below
• $S = \bigoplus_{0 \le i} S_i$ – the graded symmetric algebra of the space of virtual Banach sets on the manifold X
• I – the linear functional on the space S defined by 1) $I_{S_k} = 0$ for $k \neq n$, and 2) $I(\mathcal{E}_1 \cdot \ldots \cdot \mathcal{E}_n) = \mathrm{vol}(\mathcal{E}_1, \ldots, \mathcal{E}_n)$
• $L(x, y) = I(x \cdot y)$ – the symmetric bilinear form on the vector space S
• J – the kernel of the form L.

Lemma 1. *J is a homogeneous ideal of the graded ring S.*

We will call ring $\mathfrak{S} = S/J$ the *ring of virtual Banach sets.*

Corollary 1. *The following statements hold:*
(i) $\mathfrak{S}_0 = \mathbb{R}$
(ii) $\dim \mathfrak{S}_n = 1$
(iii) *The graded ring $\mathfrak{S}$ is generated by elements of degree* 1
(iv) *The mappings $\mathfrak{S}_p \times \mathfrak{S}_{n-p} \to \mathbb{R}$, defined as $(\eta, \xi) \mapsto L(\eta, \xi)$, are non-degenerate pairings.*

Next for n virtual Banach sets $\mathcal{B}_1, \ldots, \mathcal{B}_n$, we use the notation

$$\mathrm{vol}(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_n) = I(\mathcal{B}_1 \cdot \ldots \cdot \mathcal{B}_n) = \mathrm{vol}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$$

Zeros of random sections

Transitioning to zeros of random sections of vector bundles, without formulating precise theorems, we will first briefly describe the situation in the case when considering zeros of sections of an n-dimensional vector bundle $\mathcal{F}$ on X. Just as in the case of functions we consider a finite dimensional space V of smooth sections of $\mathcal{F}$. Here we denote by $\mathfrak{M}(V; U)$ the expectation of the number of zeros of random section $s \in V$ contained in the open set $U \subset X$. By $\mathrm{Res}_U \mathcal{B}$ we denote the constraint of $\mathcal{B} \in \mathfrak{S}$ on the subvariety U.

Theorem 3. *There exists the unique element $\mathcal{B} \in \mathfrak{S}_n$, such that for any $U \subset X$*

$$\mathfrak{M}(V; U) = \frac{n!}{(2\pi)^n} \mathrm{vol}(\mathrm{Res}_U \mathcal{B})$$

Further results can be approximately described as follows. We associate to an element $\mathfrak{s}$ of degree k of the ring of Banach sets $\mathfrak{S}$ a certain k-density $d_k(\mathfrak{s})$ on X and interpret the ring $\mathfrak{S}$ as a ring of these densities. Such densities serve as analogues of Chern forms, representing Chern classes of complex vector bundles, and inherit some properties of Chern forms.

In conclusion, let us define the density $d_k(\mathfrak{s})$. An alternative construction of multiplication in the density ring is given in [1]; see also [5].

Definiton 4. Let H be the subspace of T_xX generated by tangent vectors $\xi_1, \dots, \xi_k$, $H^\perp \subset T_x^*X$ the orthogonal complement to H, and $\pi_H : T_x^*X \to T_x^*X/H^\perp$ the projection map. The volume form on $T_x^*X/H^\perp$ is defined by $\omega(x) = \xi_1 \wedge \dots \wedge \xi_m$. Let $\mathcal{B}_1, \dots, \mathcal{B}_k$ be Banach sets on X. Then $d_k(\mathcal{B}_1 \cdot \dots \cdot \mathcal{B}_k)(\xi_1, \dots, \xi_m)$ is the mixed k-dimensional volume of convex k-dimensional sets $\pi_H\mathcal{B}_1(x), \dots, \pi_H\mathcal{B}_k(x)$ in the sense of the volume form $\omega(x)$.

The following statement is an analogue of the BKK formula for Banach sets and for densities d_i.

Theorem 4. *For any Banach sets $\mathcal{B}_1, \dots, \mathcal{B}_k$ the equality*

$$d_1(\mathcal{B}_1) \cdot \dots \cdot d_1(\mathcal{B}_k) = k!\, d_k(\mathcal{B}_1 \cdot \dots \cdot \mathcal{B}_k)$$

holds.

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On Multicomponent Continued Fraction Expansions of Hypernumbers of Certain Classes

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Abstract. A new class of multicomponent continued fractions is investigated. It is assumed that on the set of approximated elements some algebraic structures can be defined that allow the elements of these original sets to be considered as hypernumbers. This makes it possible to significantly simplify the research and transfer many well-known properties of classical continued fractions to the case under consideration. In this paper, conditions for approximating elements of the original sets by multicomponent continued fractions and estimating the rate of convergence are obtained.

Introduction

The theory of representing of real numbers by classical continued fractions is given in many works. There are known results on continued fraction expansions for complex numbers, see for example [1]. There are also some results on generalized continued fractions. Many needs of applied sciences lead to the problem of approximation of multidimensional real parameters by a set of rational numbers with the same denominators. To solve such problems, multicomponent continuous fractions can be used. In this paper which continues papers [2], [3] and others, a new class of multicomponent continued fractions is investigated. Multicomponent continued fractions are interpreted as multicomponent scalars. They are elements of some algebra which is a set of elements with two arithmetic operations. Quaternions are considered as a basic example of such algebra.

1. Basic concepts of the theory of continued fractions

Let $a_0, a_1, \dots, a_n, \dots$ be some sequence of characters. A finite continued fraction is given in the form

$$a_0 + (a_1 + \dots + (a_{n-1} + (a_n)^{-1})^{-1} \dots)^{-1}, \tag{1}$$

which can also be written as an ordinary fraction.

An infinite continued fraction is given in the form

$$a_0 + (a_1 + \cdots + (a_{n-1} + (a_n + \dots)^{-1})^{-1} \dots)^{-1}. \tag{2}$$

In the sequel, we intend to consider continued fraction expansions for a certain set of scalars. Note that commutativity is not assumed for multiplication. Further, we assume that in the set of scalars there is a certain lattice, the elements of which are «integers». Thus, the numbers a_n are elements of the lattice $\mathfrak{G}$. We also assume that the set of scalars is some Euclidean space, i.e. a norm of an element is determined. Below, we give estimates of the rate of convergence of convergent fractions to decomposed elements.

1.1. Euclidean Algorithm and Iteration sequences

The continued fraction expansion of z can be obtained by applying of the Euclidean algorithm. For $n = 0, 1, 2, \dots$, a recurrent sequence α_n is constructed, where $\alpha_0 = z$. For any n, $a_n = [\alpha_n]$ is the whole part of the element α_n. It may depend on the method of rounding and on number n. Next, the fractional part is determined by formula $\langle \alpha_n \rangle = \alpha_n - a_n$.

The previous formula can be rewritten as

$$a_n = \alpha_n - (\alpha_{n+1})^{-1} \in \mathfrak{G}, \quad n = 0, 1, 2, \dots. \tag{3}$$

The sequence α_n is called an iteration sequence.

1.2. Convergent fractions

A finite continued fraction of the form (1) can be written as a common fraction. It is called a suitable fraction for the number $\alpha \in \mathbb{T}$, where $\mathbb{T}$ is some algebra. Since we do not assume that multiplication is commutative, there are two possible representations for a convergent fraction

$$r'_n = (q'_n)^{-1} p'_n, \quad r''_n = p''_n (q''_n)^{-1}, \tag{4}$$

which give the same value, i.e.

$$r'_n = r''_n =: r_n, \quad n = 0, 1, 2, \dots.$$

The (finite or infinite) sequence r_n, $n = 1, 2, \dots$ is called a sequence of convergent fractions.

It can be shown that the quantities p'_n and q'_n, as well as p''_n and q''_n, satisfy the Euler equations where, respectively, the left and right pairs of equations for $n \geq 1$ have the following forms

$$p'_{-1} = 1,\ p'_0 = a_0, \quad p'_{n+1} = a_{n+1} p'_n + p'_{n-1}, \tag{5}$$

$$q'_{-1} = 0,\ q'_0 = 1, \quad q'_{n+1} = a_{n+1} q'_n + q'_{n-1}, \tag{6}$$

and

$$p''_{-1} = 1,\ p''_0 = a_0, \quad p''_{n+1} = p''_n a_{n+1} + p''_{n-1}, \tag{7}$$

$$q''_{-1} = 0,\ q''_0 = 1, \quad q''_{n+1} = q''_n a_{n+1} + q''_{n-1}. \tag{8}$$

1.3. Auxiliary statements

Next, we show that the pairs (p'_n, p''_n) and (q'_n, q''_n) change consistently. Thus, the following statements are true.

Lemma 1. *The relations are valid:*

$$V_{n-1,n} := p'_{n-1}q''_n - q'_{n-1}p''_n = (-1)^n, \tag{9}$$

$$V_{n.n-1} := p'_n q''_{n-1} - q'_n p''_{n-1} = (-1)^n. \tag{10}$$

Lemma 2. *The relations are valid:*

$$P'_n = P''_n, \quad Q'_n = Q''_n, \tag{11}$$

where $P'_n = p'_n(p'_{n-1})^{-1}$ *and* $Q'_n = q'_n(q'_{n-1})^{-1}$.

Let $q_n^2 := |q'_n|^2 = |q''_n|^2$. Then $|q_n| := \sqrt{q_n^2}$.

2. The main theorems on continued fractions

By formulas (9) and (10), we have the following relations:

$$zq''_n - p''_n = (-1)^n(\alpha_{n+1}q''_n + q''_{n-1})^{-1} = (-1)^n(q''_n)^{-1}(\alpha_{n+1} + (Q''_n)^{-1}), \tag{12}$$

$$q'_n z - p'_n = (-1)^n(q'_n\alpha_{n+1} + q'_{n-1})^{-1} = (-1)^n(\alpha_{n+1} + (Q'_n)^{-1})(q'_n)^{-1}. \tag{13}$$

Theorem 1. *For residuals, the following relations are valid*

$$|zq''_n - p''_n| = |q'_n z - p'_n| = \frac{1}{|\alpha_{n+1} + (Q_n)^{-1}|}\frac{1}{|q_n|}, \tag{14}$$

$$|z - r_n| = \frac{1}{|\alpha_{n+1} + (Q_n)^{-1}|}\frac{1}{|q_n|^2}. \tag{15}$$

2.1. Some special conditions for the convergence of continued fractions

In formulas (14) and (15), the inequalities $|\alpha_{n+1}| > 1$ and $|(Q_n)^{-1}| < 1$ are true for all n. However, this does not guarantee that the denominators of the fractions are separated from zero. To ensure this condition, we introduce additional enhanced restrictions:

$$|\alpha_{n+1}| \geq \alpha > 1\,\&\,|(Q_n)^{-1}| < 1, \tag{16}$$

or

$$|(Q_n)^{-1}| \leq 1 - c^{-1} < 1\,\&\,|\alpha_{n+1}| > 1. \tag{17}$$

Remark 1. *The inequalities* $\alpha_n \geq \underline{\alpha} > 1$ *from formula* (17) *can be interpreted as conditions of strong non-degeneracy of a iteration sequence. In some cases, these may be provided by selecting of a rounding function. Inequalities* $|Q_n^{-1}| \leq c^{-1} < 1$ *from formula* (17) *is equivalent to the inequalities* $|q_n| > c|q_{n-1}|$. *These relations are strong conditions for exponential growth of denominators of a convergent fraction.*

By the triangle inequality, for $C = \min\{\underline{\alpha} - 1, 1 - c^{-1}\}$, $c > 1$, the following condition is satisfied:

$$0 < C \leq |\alpha_{n+1} + (Q_n)^{-1}|, \quad n \geq 0. \tag{18}$$

Theorem 2. *Let condition* (17) *be satisfied. Then the following estimates for the residuals are valid:*

$$|q_n' z - p_n'| = |zq_n'' - p_n''| \leq \frac{1}{C|q_n|^2}, \tag{19}$$
$$|z - r_n| \leq \frac{1}{C|q_n|^2}.$$

Conclusion

The results obtained in this work were used to solve some control theory problems related to determining switching instants at discrete times, [3].

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On the parameterizations of the special unitary group SU(4) and related double cosets

Arsen Khvedelidze, Dimitar Mladenov and Astghik Torosyan

A nice structure of Lie groups due to their dual nature of being a group and a manifold simultaneously reveals in a rich variety of different types of coordinates on the group manifold. Among them traditionally used parameterizations in mathematics and theoretical physics are those related to the case when coordinates on the group manifold provide the projected coordinates on the corresponding quotients or bi-quotients obtained as a result of group action. There is a big amount of general results on the topological properties [1] and explicit constructions of different parameterization of compact Lie groups well adapted to either cosets or the double cosets description. Among them one can point out the classical results based on the Cartan involution [2], their generalization associated to a pair of the non-commutative involutions [3], and classification of all possible Kobayashi triples of the form $(\mathrm{U(n)},\ \mathrm{U(p)}\times\ \mathrm{U(q)}\,,\ \mathrm{U(n_1)}\times\mathrm{U(n_2)}\times\cdots\times\mathrm{U(n_l)})$, with $\mathrm{n}=\mathrm{p}+\mathrm{q}=\mathrm{n}_1+\mathrm{n}_2+\cdots+\mathrm{n}_\mathrm{l}$ [5].

In the present report we are not aiming to discuss the general theory of double coset description, but our less ambitious goal is to tackle the issue of parameterizations for the special unitary group SU(4) adapted to the description of double coset related to the triplet $(\mathrm{SU(4)},\ \mathrm{SU(2)}\times\mathrm{SU(2)},\ \mathrm{T}^3)$ comprised of the maximal torus T^3 and the subgroup $\mathrm{SU(2)}\times\mathrm{SU(2)}\subset\mathrm{SU(4)}$. Based on the suggested "coordinatization", the Riemannian bi-invariant metric on group manifold is derived and the corresponding volume 15-form is calculated.

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Finite groups and quantum mechanics: evolution and decomposition of quantum systems

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Abstract. Quantum mechanics is based on two main points: (1) the assumption that the evolution of a closed system is described by unitary transformations in Hilbert space, and (2) the idea of observation, formalized in the concept of an observable and canonical commutation relations between pairs of conjugate observables. We call such conjugate pairs complementary, since they form the basis of Borh's complementarity principle.

The combined use of complementary observables allows us to obtain the maximum available information about the state of a quantum system. Complementary observables are related to such issues as the uncertainty principle, the principle of least action, the path integral formulation of quantum mechanics, mutually unbiased bases etc.

Replacing a continuous unitary group with a finite permutation group in the quantum formalism [1–4] allows us to reduce the description of evolution to the group of cyclic permutations $\mathbb{Z}_N$. The product of $\mathbb{Z}_N$ and its Pontryagin dual, $\widetilde{\mathbb{Z}}_N$, has a nontrivial projective representation, which allows to describe quantum interferences taking into account phase differences.

Thus, by starting with just a cyclic permutation, we obtain a complete finite version of quantum mechanics, including unitary evolution and the complementarity principle. Finite structures that stem from cyclic permutations are naturally found in various fields, including quantum computer science and signal processing. These finite structures were first discovered, within the framework of continuous quantum mechanics, by Weyl when he constructed an analogue of the Heisenberg canonical commutation relations suitable for finite-dimensional Hilbert spaces.

The generator of the *regular representation* of $\mathbb{Z}_N$ on the N-dimensional Hilbert space $\mathcal{H}_N$ is the cyclic permutation matrix

$$X = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} .$$

The matrix X is related to the basis $B_X = \{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$, called *the position basis*. This basis has other names, such as *ontic* (G. 't Hooft) or *initial*, or *computational* (quantum computer science).

In this basis, the position operator has the diagonal form $\widehat{x} = \sum_{x=0}^{N-1} x\,|x\rangle\langle x|$. If $\gcd(v, N) = 1$, then the matrix $X_v = X^v$ defines a cyclic evolution on the eigenvalues of the position operator $\widehat{x}_t = X_v^t \widehat{x}_0 X_v^{-1}$. In the components we have

$$x_t = x_0 + vt \mod N.$$

Therefore, the parameter v can be interpreted as "velocity".

The generator of the Pontryagin dual group $\widetilde{\mathbb{Z}}_N$ is

$$Z = \widetilde{X} = FXF^* = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{N-1} \end{pmatrix},$$

where F is the Fourier transform and $\omega = \mathrm{e}^{2\pi\mathbf{i}/N}$ is the Nth base root of unity. The basis $B_Z = \left\{\left|\widetilde{0}\right\rangle, \left|\widetilde{1}\right\rangle, \ldots, \left|\widetilde{N-1}\right\rangle\right\}$ formed by the eigenvectors of Z is called *the momentum basis*.

The bases B_X and B_Z are interconnected by the Fourier transform, and they are *mutually unbiased*.

A direct calculation reveals that $XZ = \omega ZX$. This is precisely *the Weyl canonical commutation relation*. The operators X and Z generate a non-trivial *projective representation* of the group $\mathbb{Z}_N \times \widetilde{\mathbb{Z}}_N \cong \mathbb{Z}_N \times \mathbb{Z}_N$ on the space $\mathcal{H}_N$.

The main constructs derived from the matrices X and Z are:

- *Weyl–Heisenberg group*

  $$\mathrm{H}(N) = \left\{\tau^k X^v Z^m\right\},$$

  where $\tau = -\omega^{1/2} = -\mathrm{e}^{\pi\mathbf{i}/N}$, $v, m \in \mathbb{Z}_N$, $k \in \mathbb{Z}_{\overline{N}}$, $\overline{N} = \begin{cases} N, & N \text{ is odd}, \\ 2N, & N \text{ is even}. \end{cases}$
- *Finite position-momentum phase space* T^2 is a 2D discrete torus of size $N \times N$.
- *Symplectic group* $\mathrm{Sp}(2, \mathbb{Z}_N)$ is the group of symplectic transformations of the phase space T^2.
- *Clifford group* $\mathrm{C}\ell(N) \cong \mathrm{H}(N) \rtimes \mathrm{Sp}(2, \mathbb{Z}_N)$ is the normalizer of $\mathrm{H}(N)$ in $\mathrm{U}(N)$. It is the group of all symmetries of the group $\mathrm{H}(N)$: $\mathrm{C}\ell(N) \cong \mathrm{Aut}(\mathrm{H}(N))$.

The properties of the operators X and Z and the constructions derived from them, as well as the possibility of decomposing the corresponding quantum system into subsystems, are determined by the structure of the group $\mathbb{Z}_N$. A decomposition of a cyclic group into smaller groups has the form

$$\mathbb{Z}_N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m}, \tag{1}$$

where $N = n_1 \cdot n_2 \cdot \ldots \cdot n_m,\ \gcd(n_i, n_j) = 1$. The canonical decomposition takes the form $\mathbb{Z}_N \cong \mathbb{Z}_{p_1^{\ell_1}} \times \cdots \times \mathbb{Z}_{p_m^{\ell_m}}$, where $N = p_1^{\ell_1} \cdots p_m^{\ell_m}$ is the prime factorization.

Mappings that provide isomorphism (1), which can be considered as an isomorphism of rings, can be calculated using the Chinese remainder theorem. Namely, for $k \in \mathbb{Z}_N$ we have the mappings

$$k \mapsto (r_1, \ldots, r_m)\,,$$
$$(r_1, \ldots, r_m) \mapsto k = \sum_{i=1}^{m} r_i N_i^{-1} N_i \mod N,$$

where $r_i = k \mod n_i \in \mathbb{Z}_{n_i}$, $N_i = N/n_i \in \mathbb{Z}_N$, $N_i^{-1} \in \mathbb{Z}_{n_i}$ is the *multiplicative inverse* of N_i within $\mathbb{Z}_{n_i}$.

Dual mappings, which are more useful for many problems, have the form

$$k \mapsto (k_1, \ldots, k_m)\,,$$
$$(k_1, \ldots, k_m) \mapsto k = \sum_{i=1}^{m} k_i N_i \mod N, \tag{2}$$

where $k_i = r_i N_i^{-1} \in \mathbb{Z}_{n_i}$. For example, the equation

$$\frac{k}{N} = \sum_i \frac{k_i}{n_i} \mod 1, \tag{3}$$

which follows from (2), helps us to understand the additivity of the energy in a composite quantum system: in representing the frequency of a system as a sum of frequencies of subsystems, frequencies can be interpreted as corresponding energy levels in accordance with the Planck relation, which states the equivalence of energy and frequency.

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On coefficients of the Berenstein-Kazhdan decoration functions for classical groups

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Abstract. For type A_n, $SL_{n+1}(\mathbb{C})$, all coefficients of the BK decoration function are equal one, because all weights are minuscule. This is not the case for other types. For types BCD, we prove that the coefficients are powers of two (zero power is one). We propose an algorithm which computes these coefficients. The complexity of the algorithm is comparable with the complexity of writing the BK decoration function.

1. i-trails and generalized minors

1.1. Notation

Let G be a simply connected connected simple algebraic group or rank r, B, $B^- \subset G$ its Borel subgroups, $T := B \cap B^-$ the maximal torus, $W = \mathrm{Norm}_G(T)/T$ Weyl group, U, U^- be unipotent radicals of B, B^-, $A = (a_{i,j})$ the Cartan matrix of G with an index set $I = \{1, 2, \cdots, n\}$. We define $\mathfrak{g} = \mathrm{Lie}(G)$ with Chevalley generators e_i, f_i, h_i $(i \in I)$, a Cartan subalgebra $\mathfrak{h}$ and the canonical pairing $\langle , \rangle$ between $\mathfrak{h}$ and $\mathfrak{h}^*$. Let Λ_i denote the i-th fundamental weight, that is, $\langle h_j, \Lambda_i \rangle = \delta_{j,i}$ and $P = \oplus_{i \in I} \mathbb{Z}\Lambda_i$ be the weight lattice, $P_+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0}\Lambda_i$ the positive weight lattice, $P^* = \oplus_{i \in I} \mathbb{Z}h_i$ the dual weight lattice, $\{\alpha_i\}$ $(i \in I)$ the set of simple roots. For each $\lambda \in P_+$, let $V(\lambda)$ denote the finite dimensional irreducible $\mathfrak{g}$-module with highest weight λ. Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra with generators E_i, F_i $(i \in I)$ and K_λ $(\lambda \in P)$ and $U_q(\mathfrak{g})^- \subset U_q(\mathfrak{g})$ be the subalgebra generated by $\{F_i\}_{i \in I}$. It is well-known that $U_q(\mathfrak{g})^-$ has the crystal base $(L(\infty), B(\infty))$. For two integers l, $m \in \mathbb{Z}$ such that $l \leq m$, one sets $[l, m] := \{l, l+1, \cdots, m-1, m\}$.

1.2. A birational map

Let us recall a definition of $B^-_{w_0}$, where w_0 is the longest element in W, and an open embedding $(\mathbb{C}^\times)^N \hookrightarrow B^-_{w_0}$ associated with a reduced word $\mathbf{i} = (i_1, i_2, \cdots, i_N)$ of w_0.

First, for $i \in I$ and $t \in \mathbb{C}$, we put

$$x_i(t) := \exp(te_i),\ y_i(t) := \exp(tf_i) \in G.$$

There exists the canonical embedding $\phi_i : SL_2(\mathbb{C}) \to G$ satisfying

$$x_i(t) = \phi_i\left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right), \quad y_i(t) = \phi_i\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right).$$

Using the embedding, one puts

$$t^{h_i} := \phi_i\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \in T$$

and

$$x_{-i}(t) := y_i(t)t^{-h_i} = \phi_i\left(\begin{pmatrix} t^{-1} & 0 \\ 1 & t \end{pmatrix}\right) \in G$$

for $i \in I$ and $t \in \mathbb{C}^\times$. One can construct a representative of a simple reflection $s_i \in W = \mathrm{Norm}_G(T)/T$ by

$$\overline{s_i} := x_i(-1)y_i(1)x_i(-1) \in \mathrm{Norm}_G(T)$$

for each $i \in I$. For $w \in W$, one can define a representative $\overline{w} \in \mathrm{Norm}_G(T)$ by the rule

$$\overline{uv} = \overline{u} \cdot \overline{v} \quad \text{ if } l(uv) = l(u) + l(v),$$

where l is the length function on W. We define a variety $B^-_{w_0} := B^- \cap U\overline{w_0}U$. One defines a map $\theta^-_{\mathbf{i}} : (\mathbb{C}^\times)^N \to G$ associated with a reduced word $\mathbf{i} = (i_1, \cdots, i_N)$ of $w_0 \in W$ by

$$\theta^-_{\mathbf{i}}(t_1, \cdots, t_N) := x_{-i_1}(t_1) \cdots x_{-i_N}(t_N). \tag{1.1}$$

Proposition 1.1 ([2]). *The map $\theta^-_{\mathbf{i}}$ is an open embedding from $(\mathbb{C}^\times)^N$ to $B^-_{w_0}$.*

1.3. Generalized minors and i-trails

Let $G_0 := U^-TU \subset G$ denote the open subset whose elements $x \in G_0$ are uniquely decomposed as $x = [x]_-[x]_0[x]_+$ with some $[x]_- \in U^-$, $[x]_0 \in T$ and $[x]_+ \in U$.

Definition 1.2 ([4]). For $u, v \in W$ and $i \in I$, the generalized minor $\Delta_{u\Lambda_i, v\Lambda_i}$ is defined as the regular function on G such that

$$\Delta_{u\Lambda_i, v\Lambda_i}(x) = ([\overline{u}^{-1}x\overline{v}]_0)^{\Lambda_i}$$

for any $x \in \overline{u}G_0\overline{v}^{-1}$. Here, for $t \in \mathbb{C}^\times$ and $j \in I$, we define $(t^{h_j})^{\Lambda_i} = (t^{\Lambda_i(h_j)})$ and extend it to the group homomorphism $T \to \mathbb{C}^\times$.

For calculations of generalized minors, one can use **i**-*trails* [2]. Here in this subsection, we take $\mathbf{i} = (i_1, \cdots, i_l)$ as a sequence of indices from I. Let us review pre-**i**-trails and **i**-trails.

Definition 1.3. For a finite dimensional representation V of $\mathfrak{g}$, two weights γ, δ of V and a sequence $\mathbf{i} = (i_1, \cdots, i_l)$ of indices from I, a sequence $\pi = (\gamma = \gamma_0, \gamma_1, \cdots, \gamma_l = \delta)$ is said to be a *pre*-**i**-*trail* from γ to δ if $\gamma_1, \cdots, \gamma_{l-1} \in P$ and for $k \in [1, l]$, it holds $\gamma_{k-1} - \gamma_k = c_k\alpha_{i_k}$ with some nonnegative integer c_k.

We can easily check that for $k \in [1,l]$, it holds

$$c_k = \frac{\gamma_{k-1} - \gamma_k}{2}(h_{i_k}). \tag{1.2}$$

Definition 1.4 ([2]). We consider the setting of Definition 1.3. If a pre-**i**-trail π from γ to δ satisfies the condition

- $e_{i_1}^{c_1} e_{i_2}^{c_2} \cdots e_{i_l}^{c_l}$ is a non-zero linear map from V_δ to V_γ,

then π is said to be an **i**-*trail* from γ to δ, where $V = \oplus_\mu V_\mu$ is the weight decomposition of V.

For a pre-**i**-trail $\pi = (\gamma_0, \gamma_1, \cdots, \gamma_l)$ and $k \in [1,l]$, we put

$$d_k(\pi) := \frac{\gamma_{k-1} + \gamma_k}{2}(h_{i_k}). \tag{1.3}$$

One obtains $d_k(\pi) = c_k + \gamma_k(h_{i_k}) \in \mathbb{Z}$ by (1.2). If $\gamma_{k-1} = s_{i_k}\gamma_k$ then $d_k(\pi) = 0$.

Lemma 1.5 ([8]). *Let γ, δ be weights of a finite dimensional representation V of $\mathfrak{g}$. Let $\mathbf{i} = (i_1, \cdots, i_l)$ be a sequence of indices from I and $\pi = (\gamma_0, \gamma_1, \cdots, \gamma_l)$, $\pi' = (\gamma'_0, \gamma'_1, \cdots, \gamma'_l)$ be two pre-**i**-trails from γ to δ. If $d_k(\pi) = d_k(\pi')$ for all $k \in [1,l]$ then $\pi = \pi'$.*

For a sequence $\mathbf{i} = (i_1, \cdots, i_l)$ of indices from I and $t_1, \cdots, t_l \in \mathbb{C}^\times$, just as in (1.1), we set

$$\theta_{\mathbf{i}}^-(t_1, \cdots, t_l) := x_{-i_1}(t_1) \cdots x_{-i_l}(t_l) \in G.$$

Then the following theorem holds:

Theorem 1.6 ([2]). *For $u, v \in W$ and $i \in I$, it holds*

$$\Delta_{u\Lambda_i, v\Lambda_i}(\theta_{\mathbf{i}}^-(t_1, \cdots, t_l)) = \sum_\pi C_\pi t_1^{d_1(\pi)} \cdots t_l^{d_l(\pi)},$$

*where C_π is a positive integer and π runs over all **i**-trails from $-u\Lambda_i$ to $-v\Lambda_i$ in $V(-w_0\Lambda_i)$.*

By this theorem and Lemma 1.5, for each monomial M in $\Delta_{u\Lambda_i, v\Lambda_i}(\theta_{\mathbf{i}}^-(t_1, \cdots, t_l))$, there uniquely exists a corresponding **i**-trail π from $-u\Lambda_i$ to $-v\Lambda_i$ satisfying $M = t_1^{d_1(\pi)} \cdots t_l^{d_l(\pi)}$.

2. The Berenstein-Kazhdan decoration functions and i-trails

2.1. Geometric crystal structure on $B_{w_0}^-$

Defining maps

$$\gamma_i : B_{w_0}^- \to \mathbb{C}^\times, \quad \varepsilon_i : B_{w_0}^- \to \mathbb{C}^\times, \quad \overline{e}_i : \mathbb{C}^\times \times B_{w_0}^- \to B_{w_0}^-$$

on $B_{w_0}^- = B^- \cap U\overline{w_0}U$, we get a $\mathfrak{g}$-geometric crystal $(B_{w_0}^-, \{\overline{e}_i\}_{i\in I}, \{\gamma_i\}_{i\in I}, \{\varepsilon_i\}_{i\in I})$ [1]. For the definition of maps, refer to Sect.3 of the paper [8].

The variety $T \cdot B_{w_0}^-$ has a positive structure $\theta_{\mathbf{i}} : T \times (\mathbb{C}^\times)^{l(w_0)} \to T \cdot B_{w_0}^-$ associated with each reduced word **i** of w_0 so that we obtain a crystal $X_*(T \times (\mathbb{C}^\times)^{l(w_0)})$

by the tropicalization functor. The *Berenstein-Kazhdan decoration function* Φ_{BK} on $T \cdot B^-_{w_0}$ is defined as

$$\Phi_{BK} = \sum_{i \in I} \frac{\Delta_{w_0\Lambda_i, s_i\Lambda_i}}{\Delta_{w_0\Lambda_i, \Lambda_i}} + \sum_{i \in I} \frac{\Delta_{w_0 s_i\Lambda_i, \Lambda_i}}{\Delta_{w_0\Lambda_i, \Lambda_i}}. \tag{2.1}$$

Here, Λ_i is the i-th fundamental weight, for u, $v \in W$, the function $\Delta_{u\Lambda_i, v\Lambda_i}$ is a generalized minor.

Let us define a regular function Φ^h_{BK} on $B^-_{w_0}$ as follows:

$$\Phi^h_{\mathrm{BK}} := \sum_{i \in I} \Delta_{w_0\Lambda_i, s_i\Lambda_i}.$$

In [13], Kanakubo and Nakashima proved that the function Φ^h_{BK} is an upper half-decoration on the geometric crystal $B^-_{w_0}$.

An open embedding $\theta^-_{\mathbf{i}} : (\mathbb{C}^\times)^N \hookrightarrow B^-_{w_0}$ in Proposition 1.1, which gives a positive structure on $(B^-_{w_0}, \Phi^h_{\mathrm{BK}})$. Thus, one obtains a crystal $\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}$

$$\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}} := \{z \in X_*((\mathbb{C}^\times)^N) | \mathrm{Trop}(\Phi^h_{\mathrm{BK}} \circ \theta^-_{\mathbf{i}})(z) \geq 0\},$$

$$\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}} = (\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}, \{\tilde{e}_i\}_{i \in I}, \{\tilde{f}_i\}_{i \in I}, \{\tilde{\varepsilon}_i\}_{i \in I}, \{\tilde{\varphi}_i\}_{i \in I}, \{\tilde{\gamma}_i\}_{i \in I}). \tag{2.2}$$

Here, we omitted the notation of restrictions $|_{\tilde{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}}$ for $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{\varepsilon}_i$, $\tilde{\varphi}_i$ and $\tilde{\gamma}_i$.

Theorem 2.1 ([13]). *For each reduced word* $\mathbf{i}$ *of the longest element* w_0*, the set* $\mathbb{B}_{\theta^-_{\mathbf{i}}, \Phi^h_{\mathrm{BK}}}$ *is a* ${}^L\mathfrak{g}$*-crystal isomorphic to the crystal* $B(\infty)$*.*

2.2. i-trails and BK decoration functions

The main result of ([9], Theorem 4.4) allows us, for all reduced words $\mathbf{i}$, to get all monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$ explicitly in the following cases (the numbering of Dynkin diagram is same as in [6]), which covers a significantly wide range of indices $i \in I$ comparing with [8]. Due to this theorem is computed an

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
i	all $i \in I$	all $i \in I$	all $i \in I$	all $i \in I$	$1, 2, 4, 5, 6$	$1, 5, 6, 7$	$1, 7$	$1, 4$	all $i \in I$

edge-colored directed graph $\overline{DG}$ whose vertices are labelled by the monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta^-_{\mathbf{i}}(t_1, \cdots, t_N)$, and edges are colored by letters of $\{1, 2, \cdots, N\}$. We only use easy computations of the Weyl group action on simple roots and weights and multiplications of Laurent monomials. In particular, in case of $\mathfrak{g}$ is of classical type (A_n, B_n, C_n or D_n) or type G_2, by the tropicalization, we get an explicit form of the crystal

$$\{z \in X_*((\mathbb{C}^\times)^N) | \mathrm{Trop}(\Phi^h_{BK} \circ \theta^-_{\mathbf{i}})(z) \geq 0\}, \tag{2.3}$$

for any reduced word $\mathbf{i}$.

2.3. Algorithm for computing coefficients of BK decoration functions for classical types ABCD

A problem on computing coefficients of Berenstein-Kazhdan decoration arises in study of redundant inequalities defining the cone (2.3) (see [10]).

In case of $\mathfrak{g}$ is of classical type A_n, all weights are minuscule, and because of that all coefficients are equal one [8], and there are no redundancy [10].

In case of $\mathfrak{g}$ is of classical type B_n, C_n or D_n this is not the case. We have the following

Theorem 2.2. *In case of* $\mathfrak{g}$ *is of classical type* B_n, C_n *or* D_n, *coefficients with* $\mathbf{i}$*-trails take the form* 2^k, $k \geq 0$.

For any reduced decomposition $\mathbf{i}$, the portion of $\mathbf{i}$-trails π with coefficients bigger than 1 is rather small, and we have an algorithm for computing such cases.

For the proof of Theorem 3.2 we provide an algorithm to compute all coefficients in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^-(t_1, \cdots, t_N)$. Firstly, we use the algorithm of [9] (see also [11]) to get monomials in $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^-(t_1, \cdots, t_N)$, and edge-colored directed graph $\overline{DG}$. Then we apply the following procedure:

```
set S=all monomials
set k=1
while S is not empty
     S1=get all pairs (a,b) of S,
        such that a*b is perfect square Laurent monomial
     for each pair (a,b)∈S1 set coefficient of √(a*b) to 2^k
     set S=S1
```

The proof of correctness of this algorithm is essentially the proof of Theorem 3.2. To elaborate why this algorithm always halts we use correspondence between monomials with coefficients 2^k and k-dimensional faces of Newton polytope of $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^-(t_1, \cdots, t_N)$ ([3]), so it runs no more than length of w_0 cycles.

This procedure can also be used to compute Gross-Hacking-Keel-Kontsevich potential with proper coefficients [11] (set same coefficients for corresponding monomials) and prove that coefficients of Gross-Hacking-Keel-Kontsevich potential take the form 2^k, $k \geq 0$.

2.4. Algorithm complexity

From [11] we know that complexity of computing $\Delta_{w_0\Lambda_i, s_i\Lambda_i} \circ \theta_{\mathbf{i}}^-(t_1, \cdots, t_N)$ consisting of K monomials is

$$O(r^4 K) \sim O(r^2 * \textbf{length of string representation})$$

where **length of string representation** $\sim O(r^2 K)$. Overall complexity of computing coefficients is bounded by product of number of cycles (length $w_0 \sim r^2$) and square of number of monomials

$$O(r^2 * K^2) \leq O(\textbf{length of string representation}^2).$$

This means that whole Berenstein-Kazhdan decoration function and Gross-Hacking-Keel-Kontsevich potential computation algorithm is polynomial (square) in length of string representation of answer.

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Solution of Tropical Best Approximation Problems

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Abstract. We consider discrete best approximation problems in the framework of tropical algebra, which focuses on semirings and semifields with idempotent addition. Given a set of samples from input and output of an unknown function defined on an idempotent semifield, the problem is to find a best approximation of the function by tropical Puiseux polynomial and rational functions. We describe a solution approach that transforms the problem into the best approximation of linear vector equations. Application of this approach yields a direct analytical solution for the polynomial approximation problem and an iterative algorithmic solution for approximation by rational functions. As an illustration, we present results of the best Chebyshev approximation by piecewise linear functions.

Introduction

We consider a discrete approximation problem where an unknown function $f(x)$ is approximated given a set of samples (x_i, y_i) of function values $y_i = f(x_i)$ at some points x_i. Let $F(x; \boldsymbol{\theta})$ be an approximating function that depends on the vector $\boldsymbol{\theta}$ of unknown parameters. A minimax best approximate solution to the problem is defined in the sense of a distance function d to find

$$\boldsymbol{\theta}_* = \arg\min_{\boldsymbol{\theta}} \max_i d(F(x_i; \boldsymbol{\theta}), y_i). \tag{1}$$

In this paper, we outline recent results concerning the investigation of the best approximation problem in the framework of tropical algebra, which deals with the theory and methods of semirings and semifields with idempotent addition [1, 2, 3, 4, 5, 6]. An example of the tropical semifield is an extended set of reals, where the addition is defined as maximum and the multiplication as arithmetic addition (max-plus algebra).

We formulate problem (1) to approximate functions defined on a tropical semifield (a semiring with idempotent addition and invertible multiplication). As approximating functions, we use tropical analogues of Puiseux polynomials and rational functions. The approximation error is defined by a generalized metric on

the tropical vector space over the semifield. We note that in the case of max-plus algebra, the Puiseux polynomials and rational functions are real piecewise linear functions, whereas the metric coincides with the Chebyshev metric.

Tropical Puiseux polynomials arise in a range of research contexts from tropical algebraic geometry to optimization problems in operations research [4, 6, 7, 8, 9]. Thus, the development of approximation techniques using tropical Puiseux polynomial and rational functions can be considered of benefit to both tropical algebra and its applications.

To solve the best approximation problems under study, we transform them into solving tropical linear vector equations with an unknown vector on one side (one-sided equations) or on both sides (two-sided equations). We handle the one-sided equation by applying the results in [10, 11], which offer a direct analytical solution to the problem. A best approximate solution of the two-sided equation is obtained by using the iterative alternating algorithm proposed in [12]. Further details on the solution approach and its implementation can be found in [13].

1. Definitions, Notation and Preliminary Results

In this section we outline basic definitions, notations and preliminary results that provide a framework for the description of the solutions of tropical approximation problems presented below. For more details on tropical (idempotent) algebra, one can consult several works, including [1, 2, 3, 4, 5, 6].

1.1. Idempotent Semifield

Let $\mathbb{X}$ be a non-empty set that is equipped with binary operations $\oplus$ (addition) and $\otimes$ (multiplication), and contains distinct elements $\mathbb{0}$ (zero) and $\mathbb{1}$ (unit). Assume that $(\mathbb{X}, \oplus, \mathbb{0})$ is an idempotent commutative monoid, $(\mathbb{X} \setminus \{\mathbb{0}\}, \otimes, \mathbb{1})$ is an Abelian group and multiplication $\otimes$ distributes over addition $\oplus$. The algebraic system $(\mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1})$ is commonly referred to as the tropical (idempotent) semifield.

The semifield has idempotent addition: for each $x \in \mathbb{X}$ the equality $x \oplus x = x$ holds, and invertible multiplication: for each $x \neq \mathbb{0}$, there exists x^{-1}, such that $xx^{-1} = \mathbb{1}$ (here and hereafter the multiplication sign $\otimes$ is omitted for brevity). It is assumed that the equation $x^p = a$ has a unique solution x for any $a \in \mathbb{X}$ and integer $p > 0$, which makes powers with rational exponents well defined.

Idempotent addition induces a partial order relation: $x \leq y$ if and only if $x \oplus y = y$. The corresponding partial order is assumed to extend to a total order.

An example of the idempotent semifield under consideration is the real semifield $\mathbb{R}_{\max,+} = (\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0)$, also known as max-plus algebra. In this semifield, we have $\oplus = \max$, $\otimes = +$, $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. The power x^y coincides with the product $x \times y$. The inverse x^{-1} of any $x \in \mathbb{R}$ corresponds to the opposite number $-x$. The order relation agrees with the usual linear order on $\mathbb{R}$.

1.2. Algebra of Matrices and Vectors

Matrix algebra over over a semifield is introduced in the usual way. Addition, multiplication and scalar multiplication of matrices follow the standard entrywise rules, where addition and multiplication are replaced by $\oplus$ and $\otimes$. A matrix without zero rows and columns is called regular.

A matrix that consists of a single column (row) is a column (row) vector. If a vector has no zero elements, it is called regular.

For any nonzero column vector $\boldsymbol{x} = (x_j)$, the multiplicative conjugate is the row vector $\boldsymbol{x}^- = (x_j^-)$ where $x_j^- = x_j^{-1}$ if $x_j \neq \mathbb{0}$, and $x_j^- = \mathbb{0}$ otherwise.

For any regular vectors $\boldsymbol{x} = (x_j)$ and $\boldsymbol{y} = (y_j)$, we define a distance function

$$d(\boldsymbol{x}, \boldsymbol{y}) = \bigoplus_j \left(x_j y_j^{-1} \oplus x_j^{-1} y_j\right) = \boldsymbol{y}^- \boldsymbol{x} \oplus \boldsymbol{x}^- \boldsymbol{y}.$$

In the context of $\mathbb{R}_{\max,+}$, this function coincides with the Chebyshev metric

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_j |x_j - y_j| = \max_j \max(x_j - y_j, y_j - x_j).$$

In the case of the arbitrary idempotent semifield $\mathbb{X}$, the distance function d can be considered as a generalized metric that takes values in the interval $[\mathbb{1}, \infty)$.

1.3. Tropical Puiseux Polynomials and Rational Functions

We consider a tropical Puiseux polynomial of n monomials in one variable $x \in \mathbb{X}$, which is written in the following form:

$$P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j}, \qquad x \neq \mathbb{0},$$

where $p_j \in \mathbb{Q}$ are exponents and $\theta_j \in \mathbb{X}$, $\theta_j \neq \mathbb{0}$, are coefficients for all $j = 1, \ldots, n$.

We note that a polynomial defined in the context of the semifield $\mathbb{R}_{\max,+}$ (max-plus algebra) is represented in terms of the usual operations as

$$P(x) = \max_{1 \leq j \leq n} (p_j x + \theta_j),$$

and therefore defines a piecewise-linear convex function on $\mathbb{R}$.

Now consider a tropical rational function that is given by

$$R(x) = \frac{P(x)}{Q(x)}, \qquad P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j}, \qquad Q(x) = \bigoplus_{k=1}^{l} \sigma_k x^{q_k}, \qquad x \neq \mathbb{0}.$$

When defined in terms of $\mathbb{R}_{\max,+}$, the rational function can be written as

$$R(x) = P(x) - Q(x) = \max_{1 \leq j \leq n} (p_j x + \theta_j) - \max_{1 \leq k \leq l} (q_k x + \sigma_k),$$

which is a difference of convex functions. We observe that any arbitrary continuous function can be represented as the difference of two convex functions [14].

1.4. Best Approximate Solution of Vector Equations

Given an $(m \times n)$-matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{b}$, consider the problem to find regular n-vectors $\boldsymbol{x}$ that solve the one-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \tag{2}$$

Since the problem may have no solution, we concentrate on finding a best approximate solution to the equation in the sense of the metric d. The next statement is a consequence of the results in [10] (see also [11]).

Theorem 1. *Let $\boldsymbol{A}$ be a regular matrix, $\boldsymbol{b}$ a regular vector and $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}$. Then the following statements hold:*

1. *The best approximate error for equation (2) is equal to $\sqrt{\Delta}$;*
2. *The best approximate solution of the equation is given by*
$$\boldsymbol{x}_* = \sqrt{\Delta}(\boldsymbol{b}^{-}\boldsymbol{A})^{-};$$
3. *If $\Delta = \mathbb{1}$, there are exact solutions; $\boldsymbol{x}_* = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is the maximum solution.*

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are given $(m \times n)$- and $(m \times l)$-matrices. The problem is to find regular $\boldsymbol{x}$ and $\boldsymbol{y}$ of order n and m to satisfy the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}\boldsymbol{y}. \tag{3}$$

To obtain a best approximate solution to the equation, we apply the alternating algorithm proposed in [12]. The algorithm implements the solution offered by Theorem 1 to solve a series of one-sided equations obtained from (3) in which the left and right sides are alternately replaced by constant vectors.

2. Discrete Best Approximation of Functions

We now describe an algebraic technique to solve the data-fitting problems of approximating an unknown function $y = f(x)$ from finitely many samples (x_i, y_i) in the tropical algebra setting. Both tropical polynomials and rational functions are used as approximants. The problems are handled by transforming them into best approximation of vector equations obtained from the sample data.

Suppose there are m samples (x_i, y_i) where x_i and y_i for $i = 1, \ldots, m$ are corresponding input and output of an unknown function $f : \mathbb{X} \to \mathbb{X}$. Consider the problem of approximating this function by polynomials of n monomials, given by

$$P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j},$$

where we assume for all $j = 1, \ldots, n$ that $p_j \in \mathbb{Q}$ are known exponents and $\theta_j \in \mathbb{X}$ are unknown coefficients. The problem consists in the determination of the unknown coefficients that make the equations

$$P(x_i) = y_i \qquad i = 1, \ldots, m,$$

hold exactly or approximately by minimizing the deviation between both sides.

With the matrix-vector notation

$$\boldsymbol{X} = \begin{pmatrix} x_1^{p_1} & \dots & x_1^{p_n} \\ \vdots & & \vdots \\ x_m^{p_1} & \dots & x_m^{p_n} \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1, \\ \vdots \\ y_m \end{pmatrix}, \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix},$$

we combine the scalar equations into the one-sided vector equation

$$\boldsymbol{X}\boldsymbol{\theta} = \boldsymbol{y},$$

where $\boldsymbol{X}$ and $\boldsymbol{y}$ are a known matrix and vector, and $\boldsymbol{\theta}$ is an unknown vector.

We find a best approximate solution of the equation by applying Theorem 1 to obtain the squared error Δ^* and vector $\boldsymbol{\theta}_* = (\theta_1^*, \dots, \theta_n^*)^T$ of coefficients,

$$\Delta_* = (\boldsymbol{X}(\boldsymbol{y}^- \boldsymbol{X})^-)^- \boldsymbol{y}, \qquad \boldsymbol{\theta}_* = \sqrt{\Delta_*}(\boldsymbol{y}^- \boldsymbol{X})^-.$$

The best approximating polynomial is then given by

$$P_*(x) = \theta_1^* x^{p_1} \oplus \dots \oplus \theta_n^* x^{p_n}.$$

Consider a rational function as an approximant, which is defined as

$$R(x) = \frac{P(x)}{Q(x)}, \qquad P(x) = \bigoplus_{j=1}^{n} \theta_j x^{p_j}, \qquad Q(x) = \bigoplus_{k=1}^{l} \sigma_k x^{q_k}.$$

We assume $p_j, q_k \in \mathbb{Q}$ to be known exponents and $\theta_j, \sigma_k \in \mathbb{X}$ unknown coefficients for $j = 1, \dots, n$ and $k = 1, \dots, l$. Given samples $x_i, y_i \in \mathbb{X}$ for $i = 1, \dots, m$ from input and output of an unknown function, the problem is to find the coefficients that achieve the best approximation of the equations

$$R(x_i) = y_i, \qquad i = 1, \dots, m.$$

To represent the problem in vector form, we introduce the notation

$$\boldsymbol{X} = \begin{pmatrix} x_1^{p_1} & \dots & x_1^{p_n} \\ \vdots & & \vdots \\ x_m^{p_1} & \dots & x_m^{p_n} \end{pmatrix}, \qquad \boldsymbol{Y} = \begin{pmatrix} y_1 & & \mathbb{0} \\ & \ddots & \\ \mathbb{0} & & y_m \end{pmatrix},$$

$$\boldsymbol{Z} = \begin{pmatrix} x_1^{q_1} & \dots & x_1^{q_l} \\ \vdots & & \vdots \\ x_m^{q_1} & \dots & x_m^{q_l} \end{pmatrix}, \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \qquad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_l \end{pmatrix}.$$

The scalar equations can be represented as the two-sided vector equation

$$\boldsymbol{X}\boldsymbol{\theta} = \boldsymbol{Y}\boldsymbol{Z}\boldsymbol{\sigma},$$

where $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are known matrices, and $\boldsymbol{\theta}$ and $\boldsymbol{\sigma}$ are unknown vectors.

We obtain a best approximate solution of the vector equation by using the alternating algorithm proposed in [12]. The algorithm yields a minimum squared error Δ_* and related coefficients $\theta_1^*, \dots, \theta_n^*$ and $\sigma_1^*, \dots, \sigma_l^*$ that define the function

$$R_*(x) = \frac{\theta_1^* x^{p_1} \oplus \dots \oplus \theta_n^* x^{p_n}}{\sigma_1^* x^{q_1} \oplus \dots \oplus \sigma_l^* x^{q_l}}.$$

We note that in the real problems, the exponents in the approximating polynomials $P(x)$ and $Q(x)$ may be unknown and thus need to be assessed along with the coefficients of monomials. Below we use a Monte Carlo random sampling technique to search for optimal values of exponents in the polynomials.

3. Numerical Examples

In this section, we offer examples in terms of the semifield $\mathbb{R}_{\max,+}$ (max-plus algebra), for which both tropical polynomials and rational functions can be represented as piecewise linear functions. We assume the polynomials to have a fixed number of monomials, while the exponents of these monomials are not given in advance.

We apply a two-level solution approach that combines direct random search to fix exponents with the best approximation by the polynomials with fixed exponents to evaluate the coefficients of monomials. To reduce the feasible set of exponents in random search, we consider only polynomials with integer exponents.

We start with a function defined on the interval $[0, 2]$ as follows:

$$f(x) = x^2 - 3x^{1/3} + 5/2, \qquad 0 \le x \le 2.$$

The problem is to find an approximate tropical polynomial from a set of $m = 21$ samples (x_i, y_i), where $x_i = (i-1)/10$ and $y_i = f(x_i)$ for $i = 1, \ldots, m$. We consider polynomials with $n = 7$ monomials where the exponents are produced by random sampling from the discrete uniform distribution over $[-15, 5]$.

For each sample set of exponents, we evaluate the coefficients that attain the minimum of the approximation error. After examining 10,000 sample sets of exponents, we arrive at the minimum squared error $\Delta_* = 0.0481$ and the polynomial, which in the conventional form is written as

$$\begin{aligned} P_*(x) = \max(&2.5240 - 15x, 1.4096 - 3x, 0.8736 - x, 0.3503, \\ &- 0.4760 + x, -1.6720 + 2x, -3.2853 + 3x). \end{aligned}$$

A graphical illustration of the solution is given in Figure 1.

Now suppose that $m = 21$ samples (x_i, y_i) are given from the function

$$g(x) = 3(x-1)^2 \sin(x) + 1/4, \qquad 0 \le x \le 2;$$

where $x_i = (i-1)/10$ and $y_i = g(x_i)$ for $i = 1, \ldots, m$.

We approximate $g(x)$ by a tropical rational function $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials with $n = 6$ and $l = 4$ monomials.

After random sampling of 10,000 pairs of sets of exponent and evaluating corresponding coefficients, we obtain a solution with $\Delta_* = 0.0701$. Figure 2 shows the obtained approximating function $R_*(x) = P_*(x) - Q_*(x)$, where

$$\begin{aligned} P_*(x) &= \max(6.9455 - 3x, 6.0860 - 2x, 4.9978 - x, 3.7461, \\ &\qquad 0.7639 + 2x, -2.6361 + 4x), \\ Q_*(x) &= \max(6.6880 - 5x, 6.2962 - 3x, 5.8009 - 2x, 2.4211). \end{aligned}$$

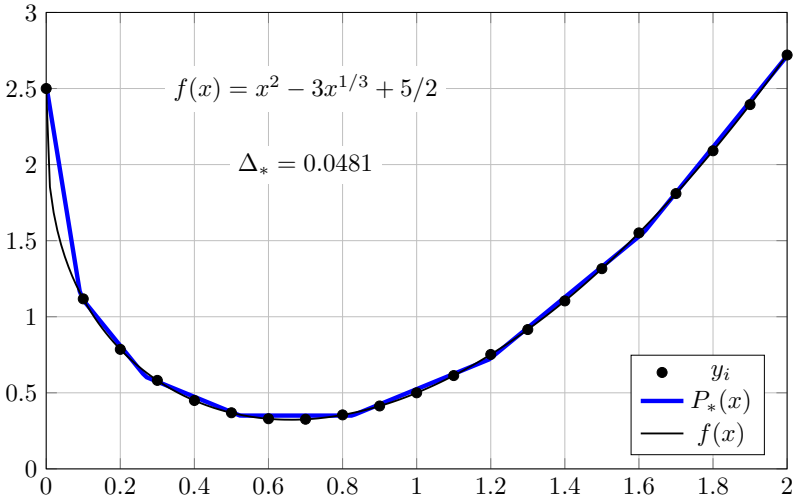

FIGURE 1. Approximation of $f(x)$ by a tropical polynomial $P_*(x)$ with $n = 7$ terms.

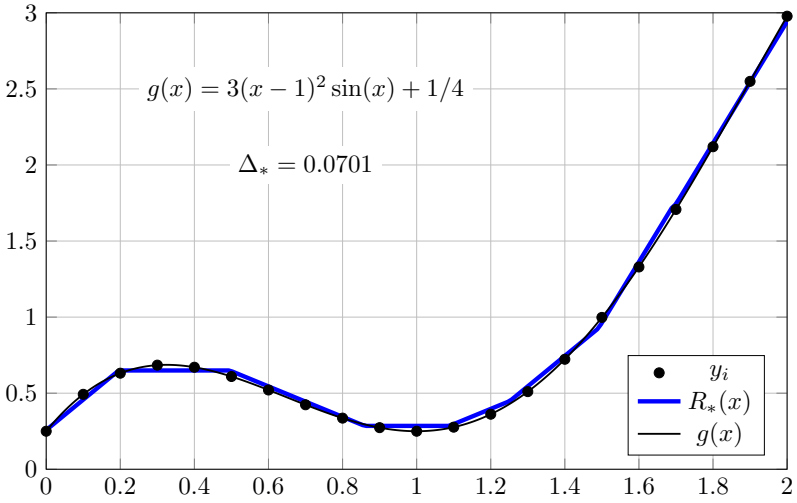

FIGURE 2. Approximation of $g(x)$ by a tropical rational function $R_*(x)$ with $n = 6$ and $l = 4$.

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On computer experiments with reversible difference schemes in Sage

Mikhail Malykh, Mark Gambaryan and Lubov' Lapshenkova

Abstract. The results of computer experiments with reversible difference schemes approximating differential equations with a quadratic right side are presented, they made in the routine called fdm for sage. The generalization of the theory of reversible difference schemes for dynamical systems with a polynomial right part was discussed. It is proposed to use for this the quadratization of the system of differential equations with polynomial part according to Appelroth. The results of computer experiments with such schemes are presented.

1. Reversible schemes

Consider an autonomous system of di erential equations

$$\frac{dx_1}{dt} = f_1(x_1, \ldots, x_n), \quad \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n). \tag{1}$$

Cremona transformations are a very interesting algebraic object. They were discovered relatively recently, in the middle of the 19th century. After the first successes in studying their properties, undertaken by Cremona, Netter, Rosanes and Mlodsijewski, there was a long pause due to the unexpected breadth of the question. Now this vastness is clear — any dynamical system with a quadratic right-hand side is described using the Cremona transformation.

The algorithm for constructing a reversible scheme, presented at PCA'2021 and described in [1], is implemented in our fdm for sage system in the Sage computer algebra system [2]. In this system, the initial problem is specified separately from the method for solving it. Its solution, say, according to the explicit Euler scheme is specified as **`erk(pr)`**, and according to the reversible scheme as **`cremona_scheme(pr)`**. Both functions return an element belonging to the **`Numsol`** class, so you can work with the new scheme in this system in the same way as with Runge-Kutta schemes. Of course, the externally obtained solutions using these schemes di er significantly (see Fig. 1). Using this implementation, several

computer experiments were carried out, during which very unexpected properties of these schemes were found. Later, it turned out to be possible to substantiate some of them.

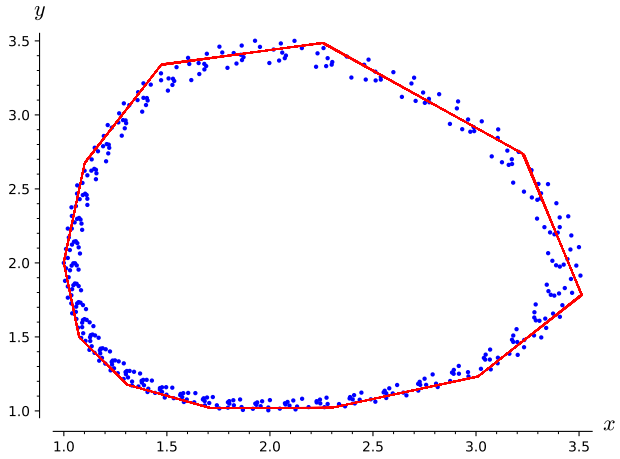

FIGURE 1. Solution of the initial value problem for the Volterra–Lotka system of equations: dots — the solution found by the Runge–Kutta scheme, polygon — the solution found by the reversible scheme.

2. Quadratization of dynamical systems

Transferring the developed technique to the case of equations with a polynomial right-hand side does not cause significant di culties, since back at the beginning of the 20th century G.G. Appelroth proposed a method that allows, by increasing the number of unknowns, to reduce a system with a polynomial right-hand side to a system with a quadratic right-hand side. This procedure was later called quadratization; Ref. [6] describes the software that allows performing such quadratization of any system with a quadratic right-hand side. This makes it possible to construct a reversible di erence scheme for any dynamical system with a polynomial right-hand side.

Computer experiments have shown that the relationships between new and old variables, which are valid for the exact solution, are no longer valid for the approximate solution, which is especially noticeable near moving singular points of the solution. The appearance of moving algebraic singular points is typical for nonlinear systems. In the case of poles, the solution found using a reversible scheme passes through the pole without distortion and, after the pole, fits perfectly on the

integral curve [1]. However, in the case of an algebraic singular point, the situation inevitably changes: there may not be a real integral curve behind the singular point. However, the approximate solution continues beyond such singular points while remaining real. In the future, we plan to combine software for squared di erential equations and our system for numerical integration of di erential equations [2].

3. Properties of reversible circuits

Classical nonlinear oscillators integrable in elliptic functions, are dynamic systems with quadratic right-hand side; a top fixed in its center of gravity is an example. In this case, the new discrete theory completely repeats the continuous theory: i)the points of the approximate solution lie on a certain elliptic curve, which at $\Delta t \to 0$ transforms into an integral curve [3]; ii) the di erence scheme allows a quadrature representation [4]; iii) the approximate solution can be presented by means of an elliptic function of discrete argument [4]. All the di erence reduces to the fact that the place of birational transformations on an integral curve is occupied by the Cremona transformations of the entire three-dimensional space of velocities.

In the case of nonlinear oscillators nonintegrable in elliptic functions, e.g., in the case of the Volterra–Lotka system, the points are arranged along some closed curve, which, however, is not algebraic. In contrast to the case of elliptic oscillators, here it is impossible to choose the step Δt such that the points of the trajectory would form a periodic sequence. In particular, the polygon shown in Fig. 1, actually changes with time, but very slowly.

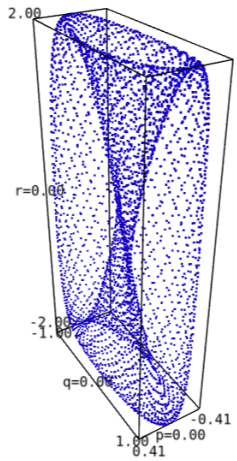

FIGURE 2. Typical solution of the initial-value problem for an asymmetric top.

More complex conservative systems, e.g., the system describing the motion of an asymmetric top, are also described by a system of the form (1) with the quadratic right-hand side. Computer experiments show that the points of one trajectory in this case fill everywhere densely some surfaces, see Fig. 2.

In the case of dissipative systems, the trajectory already cannot fill densely the entire surface. However, here it is of interest that by choosing Δt it turns out to be possible to transform complex limit structures into multiple loops.

We believe that the development of methods for studying trajectories obtained using reversible schemes will make it possible to look at non-integrable systems from a new angle. The advantage of this point of view is that it is always possible to calculate arbitrarily many solution points using a reversible scheme, and therefore to see the structures into which the solution points are arranged in phase space.

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Divergent Fourier series and summation in finite terms using the A.N. Krylov method in CAS

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Abstract. New procedures of the «Kryloff for Sage» software package, created to accelerate the convergence of Fourier series, are presented. We implement divergent Fourier series into our package, considered as generalized functions from the space of distributions $D'(-\pi, \pi)$. This not only expands the class of series that can be studied using symbolic calculations, but also creates the possibility of symbolically summing the Fourier series for which in the previous version of the «Kryloff for Sage» package it was only possible to speed up the convergence, without finding an expression in a closed form.

Introduction

This work is a continuation of previous ones aimed at creating programs for symbolic analysis of Fourier series. We consider the formulation of the problem in which the function represented by the Fourier series is not known, but must be determined from its Fourier coefficients. Fourier coefficients are given in symbolic form. This problem can be solved by standard series summation functions built into modern computer algebra systems for a wide class of symbolic expressions of Fourier coefficients. However, some difficulties arise along this path: when applied to Fourier series, CAS usually produce results in terms of complex-valued higher transcendental functions [1]. At the same time, often the indicated Fourier series represent piecewise polynomial and other piecewise elementary expressions. Therefore, it is relevant to develop symbolic algorithms capable of finding finite expressions for the sums of Fourier series precisely among real-valued piecewise elementary functions.

In our previous report [2], we talked about A.N. Krylov accelerating the convergence of some special Fourier series, the coefficients of which are rational functions of the harmonic number. For them, a simple version of the A.N. Krylov method was implemented in the CAS `Sage` [3] in the form of several functions of the «Kryloff for Sage» software package. In that version of the program the

acceleration of convergence led to a closed-form expression for a very narrow, but not empty class of Fourier series. In the proposed work, we complement these considerations with new functions that can solve some of the summation problems that were previously beyond our capabilities.

We consider Fourier series as elements of the space of distributions $D'(-\pi,\pi)$ [4]. This is correct provided there is no more than a power-law increase in Fourier coefficients [5]. Then the series under consideration can be differentiated term by term any number of times: if a series diverging in the classical sense arises, then we treat it as an element of the space $D'(-\pi,\pi)$ in which this series is convergent. By differentiating the Fourier series term by term, one can try to look for a differential equation of which it may be a solution, cf. [6, p.224-226, 235], cf. [7] and references therein. On this path, it is possible to formulate several simbolic procedures that allow one to reconstruct, using the Fourier series, an inhomogeneous boundary value problem that it satisfies as an element of the space $D'(-\pi,\pi)$. The inhomogeneity of the differential equation will be an element of the space $D'(-\pi,\pi)$. To obtain an expression for the sum, it is necessary to solve this boundary value problem. The Fourier series is represented as a convolution of distributions. In some cases, this approach makes it possible to obtain the desired expressions for the sums of Fourier series in the form of elementary real-valued functions.

The report will present the implementation of the outlined strategy using CAS `Sage`, and experiments on it in computer algebra systems `WolframAlpha` [8] and `Maple` [9].

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The rotation number integer quantization effect in groups acting on the circle

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Abstract. V.M. Buchstaber, O.V. Karpov, and S.I. Tertychnyi initiated the study of the rotation number integer quantization effect for a class of dynamical systems on a torus that includes dynamical systems modeling the dynamics of the Josephson junction. Based on this approach, we study the rotation number integer quantization effect in Artin braid groups and other finitely generated groups acting on the circle. In this case, we find the following manifestation of the quantization effect. Assume that a finitely generated group G acts proximally on the circle by orientation-preserving homeomorphisms. Then for almost every path of any non-degenerate random walk on G, the proportion of elements with integer rotation number in the initial section of the path tends to 1 as the length of the section approaches infinity.

Introduction

We will discuss a new counterintuitive effect for groups acting on the circle. In order to describe this effect, we introduce a series of definitions.

We begin with the concepts of translation and rotation numbers introduced by Henri Poincaré [24]. Let $\mathbb{R}$ be the real line, $\mathbb{Z}$ be the set of integers, and S^1 be the circle $\mathbb{R}/\mathbb{Z}$. The quotient map $\pi\colon \mathbb{R} \to S^1$ is the universal covering map. If $f\colon S^1 \to S^1$ is an orientation-preserving autohomeomorphism, $F\colon \mathbb{R} \to \mathbb{R}$ is a lift of f to $\mathbb{R}$ (that is, $\pi \circ F = f \circ \pi$), and x is a point in $\mathbb{R}$, then the sequence

$$\frac{F(x)-x}{1}, \quad \frac{F^2(x)-x}{2}, \quad \frac{F^3(x)-x}{3}, \quad \dots$$

converges. The limit

$$\tau(F) \;=\; \lim_{k\to\infty} \frac{F^k(x)-x}{k} \;=\; \lim_{k\to\infty} \frac{F^k(x)}{k} \tag{1}$$

does not depend on x and is called the *Poincaré translation number* of F. The value

$$\rho(f) = \tau(F) \mod \mathbb{Z}$$

is independent of the choice of F and is called the *Poincaré rotation number* of f. The translation and rotation numbers are naturally defined for circle isotopies, torus foliations, etc. It is often the case that the term "rotation number" is used to refer to the translation number as well.

Another concept we use is that of proximal group actions. An action of a group G on a space X is said to be *proximal* if, for any two points x and y in X, there exists a sequence $\{g_k\}$ in G such that the sequences $\{g_k(x)\}$ and $\{g_k(y)\}$ converge to one and the same point.

Theorem 1. *Assume that a finitely generated group G acts proximally on the circle by orientation-preserving homeomorphisms. Then for almost every path of any non-degenerate random walk on G, the proportion of elements with integer rotation number in the initial section of the path tends to* 1 *as the length of the section approaches infinity.*

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The software GInv for calculating involutive bases of polynomial ideals

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Abstract. A new version of the softwarefor finding the involutive ideals of polynomial and differential ideals is presented, it called GInv. A brief description of the improvements is given and the results of its testing on real problems are presented, including the problem about the nodes and weights of cubature formulas on the sphere.

Introduction

The study of problems in mechanics and mathematical physics is often reduced to solving systems of polynomial equations. However, as rightly noted by the authors of Numerical recipes [1], this problem does not admit a universal numerical solution method in the cases of two and more unknowns. Fortunately, in the 1990s, an implementation of the Buchberger's algorithm for finding Gröbner bases of ideals of polynomial rings appeared in computer algebra systems (SKA). Finding the Gröbner basis in the lexicographic order of monomes allows us to reduce the solution of a system of nonlinear equations with a finite number of solutions to the solution of one equation with one unknown [2]. Although the Buchberger's algorithm allows to find the Gröbner basis in a finite number of steps, in practice on a modern computer it can be used to solve systems of a small degree and with no more than a dozen unknowns. Several improvements to this algorithm were proposed in the 1990s, some of which remain commercial. At the same time, Gerdt, Zharkov and Blinkov [3] proposed a new approach to finding the bases of polynomial ideals based on the original concept of involutive division. In the 2000s, this algorithm was implemented in the form of GInv software and applied to the study of a number of problems in mathematical physics [4]. Recently, this software has been completely redesigned and transferred to the public domain [5].

1. New version

The new version of GInv is written as a library in C++11. For dynamic data structures such as lists, red-black and binary trees, GMP libraries are used. This allows to complete calculations with arbitrary precision integers and object-oriented memory redistribution. The package interface is designed as an additional module, written in Python3 language. Attention was paid to optimizing the use of memory and processor cache. At the same time, the well-known disadvantages of the standard malloc/free and the garbage collection approach were taken into account. The original approach turned out to be easy to implement compared to standard malloc/free and convenient when searching for errors related to memory leaks.

Improving the methods of calculating the Gröbner basis has two goals: speeding up calculations and reducing the resources used in calculations. These goals are not always possible to combine. From general considerations, it is obvious that very frequent garbage collection can slow down calculations, and on the contrary, the willingness to provide the system with an unlimited number of clean memory pages can lead to a very rapid exhaustion of resources. Moreover, it is not always clear whether a particular change will lead to an actual improvement, that is, the achievement of one of these goals. Therefore, it is extremely important to conduct systematic testing at every stage of the development of such systems. The new version of GInv was tested on a real problem – the problem of calculating nodes and weights of cubature formulas on a sphere cite [6]. This problem leads to a very complex system of nonlinear equations, which previously could only be solved numerically. The economical attitude to memory made it possible to solve it analytically.

2. Testing

A special software tool has been developed that allows to test various versions of GInv and similar computer algebra systems in automatic mode and get the results in graphical form. To proceed with testing a fairly large and representative set of 135 equations was assembled. This test database was updated to use JSON format, suitable for use in various computer algebra systems (GInv, SymPy, Sage).

This article will describe this tool and present the results of testing the current version of GInv. The source files of the testing are publicly available and available for download in a separate GitHub repository at [7].

The system was tested on a server platform consisting of two 4-core Intel Xeon L5630 processors. Each processor had 4 computing cores with support for Hyper-Threading technology, which allowed running 2 threads on each physical core. Thus, the total number of logical cores (processing threads) was 8. The base clock frequency of each processor core was 2134 MHz. Some results of this testing are displayed on following table 1. Full terults available at [7].

Test	Dimension	Length of the Basis	Reduction	Time
ilias13	7	1	0	0,53
comb3000	10	35	503	11,70
hcyclic6	7	221	18634	308,41
eco9	9	189	159992	4322,05
hcyclic7	8	1182	542213	56018,99

TABLE 1. Sample of test results

Conclusion

A new version of software GInv was developed and tested. This new version and dataset for testing are now publicly available on open sources. The new version allowed to solve some problems analytically, which were previously solvable only numerically.

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Computing of tropical sequences associated with Somos sequences in Gfan package

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Abstract. The main objective of this work is to study tropical recurrent sequences associated with Somos sequences. For a set of tropical recurrent sequences, D. Grigoriev put forward a hypothesis of stabilization of the maximum dimensions of solutions to systems of tropical equations given by polynomials, which depend on the length of the sequence under consideration. The validity of such a hypothesis would make it possible to calculate the dimensions of these solutions for systems of arbitrary length. The main purpose of this work is to compute tropical sequences associated with Somos sequences using the Gfan package and to test the Grigoriev hypothesis.

Introduction

Tropical mathematics is a young area of modern mathematics related to the study of semirings with idempotent addition. Despite its novelty, it has already found its application in algebra, geometry, mathematical physics, biology, economics, neural network theory, dynamic programming, and other areas.

This work is a continuation of the work [1], which was devoted to tropical linear recurrent sequences. As part of this work, tropical sequences associated with Somos sequences are computed in the Gfan package.

Gfan is a software package for computing universal Gröbner bases, some related geometric objects (Gröbner fans) and tropical varieties, developed in 2005 by A. Jensen, based on the algorithms described and developed in his dissertation [2].

1. Formulation of the problem

One of the main objects of tropical mathematics is the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$, where $x \oplus y := \max\{x, y\}$, $x \otimes y := x + y$. Tropical mathematics has its analogues of polynomial algebra, linear algebra and other areas of mathematics

[3]. Taking the minimum can be considered as tropical addition, then the additional element to the set of real numbers will be plus infinity.

Let $k \geq 2$ be a natural number and

$$\alpha = \{\alpha_i | 1 \leq i \leq [k/2]\}, \quad x = \{x_j | -k/2 < j \leq [k/2]\}$$

- two sets of independent formal variables in the amount of $[k/2]$ in the first case and k in the second. The sequence of rational functions Somos-k of variables from α and x, $S_k(n) = S_k(n; \alpha; x)(n \in \mathbb{Z})$, is defined by the recursive relation

$$S_k\left(n + \left[\frac{k+1}{2}\right]\right) S_k\left(n - \left[\frac{k}{2}\right]\right) = \sum_{1 \leq i \leq k/2} \alpha_i S_k\left(n + \left[\frac{k+1}{2}\right] - i\right) S_k\left(n - \left[\frac{k}{2}\right] + i\right).$$

In this work, we study the tropical sequences $p_k(n)$ associated with $S_k(n)$ that satisfy the recurrent relation

$$p_k\left(n + \left[\frac{k+1}{2}\right]\right) + p_k\left(n - \left[\frac{k}{2}\right]\right) = \min_{1 \leq i \leq k/2} \left\{p_k\left(n + \left[\frac{k+1}{2}\right] - i\right) + p_k\left(n - \left[\frac{k}{2}\right] + i\right)\right\}.$$

An interesting fact is that the tropical analogue of such sequences is related to the classical Somos sequences by some relation. It was proved in [4] that $S_k(n)$ is a Laurent polynomial in the initial variables x_j and an ordinary polynomial in α_i. Therefore, it can be written as

$$S_k(n) = \left(\prod_{-k/2 < j \leq [k/2]} x_j^{p_k^{(j)}(n)}\right) P_k(n),$$

where $P_k(n) = P_k(n; \alpha; x)$ are polynomials with integer coefficients and $p_k^{(j)}(n)$ are integer sequences.

In this work, we will consider all solutions of the finite sequences $p_k(n)$ with $0 \leq n \leq s$ for $k = 4$ and $k = 5$.

2. Computations of Somos-4 sequences in the Gfan package

To compute the sequences $p_4(n)$, we consider the sequences

$$q_4(n) = \Delta^2 p_4(n) = \Delta p_4(n+1) - \Delta p_4(n) = p_4(n+2) - 2p_4(n+1) + p_4(n).$$

Then the tropical relations will look like

$$q_4(n-1) + q_4(n) + q_4(n+1) + \max\{0, q_4(n)\} = 0$$

For computation in the Gfan package, we reduce this relation to a tropical polynomial. Let $y_n = q_4(n)$. Then we get

$$\max\{y_{n-1} + y_n + y_{n+1}, y_{n-1} + 2y_n + y_{n+1}\} = y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1}$$

To find solutions to this relation, we find tropical prevarieties. Since tropical prevarieties are the set of nonsmoothness of a tropical polynomial, the difficulty

for this is that this polynomial is equal to zero. To solve this problem, add 0 as a term to the tropical polynomial

$$y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1} \oplus 0.$$

We can notice that the system of tropical polynomials $\max\{y_{n-1} + y_n + y_{n+1}, y_{n-1} + 2y_n + y_{n+1}\}$ for $1 \leq n \leq s-1$ reaches a maximum greater than zero only in two cases: $y_0 > 0$ and $y_s > 0$. Because of this, the addition of the term 0 does not affect the dimension of the tropical prevariety. Therefore, to compute the dimensions of the solution space, these cases were excluded. This idea was verified experimentally in the Gfan package for computed finite sequences.

Tropical prevarieties can be computed using the function **gfan_tropicalintersection** of the Gfan package [5]. Denote the dimension of the solution space by d_s. The obtained dimensions of the solution space are presented in Table. 1. The obtained solutions correspond to the calculations carried out in [6].

TABLE 1. Dimensions of the Somos-4 solution space

s	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d_s	2	2	2	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6

3. Computations of Somos-5 sequences in the Gfan package

The tropical relations in this case look like

$$q_5(n-2) + q_5(n-1) + q_5(n) + q_4(n+1) + \max\{0, q_5(n-1) + q_5(n)\} = 0.$$

Let $y_n = q_5(n)$. Then we get

$$\max\{y_{n-2} + y_{n-1} + y_n + y_{n+1}, y_{n-2} + 2y_{n-1} + 2y_n + y_{n+1}\} = 0.$$

Then we consider tropical prevarieties for the following polynomial

$$y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1} \oplus 0.$$

We can notice that the system of tropical polynomials $\max\{y_{n-2} + y_{n-1} + y_n + y_{n+1}, y_{n-2} + 2y_{n-1} + 2y_n + y_{n+1}\}$ for $2 \leq n \leq s-1$ reaches a maximum greater than zero only in three linear cases: $y_0 > 0$, $y_s > 0$ and $y_n = (-1)^n$. Because of this, the addition of the term 0 does not affect the dimension of the tropical prevariety. Therefore, to compute the dimensions of the solution space, these cases were excluded.

The obtained dimensions of the solution space are presented in Table. 2.

TABLE 2. Dimensions of the Somos-5 solution space

s	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
d_s	3	3	3	4	4	4	4	4	5	5	6	6	6	6	6	7	7	8

Conclusion

Based on the computed tropical prevarieties, we can make the assumption that for Somos-4 sequences $d_s = \left[\frac{s-2}{4}\right]+2$. Then for such sequences the tropical entropy [1] takes the value $H = 1/4$. For systems of tropical polynomials $y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-1} \otimes y_n^{\otimes 2} \otimes y_{n+1}$ for $1 \leq n \leq s-1$ without addition 0, it is obtained that $d_s = 2$ for any s. Then for such sequences the tropical entropy takes the value $H = 0$.

Based on the computed tropical prevarieties, we can make the assumption that for Somos-5 the tropical entropy takes the value $H = 2/7$. For systems of tropical polynomials $y_{n-2} \otimes y_{n-1} \otimes y_n \otimes y_{n+1} \oplus y_{n-2} \otimes y_{n-1}^{\otimes 2} \otimes y_n^{\otimes 2} \otimes y_{n+1}$ for $2 \leq n \leq s-1$ without addition 0, it is obtained that $d_s = 3$ for any s. Then for such sequences the tropical entropy takes the value $H = 0$.

For the Somos-6 and Somos-7 cases, it is more difficult to find the dimension of the solution space using the computation of tropical prevarieties. The problem is that before adding zero as a tropical monomial to tropical polynomials, the solution space of finite sequences increases.

The results obtained are consistent with Grigoriev's hypotises on the stabilization of the maximum dimensions of solutions to systems of tropical sequences.

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Murnaghan-Nakayama rule for complete flag variety

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Abstract. Schubert polynomials were introduced by A. Lascoux and M. P. Schützenberg to describe the cohomology ring of complete flag varieties. The famous Schur functions are special cases of Schubert polynomials. In this work we study generalization of Murnaghan-Nakayama rule for Schubert polynomials. We found a symmetric formula for this rule using the Fomin-Kirillov algebra.

Introduction

The cohomology ring of the complete flag variety $H^*(\mathcal{F}\ell_n)$ admits a special linear basis $\{\sigma_w\}$ indexed by permutations on n elements. The elements of the basis are called Schubert classes. For any $u, v \in S_n$, we have

$$\sigma_u \sigma_v = \sum_{w \in S_n} c_{u,v}^w \sigma_w$$

for some $c_{u,v}^w \in \mathbb{R}, u, v, w \in S_n$. The numbers $c_{u,v}^w$ are called the structure constants for $H^*(\mathcal{F}\ell_n)$. By algebro-geometric reasons, the structure constants are always non-negative integers. To provide a combinatorial interpretation for these structure constants is a long standing open problem in algebraic combinatorics. The constants are generalizations of famous Littlewood-Richardson coefficients ([9]), which correspond to the case when both permutations are Grassmannian of the same descent.

Study of cohomology rings of flag varieties started long ago and the first multiplication rule was constructed by D. Monk [12] in 1959. I. N. Bernstein, I. M. Gelfand, S. I. Gelfand [1] and M. Demazure [3] gave a description of the cohomology ring of the complete flag variety $\mathcal{F}\ell_n$ in 70th. Later in 1982 A. Lascoux and M. P. Schützenberg [7, 8] defined Schubert polynomials recursively using divided differences operators. For the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, \ldots]$, the i-th divided differences operator is given

by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}.$$

It is easy to see that these operators send polynomials to polynomials, furthermore, if f has integer coefficients, then $\partial_i f$ also has integer coefficients.

Definition 1 (c.f. [7, 8]). *For a permutation $w_0 = (n, n-1, \dots, 1) \in S_n$, its Schubert polynomial is given by*

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \in \mathbb{Q}[x_1, x_2, \dots].$$

For a permutation $w \in S_n$ s.t. $w \neq w_0$,

$$\mathfrak{S}_w = \partial_i \mathfrak{S}_{ws_i} \quad \text{for } i \text{ such that } \ell(ws_i) = \ell(w) + 1.$$

These polynomials are well defined and the definition above agrees with the following inclusion $S_1 \subset S_2 \subset S_3 \subset \ldots \subset S_{\mathbb{N}}$.

Theorem 1 (c.f. [7, 8]). *For any $u \in S_{\mathbb{N}}$, its Schubert polynomial $\mathfrak{S}_u$ is well defined and $\mathfrak{S}_u$ is a homogeneous polynomial of degree $\ell(u)$.*

The set $\{\mathfrak{S}_u, u \in S_{\mathbb{N}}\}$ of all Schubert Polynomials is a linear basis of $\mathbb{Q}[x_1, x_2, x_3, \dots]$.

The closed formula for each Schubert polynomials in terms of the reduced decompositions was given by S. Billey, W. Jockusch, and R. Stanley [2] and using rc-graphs by S. Fomin and A. N. Kirillov [4], see also [6]. Schubert polynomials are generalizations of famous Schur functions, see the book [10].

Since $\{\mathfrak{S}_u, u \in S_{\mathbb{N}}\}$ is a linear basis of $\mathbb{Q}[x_1, x_2, x_3, \dots]$, we have unique coefficients $c_{u,v}^w, u, v, w \in S_{\mathbb{N}}$ such that, for any $u, v \in S_{\mathbb{N}}$,

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_{\mathbb{N}}} c_{u,v}^w \mathfrak{S}_w.$$

These coefficients $c_{u,v}^w, u, v, w \in S_{\mathbb{N}}$ are exactly the structure constants for flag varieties.

The following rule was proven for the original problem.

Theorem 2 (Monk's rule, c.f. [12]). *For $u \in S_{\mathbb{N}}$ and $k \in \mathbb{N}$, we have*

$$\mathfrak{S}_u \mathfrak{S}_{s_k} = \mathfrak{S}_u \cdot (x_1 + x_2 + \ldots + x_k) = \sum_{a \leq k < b:\ \ell(ut_{a,b}) = \ell(u)+1} \mathfrak{S}_{ut_{a,b}},$$

where $t_{a,b}$ is a transposition of a and b.

Later Pieri's rule and a more general rule for rim hooks were given by F. Sottile in 1996 [14]. K. Mészáros, G. Panova, and A. Postnikov in 2014 [11] rewrote and gave a new prove of the rule for rim hooks (and proved that this way works for hooks with extra square) in terms of Fomin-Kirillov algebra [5]. We will define Fomin-Kirillov algebra and formulate Pieri's rule in the next section. There are also some other rules, but unfortunately they have restrictions on both permutations. A. Morrison and F. Sottile found the Murnaghan-Nakayama rule for Schubert polynomials, see [13] and below we develop Murnaghan-Nakayama rule

in Fomin-Kirillow algebra. Our formula has extra symmetries unlike Morrison-Sottile's rule and it is better in sense of Bruhat orders, see Proposition 2.

1. Fomin-Kirillov algebra

Denote by $\mathcal{FK}_{\mathbb{N}}$ the algebra with generators $[i,j]$, where $i \neq j \in \mathbb{N}$ and relations

- $[i,j] = -[j,i]$;
- $[i,j]^2 = 0$;
- $[i,j][k,\ell] = [k,\ell][i,j]$ for distinct i,j,k,ℓ;
- $[i,j][j,k] + [j,k][k,i] + [k,i][i,j] = 0$.

The last equation is know as associate Yang-Baxter equation. The classical $\mathcal{FK}_n$ is generated only by $[i,j], i,j \in [n]$. The Fomin-Kirillov algebra acts on Schubert polynomials (on the cohomology ring) as the following one side operators

$$\mathfrak{S}_w[a,b] = \begin{cases} \mathfrak{S}_{wt_{a,b}} & \text{if } \ell(wt_{a,b}) = \ell(w)+1 \text{ and } a<b, \\ -\mathfrak{S}_{wt_{a,b}} & \text{if } \ell(wt_{a,b}) = \ell(w)+1 \text{ and } a>b, \\ 0 & \text{otherwise.} \end{cases}$$

We define Dunkl elements in $\mathcal{FK}_{\mathbb{N}}$ as

$$\theta_k = -\sum_{i<k}[i,k] + \sum_{j>k}[k,j] = \sum_i [k,i].$$

Dunkl elements are commute pairwise, i.e., $\theta_i\theta_k = \theta_k\theta_i$, see [5]. As corollary of Monk's rule we get

Proposition 1 (c.f. [5]). *For any permutation $u \in S_n$ and $k \in \mathbb{N}$, we have*

$$\mathfrak{S}_u x_k = \mathfrak{S}_u \theta_k.$$

Theorem 3 (Pieri's rule [11]). *For $u \in S_{\mathbb{N}}$ and $k,m \in \mathbb{N}$, we have*

$$\mathfrak{S}_u \cdot h_k(x_1,x_2,\ldots,x_m) = \mathfrak{S}_u \cdot \left(\sum_{i_1 \leq i_2 \leq \ldots \leq i_k \leq m} x_{i_1}x_{i_2}\cdots x_{i_k} \right) =$$

$$= \sum_{\substack{a_1 \leq \ldots \leq a_k \leq m \\ m<b_1,\ldots,b_k \text{ are distinct}}} \mathfrak{S}_u[a_1b_1][a_2b_2]\cdots[a_kb_k]$$

and

$$\mathfrak{S}_u \cdot e_k(x_1,x_2,\ldots,x_m) = \mathfrak{S}_u \cdot \left(\sum_{i_1 < i_2 < \ldots < i_k \leq m} x_{i_1}x_{i_2}\cdots x_{i_k} \right) =$$

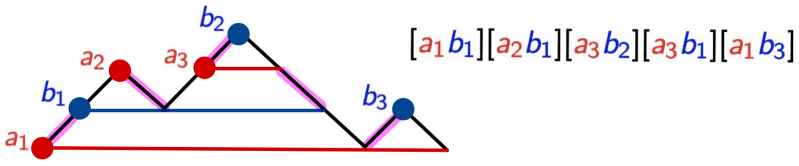
$$= \sum_{\substack{a_1,\ldots,a_k \leq m \text{ are distinct} \\ m<b_1 \leq \ldots \leq b_k}} \mathfrak{S}_u[a_1b_1][a_2b_2]\cdots[a_kb_k].$$

In this paper we extend this approach and present the formula for Murnaghan-Nakayama rule in Fomin-Kirillov algebra.

Theorem 4 (Murnaghan–Nakayama rule). *For $u \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, we have*

$$\mathfrak{S}_u \cdot p_k(x_1, x_2, \dots, x_m) = \mathfrak{S}_u \cdot (x_1^k + x_2^k + \dots + x_m^k) =$$
$$= \sum_{P \text{ is a Dyck path of length } 2k} (-1)^{u_e(P)} \sum_{\substack{a_1,\dots,a_{u_e(P)+1} \le m \\ b_1,\dots,b_{k-u_e(P)} > m \\ \text{are distinct}}} \mathfrak{S}_u \mathcal{M}_P(a,b).$$

The first summation is overall Dyck paths and the second summation is overall distinct indexes $a_1, \dots, a_{u_e(P)+1} \le m < b_1, \dots, b_{k-u_e(P)}$ corresponding to moves up on even and odd places resp. and $\mathcal{M}_P(a,b)$ is a product of $[a_i, b_j]$ as in the picture.



It is clear that our rule is symmetric on indexes $[m]$ and on indexes $\{m+1, m+2, m+3, \dots\}$, which should help in a construction of such a rule for the case of Schubert polynomials times Schur functions. Our rule is impossible to simplify, see Proposition 2.

Proposition 2. *For $u, v \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, there is at most one Dyck path with indexes $a_1, \dots, a_{u_e(P)+1} \le m < b_1, \dots, b_{k-u_e(P)}$ such that $\mathfrak{S}_u \mathcal{M}_P(a,b) = \mathfrak{S}_v$.*

In particular, $\mathfrak{S}_u p_k(x_1, x_2, \dots, x_m) = \sum \pm \mathfrak{S}_v$, where summation by some permutations.

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Combinatorial Ky Fan theorem for sphere bundles

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(By a joint work with Rade Zivaljevic)

Combinatorial statements, such as theorems of Carathéodory, Radon, Helly, Sperner, Tucker, Ky Fan, etc., are fundamental results of combinatorial (algebraic) topology, accessible to non-specialists, which are immediately applicable to mathematical economics, data science, game theory, graph theory, mathematical optimization, computational geometry, and other fields.

The Ky Fan theorem is also a disguised combinatorial counterpart of the Borsuk-Ulam theorem. Recall its usual set-up: the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$ is triangulated and the triangulation is assumed to be centrally symmetric. There is a labeling (coloring) of vertices of this triangulation

$$\lambda : Vert(S^n) \to \{\pm 1, ..., \pm N\}$$

which is

- *antipodal*, $\lambda(-v) = -\lambda(v) \ \ \forall v \in Vert(S^n)$, and
- $\lambda(v) \neq -\lambda(w)$ for each pair $\{v, w\}$ of adjacent vertices of the triangulation.

The *alternating number* $Alt(\sigma)$ of a simplex is the number of sign changes in the labels of its vertices (which are ordered by the absolute values). For example $Alt(-1, 2, 3, -4) = 2$; $\ Alt(-1, 2, -3, 4) = 3$, etc.

Clearly, the alternating numbers of a simplex and its antipodal one are equal. The maximal possible alternating number is n, and these simplices come in pairs. The Ky Fan theorem states that $n < N$, and the number of (pairs of) simplices with alternating number n is odd.

We shall address the following questions:

When is it possible to replace the triangulated sphere by some other triangulated manifold with a free $\mathbb{Z}_2$-action? What happens if one replaces a unique sphere S^n by a parameterized continuous family of spheres, that is, by the total space of some spherical bundle over a smooth manifold?

Our main results are:

- For a spherical bundle, there are "many" simplices with alternating number n; taken together, they form a closed pseudomanifold which is topologically as complicated as the base of the bundle.
- For non-trivial bundles one expects simplices with alternating numbers bigger than n. How much bigger depends on the Stiefel-Whitney classes of the bundle.
- Some explicit examples will be provided. They include spherical bundles associated to the tangent bundles of selected real and complex projective spaces.

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Exact solutions to systems of second order ordinary differential equations by Decomposion method

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Abstract. The solvability conditions and exact solutions to the system of linear second order differential equations in terms of the abstract operator equations

$$\mathbb{B}X(t) = X''(t) - S(t)X'(t) - Q(t)X(t) = F(t),$$

with nonlocal multipoint and integral boundary conditions

$$M_0X(0)+\sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}, \quad N_0X'(0)+CX(0)+\sum_{i=1}^{n}[N_iX'(t_i)+V_iX(t_i)] = \vec{0},$$

by decomposition method is proposed in this paper, where $C, M_0, N_0, M_i, N_i, V_i$ are matrices, $\Psi_i(X)$ Fredholm integrals. In the case where the fundamental solution of the first order system is known, the fundamental solution of the corresponding second order system was obtained. The technique is easy to implement to any Computer Algebra System (CAS) and is economic and efficient compared to other existing methods.

Introduction

Everewhere below we denote by $\mathcal{X}$ the space $C[0,1]$ or $L_p(0,1)$ and by $\mathcal{X}_m$ the space of column vectors $X(t) = col(x_1(t),..,x_m(t))$ with elements from $\mathcal{X}$, i.e. $\mathcal{X}_m = C_m[0,1]$ or $\mathcal{X}_m = L_{p_m}(0,1)$. Denote also by $\mathcal{X}^i$ the space $C^i[0,1]$ or the Sobolev space $W_p^i(0,1)$, and by $\mathcal{X}_m^i$ the space $C_m^i[0,1]$ or $W_{p_m}^i(0,1)$, $i = 1,2$. We will also denote by 0_m the zero and by I_m the identity $m \times m$ matrices. By $\vec{0}$ we will denote the zero column vector.

Lemma 1. *Let $P(t), T(t)$ be $m\times m$ matrices with element from $\mathcal{X}$, the operators $A_1, A_2 : \mathcal{X}_m \to \mathcal{X}_m$ be defined by*

$$A_1Y(t) = Y'(t) - P(t)Y(t), \quad Y(t) \in D(A_1) = \mathcal{X}_m^1, \tag{1}$$

$$A_2X(t) = X'(t) - T(t)X(t), \quad X(t) \in D(A_2) = \mathcal{X}_m^1, \tag{2}$$

and $Z, \mathcal{Z}$ the fundamental matrices to the homogeneous equations $A_1Y(t) = \vec{0}$, $A_2X(t) = \vec{0}$, respectively, such that $Z(0) = I_m, \det \mathcal{Z} \neq 0$. Then the operators $\widehat{A}_1, \widehat{A}_2$ corresponding to the problems

$$\widehat{A}_1Y(t) = A_1Y(t) = F(t),\ D(\widehat{A}_1) = \{Y(t) \in D(A_1) : Y(0) = \vec{0}\}, \tag{3}$$

$$\widehat{A}_2X(t) = A_2X(t) = Y(t),\ D(\widehat{A}_2) = \{X(t) \in D(A_2) : X(0) = \vec{0}\}, \tag{4}$$

are correct and their unique solutions are given by

$$Y(t) = \widehat{A}_1^{-1}F(t) = Z(t)\int_0^t Z^{-1}(s)F(s)ds, \tag{5}$$

$$X(t) = \widehat{A}_2^{-1}Y(t) = \mathcal{Z}(t)\int_0^t \mathcal{Z}^{-1}(s)Y(s)ds. \tag{6}$$

Theorem 1. *Let the operators $A_1, A_2, \widehat{A}_1, \widehat{A}_2$, vectors X, Y, F and matrices $Z, \mathcal{Z}$ be defined as in Lemma 1, the vectors $\Psi = col(\Psi_1, ..., \Psi_n) \in \mathcal{X}_n^*$, and $Y(\vec{t}) = col(Y(t_1), ..., Y(t_k))$, $0 < t_1 < ... < t_k \leq 1$, $M = (M_1, ..., M_n)$ and $N = (N_1, ..., N_k)$ be a $m \times (mn)$ and $m \times (mk)$ constant matrices, respectively, and M_i, N_j the $m \times m$ constant matrices, $i = 0, 1, ..., n$, $j = 0, 1, ..., k$. Then:*
(i) The operator $B_1 : \mathcal{X}_m \to \mathcal{X}_m$, corresponding to the problem

$$B_1Y(t) = A_1Y(t) = Y'(t) - P(t)Y(t) = F(t), \tag{7}$$

$$D(B_1) = \{Y(t) \in D(A_1) = \mathcal{X}_m^1 : N_0Y(0) + \sum_{j=1}^{k} N_jY(t_j) = \vec{0}\}$$

is injective if and only if

$$\det L_1 = \det[N_0 + \sum_{j=1}^{k} N_jZ(t_j)] \neq 0. \tag{8}$$

(ii) If the operator B_1 is injective, then it is correct and a unique solution to Problem (7) is

$$Y(t) = B_1^{-1}F(t) = \widehat{A}_1^{-1}F(t) - Z(t)L_1^{-1}\sum_{j=1}^{k} N_j(\widehat{A}_1^{-1}F)(t_j), \tag{9}$$

where $\widehat{A}_1^{-1}F(t)$ is given by (5).
(iii) The operator $B_2 : \mathcal{X}_m \to \mathcal{X}_m$, corresponding to the problem

$$B_2X(t) = A_2X(t) = X'(t) - T(t)X(t) = Y(t), \tag{10}$$

$$D(B_2) = \{X(t) \in D(A_2) = \mathcal{X}_m^1 : M_0X(0) + \sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}\}$$

is injective if and only if

$$\det L_2 = \det[M_0\mathcal{Z}(0) + M\Psi(\mathcal{Z})] \neq 0. \tag{11}$$

(iv) If the operator B_2 is injective, then it is correct and a unique solution to Problem (10) is

$$X(t) = B_2^{-1}Y(t) = \widehat{A}_2^{-1}Y(t) - \mathcal{Z}(t)L_2^{-1}M\Psi(\widehat{A}_2^{-1}Y), \tag{12}$$

where $\widehat{A}_2^{-1}Y(t)$ is given by (6).

Theorem 2. *Let a vector Ψ be defined as in Theorem 1, vectors $U = U(t) = col\,(u_1(t), ..., u_{2m}(t)) \in \mathcal{X}^1_{2m}$, $\mathcal{F} = \mathcal{F}(t) = col\,(f_1(t), ..., f_{2m}(t)) \in \mathcal{X}_{2m}$, $\mathcal{M}_0$ and $\mathcal{M}$ be the $2m \times 2m$ and $2m \times 2mn$ constant matrices, respectively, S, Q constant $m \times m$ matrices, $\det Q \neq 0$ and the operator $\mathbf{B}$ be defined by*

$$\mathbf{B}U(t) = \mathbf{A}U(t) = U'(t) - DU(t) = \mathcal{F}, \tag{13}$$
$$D(\mathbf{B}) = \{U(t) \in D(\mathbf{A}) = \mathcal{X}^1_{2m} : \mathcal{M}_0U(0) + \mathcal{M}\Psi(U) = \vec{0}\},$$

where $D = \begin{pmatrix} S & Q \\ I_m & 0_m \end{pmatrix}$. Suppose also that there exist a constant matrix T satisfying the matrix equation $T^2 - ST = Q$, $\det T \neq 0$ and the fundamental matrices $Z = Z(t), \mathcal{Z} = \mathcal{Z}(t)$ to the systems $Y'(t) - PY(t) = \vec{0}$, $A_2X(t) = X'(t) - TX(t) = \vec{0}$, respectively, where $P = S - T$, $Z(0) = I_m$, $\det \mathcal{Z}(0) \neq 0$. Then:
(i) The $2m \times 2m$ matrix

$$\mathbf{Z}(t) = \begin{pmatrix} \mathcal{Z}(t) & \widehat{A}_2^{-1}Z(t) \\ \int_0^t \mathcal{Z}(s)ds + T^{-1}\mathcal{Z}(0) & \int_0^t \widehat{A}_2^{-1}Z(s)ds - (PT)^{-1} \end{pmatrix} \tag{14}$$

is a fundamental matrix to $U'(t) - DU(t) = \vec{0}$, where $S = P + T$, $Q = -PT$, $\widehat{A}_2$ as in Lemma 1.
(ii) Problem (13) is uniquely solvable if and only if

$$det\mathbf{L} = det[\mathcal{M}_0\mathbf{Z}(0) + \mathcal{M}\Psi(\mathbf{Z})] \neq 0, \tag{15}$$

and the unique solution to Problem (13) is given by

$$U(t) = \widehat{\mathbf{A}}^{-1}\mathcal{F}(t) - \mathbf{Z}\mathbf{L}^{-1}\mathcal{M}\Psi\left(\widehat{\mathbf{A}}^{-1}\mathcal{F}(t)\right), \tag{16}$$

where $\widehat{\mathbf{A}}^{-1}\mathcal{F}(t) = \mathbf{Z}(t)\int_0^t \mathbf{Z}^{-1}(s)\mathcal{F}(s)ds$, $\mathbf{Z}(0) = \begin{pmatrix} \mathcal{Z}(0) & 0_m \\ T^{-1}\mathcal{Z}(0) & -(PT)^{-1} \end{pmatrix}$.

Theorem 3. *Let the operator A, the matrices $S(t), Q(t), M_0, M = (M_1, ..., M_n)$, the vectors X, F, Ψ be defined as in Theorem 1 and $N = (N_1, ..., N_n)$, $V = (V_1, ..., V_n)$ be the $m \times mn$ matrices with $m \times m$ constant matrices N_i, V_i. Suppose also that N_0, C are $m \times m$ constant matrices, the vectors $X(\vec{t}) = col(X(t_1), ..., X(t_n))$, $X'(\vec{t}) = col(X'(t_1), ..., X'(t_n))$, where $0 < t_1 < ... < t_n \leq 1$, and the operator $\mathbf{B} : \mathcal{X}_m \to \mathcal{X}_m$*

is defined by

$$\begin{aligned} \mathbf{B}X(t) &= AX(t) = X''(t) - S(t)X'(t) - Q(t)X(t) = F(t), \qquad (17) \\ D(\mathbf{B}) &= \{X(t) \in D(A) = \mathcal{X}_m^2 : M_0X(0) + \sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}, \\ &\quad N_0X'(0) + CX(0) + \sum_{i=1}^{n}[N_iX'(t_i) + V_iX(t_i)] = \vec{0}\}. \end{aligned}$$

If there exists a differentiable $m \times m$ matrix $T = T(t)$, such that

$$T'(t) - S(t)T(t) + T^2(t) = Q(t),\ C = -N_0T(0),\ V_i = -N_iT(t_i),\ i = 1, ..., n, \quad (18)$$

then there exist the matrix $P(t) = S(t) - T(t)$ and the operators $A_1, A_2, \widehat{A}_1, \widehat{A}_2$ defined by (1), (2), (3), (4), respectively, such that:
(i) The operator $\mathbf{B}$ is decomposed in $\mathbf{B} = B_1B_2$, where the operators $B_1, B_2 : \mathcal{X}_m \to \mathcal{X}_m$ are given by

$$\begin{aligned} B_1Y(t) = A_1Y(t) = Y'(t) - P(t)Y(t) = F(t), \qquad (19) \\ D(B_1) = \{Y(t) \in D(A_1) : N_0Y(0) + \textstyle\sum_{i=1}^{n} N_iY(t_i) = \vec{0}\}, \\ B_2X(t) = A_2X(t) = X'(t) - T(t)X(t) = Y(t), \qquad (20) \\ D(B_2) = \{X(t) \in D(A_2) : M_0X(0) + \textstyle\sum_{i=1}^{n} M_i\Psi_i(X) = \vec{0}\}. \end{aligned}$$

(ii) The operator $\mathbf{B}$ is injective if and only if

$$\det L_1 = \det[N_0 + \sum_{i=1}^{n} N_iZ(t_i)] \neq 0,\ \det L_2 = \det[M_0\mathcal{Z}(0) + M\Psi(\mathcal{Z})] \neq 0, \quad (21)$$

where $Z, \mathcal{Z}$ are the fundamental matrices of the equations $A_1Y(t) = \vec{0}$, $A_2X(t) = \vec{0}$, respectively.
(iii) If the operator $\mathbf{B}$ is injective, then it is correct and a unique solution to Problem (17) is given by

$$\begin{aligned} X(t) = \mathbf{B}^{-1}F(t) = \widehat{A}_2^{-1}Y(t) - \mathcal{Z}(t)L_2^{-1}M\Psi(\widehat{A}_2^{-1}Y), \quad \textit{where} \qquad (22) \\ Y(t) = \widehat{A}_1^{-1}F(t) - Z(t)L_1^{-1}\textstyle\sum_{i=1}^{n} N_i(\widehat{A}_1^{-1}F)(t_i), \qquad (23) \end{aligned}$$

and $\widehat{A}_1^{-1}F(t)$, $\widehat{A}_2^{-1}Y(t)$ are given by (5), (6), respectively.

Conclusion

The solvability conditions and exact solutions to the nonlocal boundary value problems (BVPs) for the systems of first and second order ordinary differential equations were obtained in the terms of abstract operators. BVP for the system of second order were solved by Decomposition method. Some examples in [1], [2], [3] can be solved more easily by the proposed method for the systems of first order.

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Polynomial method for point configurations

Fedor Petrov

Abstract. A celebrated theorem due to Bannai–Bannai–Stanton says that if A is a set of points in $\mathbb{R}^d$, which determines two distinct distances, then

$$|A| \leqslant \binom{d+s}{s}.$$

I want to discuss a short proof of this result which combines Sylvesters Law of Inertia for quadratic forms with the proof of the so-called Croot–Lev–Pach Lemma [1] from additive combinatorics. Based on a joint work [2] with C. Pohoata.

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Algorithmic mathematics in a technical university: different ways to comprehend the material

Egor Malyutin and Sergei Pozdniakov

Abstract. The problem of a student's comprehension of mathematical concepts is becoming increasingly relevant in modern conditions, when mental activity to comprehend the material is often replaced on the student's initiative by searching for ready-made answers on the Internet or using symbolic algebra systems . The report presents a number of pedagogical experiments aimed at clarifying the conditions and methods of supporting the processes of comprehending new material in a rich external information environment.

Introduction

Bourbaki's traditions [1] in the presentation of mathematics influenced the teaching of mathematics in technical universities. Implicitly, and sometimes explicitly, the idea was postulated that formal mathematical structures correspond to analogous intellectual structures in the learner's brain. From which the conclusion was drawn: if abstract mathematical structures are strictly consistently presented, moving from the general to the particular, then in the student's head there will be a structure of concepts and mental operations that is adequate to Bourbaki's books. However, the process of acquiring knowledge is much more complex and formal structures do not become a thinking tool if they are not based on already existing mental structures. Marvin Minsky [2] called this the investment principle. In other words, understanding the material is not so much a function of correctly organized mathematical material, but rather a function of all the experience accumulated up to a given moment.

It can be assumed that the effect of understanding is associated with prediction (the development of the idea of the area of proximal development [3]): even before the teacher finishes the sentence, the listener already has some model in his head of what the teacher wants to convey to the student. Two options are possible: either this model does not contradict what the teacher says and noninsight comprehension occurs [4], or a contradiction between the model and a new

concept arises and then either insight occurs, that is, a new gestalt arises [5] - a new look at the material being presented (instant restructuring information [6]), or the contradiction persists for a longer period.This contradiction is a mechanism of delayed comprehension.

1. Summary of the report

The report discusses the following ways to understand new mathematical concepts:

1) The traditional way of conveying meaning is activity-based. In the process of its implementation, the teacher forms in the student pre-set skills that form a contextual environment for operating objects in the subject area. This specially formed contextual information environment is associated with semantic constructions that are objectified in the context of this environment.

2) Construction of algorithms that implement constructive descriptions of the properties of mathematical objects.

3) Setting tasks and interacting with systems that verify hypotheses using many examples of the subject area.

The report also describes an experiment in which the features of spontaneous concept formation were studied based on the subconscious construction of predictive models.

Conclusion

For future engineers, mathematics is important as a tool, but if earlier Krylov's words [7] that mathematics for an engineer is a tool "like an ax and a saw for a carpenter" could be interpreted as the presence of a set of applied skills, now the presence of intellectual tools changes the meaning of Krylov's statement. Now the "Krylov's tool" becomes an intellectual toolkit based on the understanding of abstract mathematical concepts. To achieve this goal, it is necessary to replace the formally mathematical approach a la Bourbaki with an "investment" approach based on the existing knowledge, ideas and experience of students.

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On pictures related to some exponential sums

N. V. Proskurin

Abstract. By numerical experiments with exponential sums in finite fields, it is discovered relation with flat curve known as Kepler trifolium. A theoretical explanation for this observation is given in the present paper.

0. Preliminaries.

Consider the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of prime order p and a non-trivial character χ of its multiplicative group extended by setting $\chi(0) = 0$. Let e_p be the additive character

$$x \mapsto \exp(2\pi i x/p)$$

of $\mathbb{F}_p$. Given one-variable polynomials f, g over $\mathbb{F}_p$, consider the sum

$$\sum_{x\in\mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big). \tag{1}$$

That is an exponential character sum of mixed type, see [1]. Under some general assumptions on f, g and χ, one has

$$\Big| \sum_{x\in\mathbb{F}_p} \chi\big(f(x)\big)\, e_p\big(g(x)\big) \Big| \le (m+n-1)\sqrt{p} \tag{2}$$

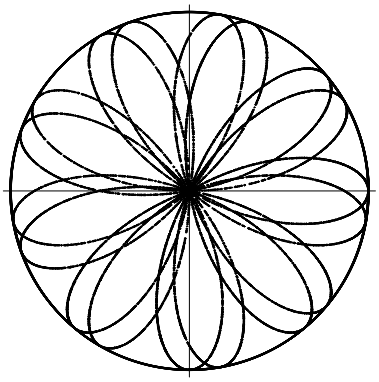
with $n = \deg(g)$ and $m = \deg(\text{radical of } f)$, see [1] and [2].
In particular, let ψ be a cubic character and let $f(x) = x$, $g(x) = x^2$. This case (2) with $\chi = \psi$ implies that the sum

$$E_p(\psi) = \frac{1}{2\sqrt{p}} \sum_{x\in\mathbb{F}_p} \psi(x) e_p(x^2) \tag{3}$$

is located in the circle $D = \big\{ z \in \mathbb{C} \;\big|\; |z| \le 1 \big\}$. We are interested in distribution of the points $E_p(\psi)$ in D. In the present paper we provide a theoretical explanation for our numerical experimental observations [4].

1. Numerical observations.

We have evaluated the sums $E_p(\psi)$ for all cubic characters ψ and for all prime $p \equiv 1 \bmod 6$ subject to $p \leq 360000$. The assumption $p \equiv 1 \pmod 6$ is included here just to ensure the existence of cubic characters. The following figure on complex plane $\mathbb{C}$ represents the results originally reported in [4].



On this figure, it is shown the circle D, its boundary $\overline{D} = \{z \in \mathbb{C} \,|\, |z| = 1\}$, the real and imaginary axis, and some 18-petals flower. The boundary of the petals are formed by the points $E_p(\psi)$ with p and characters ψ as above. In that follows, we will recognise six copies of Kepler trifolium here.

2. Gauss sums.

The Gauss sums $G(\chi)$ are the ones (1) with $f(x) = g(x) = x$, so that

$$G(\chi) = \sum_{x \in \mathbb{F}_p} \chi(x) e_p(x). \tag{4}$$

For any prime p and non-trivial character χ one has

$$|G(\chi)|^2 = p \quad \text{and} \quad G(\chi)G(\bar{\chi}) = \chi(-1)p, \tag{5}$$

where $\bar{\chi}$ is the complex conjugation of χ. One say $G(\chi)$ is a quadratic, cubic or sextic sums according to χ is a character of order 2, 3 or 6.

By Gauss, for the quadratic character κ, the sum $G(\kappa)$ is equal to $\sqrt{p}$ or $i\sqrt{p}$ according to $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$.

To deal with cubic characters, assume $p \equiv 1 \bmod 6$. This case we have two cubic characters, say ψ and $\bar{\psi}$, the quadratic character κ, and sextic characters $\kappa\psi$ and $\kappa\bar{\psi}$. The sextic sums can be evaluated (see theorem 3.1 in [3]) in terms of cubic and quadratic ones by the formula

$$G(\kappa\bar{\psi}) = \bar{\psi}(2)G(\kappa)G(\psi)^2/p. \tag{6}$$

For the cubic characters ψ one has $\psi(-1) = 1$, so that (5) implies

$$G(\bar{\psi}) = p/G(\psi). \tag{7}$$

3. Evaluation of sums (3) in terms of Gauss sums.

Consider the sum $E_p(\psi)$ in (3) with a cubic character ψ, $p \equiv 1 \bmod 6$. One has $\psi(x) = \bar{\psi}(x^2)$ for all $x \in \mathbb{F}_p$ and

$$\sharp\{x \in \mathbb{F}_p \mid x^2 = t\} = 1 + \kappa(t) \quad \text{for all} \quad t \in \mathbb{F}_p.$$

Recall, it is assumed $\kappa(0) = \psi(0) = 0$. It follows,

$$\begin{aligned} 2\sqrt{p}\,E_p(\psi) &= \sum_{t\in\mathbb{F}_p} \sharp\{x \in \mathbb{F}_p \mid x^2 = t\}\bar{\psi}(t)e_p(t) \\ &= \sum_{t\in\mathbb{F}_p} \big(1 + \kappa(t)\big)\bar{\psi}(t)e_p(t) = G(\bar{\psi}) + G(\kappa\bar{\psi}). \end{aligned}$$

This can be rewritten as $2\sqrt{p}\,E_p(\psi) = p/G(\psi) + \bar{\psi}(2)G(\kappa)G(\psi)^2/p$, see (6) and (7), and finally as follows.

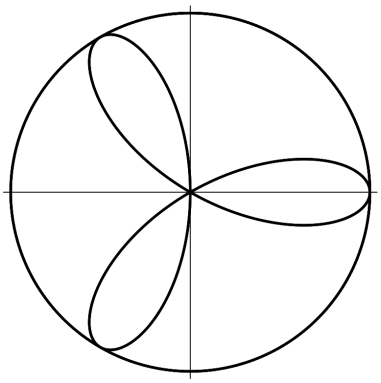
Proposition 1. *For every prime $p \equiv 1 \bmod 6$, one has*

$$E_p(\psi) = \frac{1 + QT^3}{2T} \quad \textit{with } T = G(\psi)/\sqrt{p},\ \ Q = \bar{\psi}(2)G(\kappa)/\sqrt{p}. \tag{8}$$

Here ψ and κ are cubic and quadratic characters of $\mathbb{F}_p$ and $|T| = |Q| = 1$. $\square$

4. Kepler trifolium.

Consider the complex plane $\mathbb{C}$ with the Cartesian coordinates $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $z \in \mathbb{C}$, the unit circle D centred at the origin 0, its boundary $\overline{D}$ and the curve C



defined by the equation

$$(x^2 + y^2)^2 + 3xy^2 - x^3 = 0. \tag{9}$$

This curve is known as Kepler trifolium and also as regular trifolium, three leaf/petal rose, three leaf/petal clover. It remains unchanged when rotated through the angles of $\pm 2\pi/3$ and it can be given by the polar equation $r = \cos(3\varphi)$, r being a point on the axis obtained by rotation of the real axis through the angle φ.

5. Parametrization.

Consider again the Kepler trifolium C. We intend to show that C can be parametrized by a rational function on $\overline{D}$. To be precise, we mean the complex function

$$z \mapsto \frac{1+z^3}{2z} \tag{10}$$

and its restriction to $\overline{D}$.

Proposition 2. *The function* (10) *takes any point of* $\overline{D}$ *to some point of* C. *In particular, it takes cubic roots of* -1 *to the triple point* 0 *of* C. *It takes cubic roots of* 1 *to the cubic roots of* 1. *Except for the point* 0, *every point of* C *is the image of an unique point of* $\overline{D}$. *The boundary of each petal is the image of someone arc in* $\overline{D}$ *whose endpoints are the cubic roots of* -1.

Proof. Let $z = x + iy$ with real x and y. If $z \in \overline{D}$ then $x^2 + y^2 = 1$ and

$$\frac{1+z^3}{2z} = \frac{\bar{z}+z^2}{2} = X + iY$$

$$\text{with} \quad X = (1+x)S, \quad Y = yS, \quad S = x - \frac{1}{2}.$$

It follows, $X^2 = (1+x)^2S^2$, $Y^2 = (1-x^2)S^2$, $X^2 + Y^2 = 2(1+x)S^2$, and then

$$(X^2+Y^2)^2 + 3XY^2 - X^3 = 0.$$

According to (9), that means $Z = X + iY$ is a point of the curve C, as required. In particular, if z is a cubic root of -1 then $1 + z^3 = 0$ and $Z = 0$. If z is a cubic root of 1 then $Z = \bar{z}$ and that is a cubic root of 1 as well.

Now let X and Y be the real and the imaginary parts of some point $Z \in C$. Assume, this Z is a point of the right petal on the figure and $Z \neq 1$. We have $X > 0$ and $X \neq 1$. For every such X, there is a unique $x \geq -1$ satisfying

$$(1+x)S = X \quad \text{with} \quad S = x - \frac{1}{2}.$$

This x satisfies $1/2 < x < 1$. Then we should take y satisfying $Y = yS$ and to check that $x^2 + y^2 = 1$. The point $x + iy$ is the only one of $\overline{D}$ whose image is equal to Z. This point belongs to the arc of $\overline{D}$ that passes through 1 and whose endpoints are $\exp(\pm\pi i/3)$. The points Z of another two petals can be treated similarly. □

Proposition 3. *Let* $v = \exp(it)$ *and* $w = \exp(it/3)$ *with some* $t \in \mathbb{R}$. *The image of* $\overline{D}$ *under the function*

$$z \mapsto \frac{1+vz^3}{2z} \tag{11}$$

is the curve $C' = wC$ *obtained by rotation of* C *around* 0 *through the angle* $t/3$.

Proof. As the point z runs over $\overline{D}$, the point wz runs over $\overline{D}$ and the point

$$\frac{1+vz^3}{2z} = w\frac{1+(wz)^3}{2(wz)}$$

runs over $C' = wC$ by Proposition 2, as required. $\square$

6. Distribution of the sums (3).

Now we are ready to give a theoretical explanation to the shown above distribution of the sums $E_p(\psi)$, see Section 1. Compare the formula (8) in Proposition 1 with the formula (11) in Proposition 3. It follows from the Gauss formulas for the quadratic sums that the only possible values of the coefficient Q in (8) are either 1, ω, ω^2 or i, $i\omega$, $i\omega^2$ according to $p \equiv 1 \bmod 4$ or $p \equiv 3 \bmod 4$. Here $\omega = \exp(2\pi i/3)$, so that 1, ω, ω^2 are all possible values of $\psi(2)$. We find easily that any point $E_p(\psi)$ in (3) belongs to some of six curves wC obtained by rotation of C around 0 through the angles $w = 0$, $\pm 2\pi/9$, $\pi/6$, $\pi/2 \pm \pi/9$, so that

$$E_p(\psi) \in \tilde{C} = \bigcup_w wC. \tag{12}$$

This is consistent with the figure presented in Section 1.

It remains an open question whether the countable set of all the points $E_p(\psi)$ is everywhere dense in the curve $\tilde{C}$ in (12). It seems likely that *the set of all the points $E_p(\psi)$ is everywhere dense in $\tilde{C}$* (with the topology induced by the canonical topology in $\mathbb{C}$). In the meantime, the points T in Proposition 1 forms everywhere dense subset in $\overline{D}$. That is known from research of the cubic Gauss sums related to the Kummer problem [5].

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Partially Ordered Derivations in Sequent Calculi for Nonstandard Logics

Alexander Sakharov

1. Introduction

Nonstandard logics have become the center of modern studies in mathematical logic because these logics have numerous applications especially in the area of artificial intelligence. These logics have nonstandard logical connectives requiring peculiar axioms or inference rules. These logics may also include nonlogical axioms specifying properties of concrete predicates and functions.

Sequent calculi are perhaps the most common logical formalism [4]. This formalism is simple and versatile. In addition to inference rules for logical connectives and quantifiers, sequent calculi can incorporate arbitrary axioms. Sequent calculi support both top-down and bottom-up proof search. Nonetheless, sequent calculi commonly lack normal forms of derivations. Many rule chains are permutable [2]. Reducing derivation choices arising from rule permutability is a challenging long-standing problem. For some standard logics, permutation-free sequent calculi have been crafted [3, 6, 1]. But these results have not been generalized.

The main result of this work is that inference in a variety of sequent calculi remains complete if it is restricted to derivations in which some pairs of consecutive inference rules are ordered. Additionally, weakening and contraction rules are merged with other rules which reduces choices during inference. These results are applicable to sequent calculi with non-standard inference rules and additional axioms including nonlogical ones. This research is a step forward in the quest for normal forms in various sequent calculi, in particular, in applied calculi with multiple axioms in which cut is heavily used [5].

2. Sequent Calculi

We use standard logical terminology [4]. Upper-case Latin letters are metavariables denoting formulas in inference rules and axioms. Upper-case Greek letters are

metavariables denoting formula multisets. Usually, the outcome of inference is sequents of the form $\vdash G$ where formula G is called a goal. A calculus is called consistent if sequent $\vdash$ is not derivable.

Metaformulas are built from formula metavariables, substitutions, logical connectives, and quantifiers. Expressions having the following forms are also called metaformulas: $A\theta$ and $A^*\theta$ where $\theta = \{x_1/t_1, ..., x_k/t_k\}$ is a substitution. The expression $A^*\theta$ means that the formula matching metavariable A is the only formula in its sequent where variables of the substitution θ occur. Multiset metavariables and expressions of the form $\diamond\Pi$ are called metasets. The expression $\diamond\Pi$ denotes the multiset $\{\diamond A | A \in \Pi\}$ where $\diamond$ is a unary connective. Sequents in logical rules are comprised of metaformulas and metasets.

Inference rules in sequent calculi are split into structural and logical. The structural rules are essentially universal for all of the calculi whereas logical rules vary. Logical axioms are comprised of metaformulas and possibly formulas. Nonlogical axioms may contain formulas only. Axioms that are not purely logical are also crucial for calculi specifying applied logics. The equality axiom and variants of the induction axiom are examples of such axioms. We assume that any axiom has no instances in which there are identical formula in the antecedent or in the succedent.

Definition 1. *If all metaformulas/metasets containing the same metavariable are identical, they are called context. All other metaformulas/metasets from the conclusion are called principal. All other metaformulas/metasets from premises are called active. Formulas matching metaformulas/metasets are also called principal, active, context as their respective metaformulas/metasets.*

Definition 2. *A multi-premise logical inference rule is called multiplicative if no context metavariable from one premise occurs in the other premises of the rule. A multi-premise logical inference rule is called additive if every context metavariable occurs in all premises of the rule.*

Definition 3. *A logical rule is called clear if*
- *Every metavariable from any of its premises also occurs in the conclusion.*
- *No multiset metavariable occurs in both antecedents and succedents.*
- *It is single-premise, multiplicative, or additive.*
- *It has one principal metaformula and no principal metasets.*
- *Every premise has one active metaformula if the rule has multiple premises.*
- *Every active formula is a subformula of the principal formula or a result of applying a substitution to such subformula.*
- *The context of any premise antecedent or succedent, if present, is a single multiset metavariable.*
- *There are no constraints on the application of this rule except for those given by metaformulas.*

Definition 4. *A clear rule is called simple if it has a single premise with one active metaformula or it is multiplicative.*

3. Contraction and Weakening Merging

Let $[\Gamma]$ denote the result of applying zero or more possible contractions to multiset Γ. If a calculus does not include contraction, then $[\Gamma] = \Gamma$. If a calculus includes both weakening and contraction, then the $[\ ]$ operation eliminates all duplicate formulas. If a calculus includes contraction and does not include weakening, then this operation is non-deterministic, i.e. none, some, or all contractions are applied.

Let us modify the conclusion of cut and all logical rules by applying $[\ ]$ to both the antecedent and the succedent of the conclusion of cut and all logical inference rules. The calculus obtained from calculus L by applying $[\ ]$ is denoted L'.

Proposition 1. *For every sequent calculus L, any derivation can be transformed into a L' derivation with the same endsequent and vice versa.*

Theorem 1. *The contraction rules are admissible in any L' sequent calculus.*

Let us modify any calculus L' with weakening. For any single-premise clear rule having more than one metaformula, let us add logical rules to this calculus. Each additional rule is obtained by removing one or more metaformula but not all of them from the premise. Also, additive clear rules are replaced by multiplicative rules if this calculus has both contraction and weakening. The modified calculus will be denoted L". For any calculus L' without weakening, L" is identical to L'.

Proposition 2. *For every sequent sequent calculus L, any L' derivation can be transformed into a L" derivation with the same endsequent and vice versa.*

Theorem 2. *For every consistent sequent calculus L, any L derivation of sequent $\vdash G$ can be transformed into such L" derivation with the same endsequent and without the contraction rules that every weakening rule is either followed by another weakening rule or by a logical rule that is additive or is not clear.*

4. Partially Ordered Derivations

Definition 5. *Strict order relation $\succ$ on formulas and terms is called a simplification order if it satisfies the following conditions:*
- *there is no infinite sequence of formulas $F_0 \succ F_1 \succ ...$*
- *if L/l is a particular formula/term occurrence in formula E, formula F is obtained from E by replacing this occurrence with formula/term R/r, and $L \succ R/l \succ r$, then $E \succ F$*
- *if R/r is a subformula/subterm of formula/term L/l, then $L \succ R/l \succ r$*
- *if $L, R/l, r$ are formulas/terms and $L \succ R/l \succ r$, then $L\theta \succ R\eta/l\theta \succ r\theta$ for any substitutions θ and η*

Definition 6. *Formula A is maximal (minimal) with respect to the set of formulas $\mathcal{S}$ if $B \succ A$ ($A \succ B$) does not hold for any other formula $B \in \mathcal{S}$.*

Theorem 3. *For any consistent sequent calculus L and simplification order $\succ$ on its formulas and terms, any L derivation of sequent $\vdash G$ can be transformed into a L" derivation with the same endsequent, without the contraction, with weakening rules satisfying Theorem 2, and such that the following holds for any two consecutive inference rules:*
1) If both rules are cut, then the upper cut formula is maximal with respect to the lower cut formula.
2) If the upper rule is simple and the lower rule is cut, then the cut formula is principal in the upper rule.
3) If both rules are simple, then the principal formula of the lower rule is maximal with respect to the principal formula of the upper rule.

Theorem 4. *For every consistent sequent calculus L with both weakening and contraction, Theorem 3 holds even if the word 'simple' is changed for the word 'clear'.*

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Remarks on Tarski's elimination

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Abstract. Tarski's elimination theorem (stating that, in the structure of reals, any first-order formula in the language of ordered fields is equivalent to a quantifier-free formula) is fundamental for several areas including computer algebra. In this talk, we discuss some earlier and some newer facts about possible extensions and applications of Tarski's theorem and some other results of interest for symbolic and numeric computations. In particular, we concentrate on relationships of this topic with the theory of computable models and constructive mathematics.

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On the Length of an Unsatisfiable Conjunction

Alexandr V. Seliverstov

Abstract. We consider a lower bound on the length of a conjunction of some propositional formulae such that every unsatisfiable conjunction contains an unsatisfiable subformula. In particular, our method is applicable for 2-CNF, symmetric 3-CNF, and conjunctions of voting functions in three literals. The proof is algebraic. So, a large conjunction can be reduced in a nondeterministic way. This reduction improves some upper bounds on the computational complexity.

Introduction

Let us denote by $\bot$ and $\top$ two Boolean constants. For two integers $\alpha < \beta$, the set α-or-β-in-SAT consists of CNFs such that, for some $(\bot, \top)$-evaluation, every clause contains either exactly α or exactly β true literals.

For $k < \alpha < \beta$, the set α-or-β-in-SAT contains no k-CNF.

For $k < \beta$, a k-CNF φ belongs to 1-or-β-in-SAT iff φ belongs to 1-in-k-SAT. This set consists of k-CNF such that, for some $(\bot, \top)$-evaluation, every clause contains exactly one true literal.

A 2-CNF φ belongs to 1-or-2-in-SAT iff φ is satisfiable.

A 3-CNF φ belongs to 1-or-2-in-SAT iff φ belongs to NAE-3-SAT. This set consists of 3-CNF such that, for some $(\bot, \top)$-evaluation, every clause contains both true and false literals. A 3-CNF $\varphi(p_1, \ldots, p_n)$ belongs to NAE-3-SAT iff $\varphi(p_1, \ldots, p_n) \wedge \varphi(\neg p_1, \ldots, \neg p_n)$ is satisfiable.

It is well known that both problems whether a given 3-CNF belongs to NAE-3-SAT and whether it belongs to 1-in-3-SAT are NP-complete. The formula length serves as a natural parameter for estimating the runtime. Therefore, the possibility of replacing the original 3-CNF with a subformula is interesting. On complexity upper bounds refer to [1]

Let us fix arbitrarily small $\varepsilon > 0$. If the number of clauses is less than the threshold $(1 - \varepsilon)n$, then almost all 2-CNFs in n variables are feasible. On the

contrary, if the number of clauses is greater than the threshold $(1+\varepsilon)n$, then almost all 2-CNFs in n variables are unsatisfiable [2, 3].

Such a threshold is usually called a phase transition and is also shown by random samples of several types of formulae.

A k-CNF is called d-regular when each clause contains exactly k literals and each variable appears in exactly d clauses. For any sufficiently large number k, the membership of a random d-regular k-CNF to the set NAE-k-SAT undergoes a phase transition with increasing d at some critical value d_k, which depends on k. As the number of variables increases, for $d < d_k$, the fraction of d-regular k-CNF belonging to NAE-k-SAT tends to one. For $d > d_k$, this fraction tends to zero [4]. A similar result is known for d-regular k-CNFs having exactly two true literals per clause [5].

Next, let us consider a bound that holds for all, and not just almost all, formulae in consideration. In the proof, we replace a CNF with a system of algebraic equations depending on auxiliary variables, one per clause. So, the original satisfiability problem is reduced to the problem of the incidence of an affine subspace defined by a system of linear equations and a set of points with coordinates from the set $\{0,1\}$. From a geometric point of view, removing a clause corresponds to a projection onto some coordinate subspace [6]. In turn, the projection corresponds to eliminating the auxiliary variable. The solution to a system of equations in which each variable takes values from the set $\{0,1\}$ is called a $(0,1)$-solution. The existence of a $(0,1)$-solution to a system of linear equations over the field of rational numbers is also a well-known computationally difficult problem [7].

1. Results

Theorem 1. *Given a system of m linear equations of the type*

$$y_j = \ell_j(x_1, \ldots, x_n)$$

in $m+n$ variables $y_1, \ldots, y_m, x_1, \ldots, x_n$, where ℓ_j denotes a linear function over a certain field. If this system has no $(0,1)$-solution and the inequality $m > 2n+2$ holds, then there is an equation in the system such that the subsystem obtained by removing this equation also has no $(0,1)$-solution.

Theorem 2. *Given a propositional CNF $\varphi(p_1, \ldots, p_n)$ with m clauses in n variables. If φ does not belong to α-or-β-in-SAT and the inequality $m > 2n+2$ holds, then there exists a CNF that does not belong to α-or-β-in-SAT and is obtained by removing some clause from φ.*

Proof. Let us define by induction a function f that maps a clause to a pseudo-Boolean linear function over the field of rational numbers. $f(\bot) = 0$ and $f(\top) = 1$. For variables $f(p_i) = x_i$. For the negation $f(\neg p_i) = 1 - x_i$. Next, the jth clause $\varphi_j = \ell_1 \vee \cdots \vee \ell_k$ corresponds to the expression $f(\ell_1) + \cdots + f(\ell_k) - \alpha - (\beta - \alpha)y_j$, where the new variable y_j appears only once. Note that $\alpha \neq \beta$.

Next, the conjunction of clauses φ_j corresponds to the system of linear equations $f(\varphi_j) = 0$, where $1 \leq j \leq m$. This system depends on $m + n$ variables. All m equations are linearly independent, since each one depends on its own auxiliary variable. Every $(0, 1)$-solution to the system corresponds to a $(\bot, \top)$-evaluation of propositional variables such that in each clause either exactly α or exactly β literals are true. Conversely, for such a $(\bot, \top)$-evaluation of propositional variables, there is a $(0, 1)$-solution to the system of linear equations. If $p_i = \bot$, then $x_i = 0$. If $p_i = \top$, then $x_i = 1$. If α literals are satisfied in the jth clause, then $y_j = 0$. If β literals are satisfied, then $y_j = 1$. According to Theorem 1, if the system has no $(0, 1)$-solution, then this property is preserved after eliminating some additional variable y_j, i.e., after removing the jth clause from φ.

2. Discussion

The bound on the number of clauses in an unsatisfiable subformula in 2-CNF is close to optimal. There is an unsatisfiable 2-CNF with $m = 2n$ clauses in n variables for which the subformula obtained by removing any clause is satisfiable. An example is 2-CNF

$$(\neg p_1 \vee p_2) \wedge (p_1 \vee \neg p_2) \wedge \cdots \wedge (\neg p_{n-1} \vee p_n) \wedge (p_{n-1} \vee \neg p_n) \wedge (p_n \vee p_1) \wedge (\neg p_n \vee \neg p_1),$$

where each variable enters twice positively and twice negatively. This 2-CNF is equivalent to the conjunction of formulae expressing the equivalence of the variables p_j and p_{j+1} for $j < n$, as well as the equivalence of the variable p_n and the negation of the variable p_1. It is impossible. But removing one clause from this 2-CNF corresponds to replacing some equivalence with an implication. The resulting formula is satisfiable for some $(\bot, \top)$-evaluation for which the antecedent of this implication is false.

Note that the bound on reducing the number of clauses lies in the segment where almost all 2-CNFs are unsatisfiable [2]. So, the possibility of removing some clause from an unsatisfiable 2-CNF does not impose unexpected additional restrictions on the unsatisfiable subformula. Also, the results obtained for other classes of formulae provide an upper estimate for the phase transition boundary, when it exists.

It is possible to consider conjunctions of formulae of another type. Let us denote by $\mathrm{maj}(p_1, p_2, p_3)$ the voting function (majority). Its value is equal to the most frequently occurring among $(\bot, \top)$-values of the propositional variables p_1, p_2, and p_3. For literals ℓ_{ij} the conjunction $\wedge_j \mathrm{maj}(\ell_{1j}, \ell_{2j}, \ell_{3j})$ is satisfiable if and only if 3-CNF $\wedge_j (\ell_{1j} \vee \ell_{2j} \vee \ell_{3j})$ belongs to the set 2-or-3-in-SAT. Therefore, Theorem 2 is applicable to formulae of this type.

Conclusion

Reducing the formula length leads to a decrease in some estimates of computational complexity. However, it does not guarantee a reduction in the running time of some heuristic algorithms. On the other hand, the main result is a pure existence theorem, which does not provide a fast algorithm for finding an unsatisfiable subformula. The result is consistent with the hypothesis that the satisfiability problem is computationally hard in the worst-case.

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The H-Transform in Wolfram Mathematica and Its Particular Cases

Oleg Marichev and Elina Shishkina

Abstract. Nowadays, the Fox H-function is the most important function, having accumulated almost all named functions, and yet it is widely unknown. This complicated function includes four groups of parameters inside of gamma functions, which allows it to accumulate about 150 named functions of very different types: power, exponential, logarithmic, discontinuous, etc. Each of these functions or their combinations can be a kernel of integral transform. The kernels of classical integral transforms (Laplace, Mellin, Fourier, Hilbert, Hankel and others) are the cases of Fox H-function. Each transform can be applied to the Fox H-function or its particular cases, which allows us to evaluate approximately 80% of integrals, presented in handbooks nowadays. Our talk is devoted to the Fox H-transform and its particular cases.

1. Fox H-Function in Wolfram Mathematica

Fox H-functions (introduced in [1]) are versatile special functions that enable a unified, coherent approach to various areas, such as integral transforms and fractional calculus. The Fox H-function is defined by a Mellin-Barnes type integral with an integrand involving products and quotients of Euler gamma functions. It generalizes most known elementary and special functions, allowing nearly all integral transforms to be expressed as H-transforms. Detailed information about Fox H-functions can be found in [2, 3, 4, 5]. Some applications of H-functions are given in [6, 7].

H-function is defined by

$$\mathrm{H}_{p,\,q}^{m,n}[z] \equiv \mathrm{H}_{p,q}^{m,n}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int\limits_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n}(s) z^{-s} ds,\ z\neq 0, \qquad (1)$$

where

$$\mathcal{H}_{p,\,q}^{m,n}(s) \equiv \mathcal{H}_{p,\,q}^{m,n}\left[\left.\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right|s\right] = \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}, \quad (2)$$

$m,n,p,q\in\mathbb{N}$, $m\le q$, $n\le p$, $\alpha_i,\beta_j\in\mathbb{R}$, $\alpha_i>0$, $1\le i\le p$, $\beta_j>0$, $1\le j\le q$.

Here $\mathcal{L}$ is a specially chosen infinite contour, described in [2, 3, 4, 5]. In (2) an empty product, if it occurs, being taken to be one.

Fox H-function was implemented in the Wolfram Mathematica system as

`FoxH[{{{`a_1`,`α_1`},...,{`a_n`,`α_n`}},{{`a_{n+1}`,`α_{n+1}`},...,{`a_p`,`α_p`}}},`
`{{{`b_1`,`β_1`},...,{`b_m`,`β_m`}},{{`b_{m+1}`,`β_{m+1}`},...,{`b_q`,`β_q`}}},z]`.

and introduced commands that allow users to transform many given function into an H-function or G-functions and back (if possible) [8, 9]. For example, commands `FoxHReduce[expr,z]`, `MeijerGReduce[expr,z]`, `FunctionExpand` can be used for these purposes, but currently it is better to use functions associated with the specified resource: `ResourceFunction["MeijerGForm"][expr,z]` and `ResourceFunction["FoxHForm"][expr,z]`.

2. Calculation of Mellin transform of product of two H-functions

For calculation of integrals by method described in the book of O.I.Marichev [10] we can use Mellin transform defined by the formula

$$K^*(s)=\int\limits_0^\infty K(x)x^{s-1}dx, \qquad s=\gamma+i\tau$$

is used. Then inverse Mellin transform is presented by integral

$$K(x)=\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty}K^*(s)x^{-s}ds, \qquad x>0, \qquad \mathrm{Re}(s)=\gamma$$

and the Mellin convolution of functions $K_1(x)$ and $K_2(x)$ for $x>0$ can be written thought relation

$$(K_1\circ K_2)(x)=\int\limits_0^\infty K_1(t)K_2\left(\frac{x}{t}\right)\frac{dt}{t}=K(x).$$

We have

$$(K_1\circ K_2)^*(s)=K_1^*(s)K_2^*(s).$$

In 1966, Gupta K.C. and Jain U.C. studied two integrals of Mellin convolution type derived from the product of two H-functions, and provided conditions for their convergence. These integrals are studied on pages 46-47 of the book [3] and on page 60 of the book [11]. Here, we consider a more generic integral along with

a more comprehensive set of conditions for convergence, which encompass those two integrals as specific cases.

The most general formula includes a lot of integrals is

$$\int_0^\infty t^{c-1}\mathrm{H}_{p,q}^{m,n}\left[u_2 t^{r_2}\middle|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]\cdot\mathrm{H}_{P,Q}^{M,N}\left[u_1 t^{r_1}\middle|\begin{matrix}(c_i,\gamma_i)_{1,P}\\(d_j,\delta_j)_{1,Q}\end{matrix}\right]dt=\theta\left(-\frac{r_1}{r_2}\right)\frac{u_2^{-\frac{c}{r_2}}}{|r_2|}\times$$

$$\times\mathrm{H}_{p+P,q+Q}^{m+M,n+N}\left[u_1u_2^{-\frac{r_1}{r_2}}\middle|\begin{matrix}(c_i,\gamma_i)_{1,N},(\mathfrak{A}_i,\mathfrak{B}_i)_{N+1,N+n},\left(\overline{\mathfrak{A}}_j,\overline{\mathfrak{B}}_j\right)_{n+1,p},(c_{j+N-p},\gamma_{j+N-p})_{p+1,p+P-N}\\(d_{i-m},\delta_{i-m})_{m+1,m+M},(\mathfrak{C}_i,\mathfrak{D}_i)_{1,m},\left(\overline{\mathfrak{C}}_j,\overline{\mathfrak{D}}_j\right)_{Q+1,Q+q-m},(d_j,\delta_j)_{1+M,Q}\end{matrix}\right]+$$

$$+\theta\left(\frac{r_1}{r_2}\right)\frac{u_2^{-\frac{c}{r_2}}}{|r_2|}\times$$

$$\times\mathrm{H}_{q+P,p+Q}^{n+M,m+N}\left[u_1u_2^{-\frac{r_1}{r_2}}\middle|\begin{matrix}(c_i,\gamma_i)_{1,N},(\mathfrak{E}_i,\mathfrak{F}_i)_{N+1,N+m},\left(\overline{\mathfrak{E}}_j,\overline{\mathfrak{F}}_j\right)_{m+1,q},(c_{j+N-q},\gamma_{j+N-q})_{q+1,q+P-N}\\(d_{i-n},\delta_{i-n})_{n+1,n+M},(\mathfrak{G}_i,\mathfrak{H}_i)_{1,n},\left(\overline{\mathfrak{G}}_j,\overline{\mathfrak{H}}_j\right)_{Q+1,Q+p-n},(d_j,\delta_j)_{1+M,Q}\end{matrix}\right],\tag{3}$$

where θ is the Heaviside function

$$\theta(x)=\begin{cases}1, & x\geq 0\\ 0, & x<0\end{cases},$$

$\mathfrak{A}_i=a_{-N+i}+\frac{c\alpha_{-N+i}}{r_2}$, $\mathfrak{B}_i=-\frac{r_1\alpha_{-N+i}}{r_2}$, $\overline{\mathfrak{A}_j}=a_j+\frac{c\alpha_j}{r_2}$, $\overline{\mathfrak{B}_j}=-\frac{\alpha_j r_1}{r_2}$, $\mathfrak{C}_i=b_i+\frac{c\beta_i}{r_2}$, $\mathfrak{D}_i=-\frac{r_1\beta_i}{r_2}$. $\overline{\mathfrak{C}}_j=b_{m-Q+j}+\frac{c\beta_{m-Q+j}}{r_2}$, $\overline{\mathfrak{D}}_j=-\frac{r_1\beta_{m-Q+j}}{r_2}$, $\mathfrak{E}_i=1-b_{i-N}-\frac{c\beta_{i-N}}{r_2}$, $\mathfrak{F}_i=\frac{r_1\beta_{i-N}}{r_2}$, $\overline{\mathfrak{E}}_j=1-b_j-\frac{c\beta_j}{r_2}$, $\overline{\mathfrak{F}}_j=\frac{r_1\beta_j}{r_2}$, $\mathfrak{G}_i=1-a_i-\frac{c\alpha_i}{r_2}$, $\mathfrak{H}_i=\frac{r_1\alpha_i}{r_2}$, $\overline{\mathfrak{G}}_j=1-a_{n-Q+j}-\frac{c\alpha_{n-Q+j}}{r_2}$, $\overline{\mathfrak{H}}_j=\frac{r_1\alpha_{n-Q+j}}{r_2}$.

The integral transform of the form (see [5], p. 71, formula (3.1.1))

$$(\mathbf{H}f)(x)=\int_0^\infty \mathrm{H}_{p,q}^{m,n}\left[xt\middle|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]f(t)dt$$

where $\mathrm{H}_{p,q}^{m,n}$ is the H-function defined in (1), is called the H-transform of a function $f(t)$.

H-Transform can be calculated by (3). The H-Transform is one of the most general integral transforms today is the H-Transform, which uses the Fox's H-function as a kernel. Anatoly A. Kilbas and Megumi Saigo wrote a book [5] which is fully devoted to the H-Transforms.

3. Examples

Formula (3) can be used for evaluation integral from product of power, Bessel J_ν and Mittag–Leffler functions with arbitrary power arguments. This result can be interpreted as values of Mellin transform from corresponding product of J_ν and Mittag–Leffler functions or as Hankel transform from product power and Mittag–Leffler functions or Mittag–Leffler transform from product power and Bessel J_ν

functions. Below we give value of this integral:

$$2\int_0^\infty t^{c-1}J_\nu\left(2u_2t^{r_2}\right)E_{\alpha,a}\left(-u_1t^{r_1}\right)dt=$$

$$=\theta\left(-\frac{r_1}{r_2}\right)\frac{u_2^{-\frac{c}{r_2}}}{|r_2|}\mathrm{H}_{1,4}^{2,1}\left[u_1u_2^{-\frac{r_1}{r_2}}\middle|\begin{matrix}(0,1)\\ \left(\frac{c}{2r_2}+\frac{\nu}{2},-\frac{r_1}{2r_2}\right),(0,1),(1-a,\alpha),\left(\frac{c}{2r_2}-\frac{\nu}{2},-\frac{r_1}{2r_2}\right)\end{matrix}\right]+$$

$$+\theta\left(\frac{r_1}{r_2}\right)\frac{u_2^{-\frac{c}{r_2}}}{|r_2|}\mathrm{H}_{3,2}^{1,2}\left[u_1u_2^{-\frac{r_1}{r_2}}\middle|\begin{matrix}(0,1),\left(1-\frac{c}{2r_2}-\frac{\nu}{2},\frac{r_1}{2r_2}\right),\left(1-\frac{c}{2r_2}+\frac{\nu}{2},\frac{r_1}{2r_2}\right)\\ (0,1),(1-a,\alpha)\end{matrix}\right].$$

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Pedestals: Polynomial matrices with polynomial eigenvalues

Senya Shlosman

Abstract. A construction will be presented that maps each poset X to a square matrix M^X. Its matrix elements are enumerated by pairs of linear orders P, Q on X, and are monomials of variables x_i. Our main result is that the eigenvalues of M^X are polynomials in x_i with integer coefficients.

In collaboration with Richard Kenyon, Maxim Kontsevich, Oleg Ogievetsky, Cosmin Pohoata and Will Sawin.

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Some geometric properties of shifted Young diagrams of maximum dimensions

Vasilii Duzhin and Egor Smirnov-Maltsev

The problem of finding Young diagrams of straight shape with large dimensions, i. e. those with a large number of Young tableaux, was previously studied in [1, 2]. This research is a continuation of works [3, 4] in which a similar problem was investigated for strict Young diagrams. The approaches used in the above works have in common that the original diagram λ_n is transformed into a new one λ'_n of the same size n such that the dimension of λ'_n is greater than the dimension of λ_n. Here we present two new methods for finding strict Young diagrams of larger dimensions which are based on the same idea.

Consider a strict Young diagram $\lambda = (x_1, x_2, ..., x_s)$ of size n. The first method is to move a box from the $(i-1)$th column to the ith one. Thus, the resulting diagram of size n will be $\lambda' = (x_1, ..., x_{i-2}, x_{i-1} - 1, x_i + 1, x_{i+1}, ..., x_s)$. We prove that the dimension of λ' is greater than the dimension of λ if the following condition is met:

$$x_{i-1} - x_i \geq \frac{3 + \sqrt{9 + 8x_i}}{2}. \tag{1}$$

Note that in the case when inequality (1) is not satisfied, the above transformation can lead to both a decrease in dimension and an increase in it.

Consider a strict Young diagram $\lambda = (x_1, x_2, ..., x_s)$. The *tail* of length t is the last t columns of λ. Tail size $\tilde{n}$ is the number of boxes in these t columns. The second method is to change a tail of length t of size $\tilde{n}$ to a tail of length $t+1$ of the same size. The resulting Young diagram is $\lambda' = (x_1, ..., x_{s-t}, y_1, ..., y_{t+1})$.

During the research, it was hypothesized that the dimension of the diagram λ is less than the dimension of the diagram λ' if 2 conditions are met:

1. The dimension of a diagram $\lambda_t = (x_{s-t+1}, x_{s-t+2}, ..., x_s)$ is not greater than the dimension of a diagram $\lambda'_t = (y_1, y_2, ..., y_{t+1})$.
2. There is no $t_1 < t$ such that there is a diagram $\lambda'_{t_1} = (\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_{t_1+1})$ whose dimension is not less than the dimension of the diagram

$$\lambda_{t_1} = (x_{s-t_1+1}, x_{s-t_1+2}, ..., x_s).$$

This conjecture was proven for t equal to 1 and 2. However, already for t equal to 7 a counterexample to it was found. This work was supported by grant RSF 22-21-00669.

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Paradoxes of Game Semantics

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Traditionally, game semantics was developed to confirm or complement some existing semantics of logical systems. Usually it was established that the Verifier has a winning strategy in the game associated with the formula A when A is derivable in some corresponding deductive system or true in a model-theoretical sense.

It is well known that the restrictions on the classes of strategies used by Verifier and Falsifier may break this relationship, so the question of "admissible" restrictions has been studied: for example whether it is possible to consider only computable strategies and still obtain the same "adequate" semantics.

The games with backward moves (i.e., where the Verifier is permitted to "replay" if it comes to a "bad" position) were proposed to improve correspondence between classical provability and existence of recursive winning strategies [1, 4, 5].

Proposition 9 ([4]). For every first order language L, every decidable L-structure M and every set Σ of axiom sequents validated by semantic games on M, any proof ϕ in $\vdash \Sigma$ is validated by games on M (i.e., one may construct a winning strategy for Verifier in these games).

We study the effect of restrictions from a different point of view: how much the semantics may be deformed due to some natural asymmetry between players. We consider various known kinds of semantic games, for example, games with backward moves. It turns out that in many semantic games, in particular, the games with backward moves, and under some conditions the Verifier may win even if the formula is not true in ordinary logical semantics.

The following general result may be used to describe paradoxal situations of this kind.

- Assume that Verifier can compute any general recursive function and knows (and can compute) a universal function $U(x, y)$ for the strategies f of Falsifier, i.e., every $f = U(k, -)$ for some k.
- Assume that if Verfier knows the strategy of Falsifier it can win. That is, Verifier can compute another function $W(x, y)$ such that $v_k = W(k, -)$ wins against $f_k = U(k, -)$.

- Here we may assume that $y \in N$ are the codes of partial plays (they may include backward moves).
- **Theorem.** In the conditions listed above the Verifier has a recursive strategy that wins against any strategy of the Falsifier.

  Two significant examples (not based on this theorem) are considered as well. **Example 1.**Let us consider the following (false) formula:

$$\phi = \exists x \forall y.(y \leq \mathcal{A}(x)).$$

Let here $\mathcal{A}$ be the Ackermann's function.

- In games without backward moves, since for Verifier the history is empty, the strategies of Verifier are just natural numbers (values of x). The strategies of Falsifier are functions $f : N \to N$. If Verifier chooses x then the answer of Falsifier is $f(x)$. Let the strategies of Falsifier be limited to the class of PR functions.

  The formula is *false* on N, but there is no winning strategy for Falsifier because $\mathcal{A}$ grows faster than any PR function. There is no winning strategy for Verifier as well, because for any x there exists some PR function f such that $f(x) > \mathcal{A}(x)$.
- Let us consider the games with backward moves. Now the strategies of Verifier are functions on histories (not empty after replay). And the strategy that takes the values $0, ..., n...$ (during n-th replay) is winning for Verifier against all PR-strategies of Falsifier because $\mathcal{A}(n)$ will outgrow any PR-function. (If Falsifier has memory, its strategies are PR functions on histories, but histories may be coded by numbers of the lists of values. Enumeration is PR, and with $0, ..., n...$ as values of y the resulting function is also PR.)

  Example 2. (A more extreme example.)
- **Generalized Ramsey theorem.**
- Recall [2]: A set of integers, S, is large if S is non-empty, and (if s is its least element) S has at least s elements.
- A being a set, $b \in N$, $A^{[b]}$ denotes the set of all subsets of A of cardinality b. If $F : A^{[b]} \to X$, a subset B of A is homogeneous for F if F is constant on $B^{[b]}$. Each integer n is, as usual, identified with the set of integers less than n. For $a, b, c \in N, a, b, c > 0$, $a \to (large)^b_c$ means that for every map $F : a^{[b]} \to c$ there is a large homogeneous set for F of cardinality greater than b (this relation is PR).
- $(\forall b, c \geq 1)(\exists a \geq 1)(a \to (large)^b_c)$ is the generalized Ramsey's theorem. It is not provable in Peano Arithmetic, but provable in second order arithmetic [2].
- We may consider the game associated with its classical negation

$$(\exists b, c \geq 1)(\forall a \geq 1)\neg(a \to (large)^b_c)$$

  This formula is *false* but the value of a (that gives a counterexample) grows faster that any function that is provably general recursive in Peano Arithmetic.

- Thus, like in Example 1, in games without backward moves there is no winning strategy for Falsifier in the class of provably general recursive functions.
- In games with backward moves the simple strategy for Verifier (take the values $0, 1, ...$ for subsequent replays) wins against any provably general recursive strategy for Falsifier.

Conclusion. It may be argued that completely automated verification (not checked by humans) is related to semantic games in Game Semantics. This underlines the relevance of "perversions" considered above.

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Periodic orbits in the shape space

Vladimir Titov

Abstract. Periodic orbits of the general three bodies problem with linear symmetry and $2-1$-symmetry are considered in the shape space. The moment of inertia of the orbits under consideration does not change much during the period, so it is enough to look these orbits in the shape sphere or in the space of angular coordinates.

Introduction

It is shown in [1] that the finite symmetry groups of the general planar three-body problem are exhausted by 10 groups. Two of these groups served as the basis for the search for symmetric periodic solutions [2]. The found trajectories can be mapped into the shape space, such transformation is unique up to the rotation of the original barycentric system. Since the distance from the origin varies little in the shape space for these trajectories (within $\pm 10\%$), then for qualitative analysis we can limit ourselves to their projection onto the shape sphere.

1. The shape space

The shape space is the space of congruent triangles, configurations of the general three body problem. This space is obtained by reducing the configuration space of the problem by translations, and then by reducing by rotations. The first reduction is performed by moving to the Jacobi coordinates

$$\begin{aligned} \mathbf{Q}_1 &= \mathbf{r}_2 - \mathbf{r}_1, \\ \mathbf{Q}_2 &= \mathbf{r}_3 - \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}, \end{aligned}$$

the second is the Hopf map: considering the coordinates $\mathbf{Q}_1$ and $\mathbf{Q}_2$ as points of a complex space, we introduce

$$\begin{aligned} \xi_1 &= \frac{1}{2}\mu_1|\mathbf{Q}_1|^2 - \frac{1}{2}\mu_2|\mathbf{Q}_2|^2, \\ \xi_2 + i\xi_3 &= \sqrt{\mu_1\mu_2}\,\mathbf{Q}_1\bar{\mathbf{Q}}_2. \end{aligned} \tag{1}$$

The three-dimensional space $\Xi = (\xi_1, \xi_2, \xi_3)$ is the space of oriented congruent triangles, each point of this space represents a class of such oriented congruent triangles. This space is called the *shape space*, and it is in this space that we will study the properties of the solutions to the three-body problem under consideration.

TABLE 1. Orbits with line symmetry

$m_1 = 0.99$, $m_2 = 1.01$, $m_3 = 1.0$					
A	E	$\|J\|$	ω	$[I_{\min}, I_{\max}]$	Stab
11.42286	-0.606002	1.36301	1/4	[10.095,10.108]	+
12.04740	-0.639135	1.19429	1/3	[7.508,7.557]	+
12.06332	-0.639979	1.17690	1/3	[7.489,7.538]	+
12.07962	-0.640844	1.15915	1/3	[7.471,7.520]	+
13.15385	-0.697833	0.92132	1/2	[5.474,5.590]	+
13.15566	-0.697930	0.93926	1/2	[5.474,5.590]	+
13.15748	-0.698026	0.95484	1/2	[5.534,5.591]	+
14.06146	-0.745984	0.83708	1/3	[5.156,5.393]	-
14.08066	-0.747002	0.85327	1/3	[5.114,5.347]	-
14.09948	-0.748001	0.86909	1/3	[5.098,5.332]	-
14.55725	-0.772286	0.88706	1/4	[5.253,5.574]	-
16.64808	-0.883208	1.19288	1/3	[3.830,3.964]	-
16.76479	-0.889400	1.37020	1/3	[6.487,6.492]	+
17.80747	-0.944715	2.06327	1/4	[3.189,3.517]	-
20.59152	-1.09242	1.45497	1/3	[6.276,6.278]	+

In this space the moment of inertia I is given by simple expression:

$$I = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}. \tag{2}$$

that is, in the shape space it represents the distance of a point (ξ_1, ξ_2, ξ_3) to the origin of coordinates and, if you enter in space Ξ spherical coordinates ρ, φ, θ, then the coordinate ρ is naturally considered the size of a triangle, and φ, θ are angular variables that determine its shape. Naturally, the coordinate ρ coincides with the moment of inertia. In the shape space, all properties related to the moment of inertia of the system are naturally related to the size of the triangle. You can take the square root of ρ for the size of the triangle, then the unit of such value will coincide with the unit of the length. In any case, the points of a sphere of any fixed radius, for example, $\rho = 1$ or $\rho = 1/2$, will be responsible precisely for the shape of a triangle, such a sphere is called *shape sphere*, and the entire shape space is a cone above this sphere with a vertex at the point of triple collision $(0, 0)$. Thus, a point on the shape sphere is a class of similar triangles, all points on the ray in the shape space, beginning in the origin correspond to similar configurations of three bodies and differ only by size (see, for example [3].

Evidently, the equator of the sphere of shapes (and the plane $\xi_3 = 0$) correspond to collinear configurations ($\mathbf{Q}_1 \times \mathbf{Q}_2 = 0$). Thus, all points of double

TABLE 2. Orbits with $2-1$-symmetry and "figure-eight"

$m_1 = m_2 = 0.95$, $m_3 = 1.1$					
A	E	$\|C\|$	ω	$[I_{\min}, I_{\max}]$	Stab
10.61083	-0.562922	1.73204	1/5	[13.520,18.706]	+
11.87886	-0.630193	1.34061	1/3	[7.646,7.695]	+
12.41405	-0.658586	1.22094	2/5	[6.446,6.518]	+
12.43822	-0.850687	3.17929	1/5	[13.037,13.062]	+
13.13826	-0.697007	1.09433	1/2	[5.463,5.580]	+
14.90941	-0.790968	2.76171	1/3	[6.779,6.847]	+
16.03507	-0.850687	2.61695	2/5	[5.352,5.457]	+
16.57031	-0.879082	2.44831	1/3	[3.869,3.957]	-
17.61955	-0.934746	2.43060	1/2	[3.967,4.154]	-
19.78460	-1.049610	1.57727	1/3	[6.501,6.503]	+
21.89957	-1.161810	2.58582	1/3	[6.441,6.443]	+
25.74992	-1.366082	1.65989	1/3	[6.380,6.381]	+
27.53447	-1.460752	2.51159	1/3	[6.362,6.363]	+
$m_1 = m_2 = 1.05$, $m_3 = 0.9$					
12.20094	-0.647280	0.98928	1/3	[7.276,7.323]	+
15.79177	-0.837779	2.68412	1/3	[6.275,6.341]	+
16.61662	-0.881539	2.33447	1/3	[3.779,3.943]	-
Figure-eight: $m_1 = m_2 = m_3 = 1.0$					
24.37193	-1.29297	0	-	[1.973,1.982]	+

collisions at any mass values lie on the equator, in the space of shapes, three rays correspond to double collisions (in the plane of the equator).

2. Periodic orbits

To find a periodic solution to the three-body problem, it is enough to find a (local) minimizer of the functional of the action of the problem in the space of 2π-periodic functions, periodic solutions with a period other than 2π can be found using the scale symmetry and you can search for a solution in the form

$$\begin{aligned} x_j(t) &= C_x^0 + \sum_{i=1} C_{xi}^j \cos it + S_{xi}^j \sin it \\ y_j(t) &= C_y^0 + \sum_{i=1} C_{yi}^j \cos it + S_{yi}^j \sin it, \end{aligned} \tag{3}$$

where j is the body number.

Barutello et al. [1] showed that all finite symmetry groups of the Lagrangian action functional in the planar three-body problem contain only ten elements. The corresponding 2π-periodic solutions for three groups from this list, the ¡¡linear¿¿ group, $2-1$-choreographies and the dihedral group D_{12}, are obtained. These solutions are shown in the table 1 and table 2.

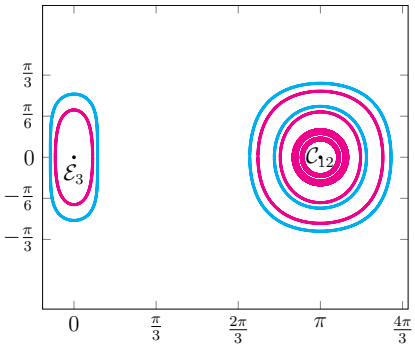

FIGURE 1. Trajectories with $2-1$-symmetry, $\omega = 1/3$

Due to the scale symmetry the solutions of the three-body problem $\mathbf{r}(t)$ can be reduced to solutions with a fixed value ($h = -1/2$ if the energy constant is negative) $\mathbf{r}(t) = \lambda\mathbf{r}(\lambda^{-1/2}t)$: $\lambda = h/h' = 2E$.

As can be seen from the tables, the moment of inertia of the obtained periodic orbits changes little, usually less than a few percent, so it is interesting to look at the obtained trajectories on the shape sphere. Figure **??** shows projections of trajectories from the table 2 with $\omega = 1/3$ on the plane of φ, θ.

Conclusion

Periodic orbits are determined by the variational method, these ones on the shape space, more precisely on the shape sphere, look in the case of $2-1$-symmetry look like small quasi-circles around the point of double collisions $\mathcal{C}_{12}$, or around the point of the corresponding Eulerian configuration $\mathcal{E}_3$; in the case of linear symmetry, where all three masses are arbitrary, quasi-circles can be located both around the points $\mathcal{C}_{12}$, $\mathcal{C}_{13}$, $\mathcal{C}_{23}$, and around the points $\mathcal{E}_1$, $\mathcal{E}_2$ or $\mathcal{E}_3$.

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Basic ideas of Topological Data Analysis

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Abstract. We give a brief outline of certain basic ideas of Topological Data Analysis. What is data? Does it have a meaningful shape? A topologist's view on the subject.

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A new proof of Maclagan's theorem

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Abstract. Maclagan's theorem [1] states that every antichain of monomial ideals is finite. We present a new proof of this theorem and discuss some of its consequences in the combinatorics of universal Gröbner bases. The proof is based on a sufficient condition that the set of ideals of a noetherian poset is again noetherian.

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